

Semilattice decompositions of trioids

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Abstract. We describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of s -simple subtrioids.

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1 Introduction

Trioids were introduced by J.-L. Loday and M. O. Ronco [1] for the study of ternary planar trees. Trialgebras, which are based on the notion of a trioid, have been studied in different papers (see, for example, [1–3]). It is well known that the notion of a trioid generalizes the notion of a dimonoid [4, 5]. Dimonoids play a prominent role in problems from the theory of Leibniz algebras. Trioids were studied in some papers of the author (see, for example, [6–8]). Note that if the operations of a trioid coincide then it becomes a semigroup. So, trioids are a generalization of semigroups.

In this work we describe semilattice decompositions of trioids. In Section 2 we give necessary definitions, auxiliary results (Proposition 1 and Lemma 1) and describe some connections between trioids and dimonoids (Lemma 2). Yamada [9] described all semilattice congruences on an arbitrary semigroup and proved that every semigroup is a semilattice of s -simple semigroups. These results were generalized to dimonoids in [10]. In Section 3 we extend results from [10] to the case of trioids (Theorems 1 and 2).

2 Preliminaries

A nonempty set T equipped with three binary associative operations \dashv , \vdash and \perp satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (T1)$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \quad (T2)$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \quad (T3)$$

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \quad (T4)$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \quad (T5)$$

$$(x \dashv y) \perp z = x \perp (y \vdash z), \quad (T6)$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \quad (T7)$$

$$(x \perp y) \vdash z = x \vdash (y \vdash z) \quad (T8)$$

for all $x, y, z \in T$, is called a trioid. If the operations of a trioid coincide, then the trioid becomes a semigroup.

Recall that a nonempty set T equipped with two binary associative operations \dashv and \vdash satisfying the axioms (T1) – (T3) is called a dimonoid (see, for example, [4, 5]).

Let (T, \perp) be an arbitrary semigroup. Define operations \dashv and \vdash on T by

$$x \dashv y = x, \quad x \vdash y = y$$

for all $x, y \in T$.

Proposition 1. ([8], Proposition 10). $(T, \dashv, \vdash, \perp)$ is a trioid.

The trioid $(T, \dashv, \vdash, \perp)$ will be denoted by T_{lr}^\perp .

Other examples of trioids can be found in [1, 6–8].

A commutative idempotent semigroup is called a semilattice.

Lemma 1. ([7], Lemma 1). The operations of a trioid $(T, \dashv, \vdash, \perp)$ coincide if (T, \dashv) is a semilattice.

Let $X = \{1, 2, 3\}$. For every pair $(x, y) \in X \times X$ let $T^{(x,y)} = (T, *_x, *_y)$ be an ordered triple, where T is a nonempty set and $*_x, *_y$ are binary operations on T . Let

$$B = \{(1, 1), (2, 2), (3, 3), (1, 2)\} \subset X \times X.$$

The following lemma describes connections between trioids and dimonoids.

Lemma 2. For any trioid $(T, *_1, *_2, *_3)$ the algebra $T^{(x,y)}$, $(x, y) \in X \times X$, is a dimonoid if $(x, y) \in B$. There exists some trioid $(T, *_1, *_2, *_3)$ for which the algebra $T^{(x,y)}$, $(x, y) \in X^2 \setminus B$, is not a dimonoid.

Proof. Let $(T, *_1, *_2, *_3)$ be a trioid. It is easy to see that the algebras $T^{(1,1)}$, $T^{(2,2)}$, $T^{(3,3)}$ and $T^{(1,2)}$ are dimonoids.

Now we shall prove the second part of the lemma.

Let $F[A]$ be the free semigroup on a set A and $F[A]_{lr}^\perp$ be a trioid (see Proposition 1) such that \perp is the concatenation on $F[A]$. Assume $(T, *_1, *_2, *_3) = F[A]_{lr}^\perp$ and show that for any $(x, y) \in X^2 \setminus B$ the algebra $T^{(x,y)}$ is not a dimonoid.

Let $w, u, \omega \in T^{(x,y)}$.

For $T^{(1,3)}$ check the axiom (T3):

$$(w *_1 u) *_3 \omega = w *_3 \omega = w\omega \neq wu\omega = w *_3 (u *_3 \omega).$$

As the axiom $(T3)$ does not hold, then $T^{(1,3)}$ is not a dimonoid.

For $T^{(2,1)}$, $T^{(2,3)}$, $T^{(3,1)}$ and $T^{(3,2)}$ check the axiom $(T1)$.

For $T^{(2,1)}$ we have

$$(w *_2 u) *_2 \omega = \omega \neq u = w *_2 u = w *_2 (u *_1 \omega).$$

For $T^{(2,3)}$:

$$(w *_2 u) *_2 \omega = \omega \neq u\omega = w *_2 (u *_3 \omega).$$

For $T^{(3,1)}$:

$$(w *_3 u) *_3 \omega = wu\omega \neq wu = w *_3 (u *_1 \omega).$$

For $T^{(3,2)}$:

$$(w *_3 u) *_3 \omega = wu\omega \neq w\omega = w *_3 (u *_2 \omega).$$

The axiom $(T1)$ does not hold for all fourth cases, so $T^{(2,1)}$, $T^{(2,3)}$, $T^{(3,1)}$ and $T^{(3,2)}$ are not dimonoids. \square

The notion of a triband of subtrioids was introduced and investigated in [7]. Recall this definition.

A trioid $(T, \dashv, \vdash, \perp)$ is called an idempotent trioid or a triband if $x \dashv x = x \vdash x = x \perp x = x$ for all $x \in T$. If $\varphi : S \rightarrow M$ is a homomorphism of trioids, then the corresponding congruence on S will be denoted by Δ_φ .

Let S be an arbitrary trioid, J be some idempotent trioid and

$$\alpha : S \rightarrow J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence Δ_α is a subtrioid of the trioid S , and the trioid S itself is a union of such trioids S_ξ , $\xi \in J$ that

$$\begin{aligned} x\alpha = \xi &\Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \mid (x, t) \in \Delta_\alpha\}, \\ S_\xi \dashv S_\varepsilon &\subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon}, \quad S_\xi \perp S_\varepsilon \subseteq S_{\xi \perp \varepsilon}, \\ \xi \neq \varepsilon &\Rightarrow S_\xi \cap S_\varepsilon = \emptyset. \end{aligned}$$

In this case we say that S is decomposable into a triband of subtrioids (or S is a triband J of subtrioids S_ξ , $\xi \in J$). If J is a band (=idempotent semigroup), then we say that S is a band J of subtrioids S_ξ , $\xi \in J$. If J is a commutative band, then we say that S is a semilattice J of subtrioids S_ξ , $\xi \in J$.

Observe that the notion of a triband of subtrioids generalizes the notion of a diband of subdimonoids [5] and the notion of a band of semigroups [11].

Examples of trioids which are decomposed into a triband of subtrioids can be found in [7].

3 Main results

In this section we describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of s -simple subtrioids.

Let $(T, \dashv, \vdash, \perp)$ be an arbitrary dimonoid. Yamada introduced the notion of a P -subsemigroup of an arbitrary semigroup (see [9]). We denote by Ω the collection of all P -subsemigroups of (T, \dashv) and by T_α, T_β, \dots the elements of Ω .

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a semilattice, then we say that ρ is a semilattice congruence.

For every subset Γ of Ω define a relation Γ_{\dashv} on $(T, \dashv, \vdash, \perp)$ by

$$a\Gamma_{\dashv}b \text{ if and only if } \{(x, y)|x \dashv a \dashv y \in T_\alpha\} = \{(x, y)|x \dashv b \dashv y \in T_\alpha\}$$

for every $T_\alpha \in \Gamma$.

Theorem 1. *The relation Γ_{\dashv} on any trioid $(T, \dashv, \vdash, \perp)$ is a semilattice congruence. Conversely, any semilattice congruence on $(T, \dashv, \vdash, \perp)$ can be obtained by this way.*

Proof. The fact that the relation Γ_{\dashv} is a semilattice congruence on a dimonoid (T, \dashv, \vdash) has been proved in [10]. Show that Γ_{\dashv} is compatible concerning the operation \perp .

Let $a\Gamma_{\dashv}b$, $a, b, c \in T$. As $a \dashv c\Gamma_{\dashv}b \dashv c$, then

$$\{(x, y)|x \dashv (a \dashv c) \dashv y \in T_\alpha\} = \{(x, y)|x \dashv (b \dashv c) \dashv y \in T_\alpha\}$$

for every $T_\alpha \in \Gamma$. By the associativity of the operation \dashv and the axiom (T4) of a trioid we obtain

$$\begin{aligned} x \dashv (a \dashv c) \dashv y &= ((x \dashv a) \dashv c) \dashv y = \\ &= (x \dashv (a \dashv c)) \dashv y = x \dashv (a \dashv c) \dashv y, \\ x \dashv (b \dashv c) \dashv y &= ((x \dashv b) \dashv c) \dashv y = \\ &= (x \dashv (b \dashv c)) \dashv y = x \dashv (b \dashv c) \dashv y. \end{aligned}$$

So, $a \dashv c\Gamma_{\dashv}b \dashv c$. Analogously, we can prove that $c \dashv a\Gamma_{\dashv}c \dashv b$. Thus, Γ_{\dashv} is a congruence on $(T, \dashv, \vdash, \perp)$.

As $(T, \dashv)/\Gamma_{\dashv}$ is a semilattice, then by Lemma 1 the operations of $(T, \dashv, \vdash, \perp)/\Gamma_{\dashv}$ coincide and so, it is a semilattice.

The converse statement follows from [9] (see also [10]). □

Theorem 1 generalizes Yamada's theorem [9] about the structure of all semilattice congruences on an arbitrary semigroup and the description [10] of all semilattice congruences on an arbitrary dimonoid.

A trioid $(T, \dashv, \vdash, \perp)$ will be called s -simple if its least semilattice congruence coincides with the universal relation on T .

Theorem 2. *The relation Ω_{\dashv} on any trioid $(T, \dashv, \vdash, \perp)$ is the least semilattice congruence. Every trioid $(T, \dashv, \vdash, \perp)$ is a semilattice of s -simple subtrioids.*

Proof. By Theorem 1 Ω_{\dashv} is a semilattice congruence. If $a\Omega_{\dashv}b$, $a, b \in T$, then it is easy to see that $a\Gamma_{\dashv}b$ for any $\Gamma \subseteq \Omega$. So, $\Omega_{\dashv} \subseteq \Gamma_{\dashv}$.

Now we shall prove the second statement of the theorem.

Since Ω_{\dashv} is a congruence on $(T, \dashv, \vdash, \perp)$ and $(T, \dashv, \vdash, \perp)/\Omega_{\dashv}$ is a semilattice, then

$$(T, \dashv, \vdash, \perp) \twoheadrightarrow (T, \dashv, \vdash, \perp)/\Omega_{\dashv} : x \mapsto [x]$$

is a homomorphism ($[x]$ is a class of the congruence Ω_{\dashv} which contains x). From [10] it follows that every class A of the congruence Ω_{\dashv} is an s -simple dimonoid concerning operations \dashv and \vdash . Hence we obtain s -simplicity of the subtrioid A of a trioid $(T, \dashv, \vdash, \perp)$. \square

Theorem 2 generalizes Yamada's theorem [9] about the structure of the least semilattice congruence on an arbitrary semigroup and the description [10] of the least semilattice congruence on an arbitrary dimonoid.

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