BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(71), 2013, Pages 125–129
ISSN 1024–7696

# The order of convexity for a general integral operator

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**Abstract.** In this paper, we consider the classes of the univalent functions denoted by $\mathcal{SH}(\beta)$, $\mathcal{SP}$ and $\mathcal{SP}(\alpha,\beta)$. On these classes we study the order of convexity of the integral operator $\int_0^z \left(te^{f(t)}\right)^{\gamma} dt$, where the function $f$ belongs to these classes.

**Mathematics subject classification:** 30C45, 30C75.
**Keywords and phrases:** Analytic function; Integral Operator; Starlike function; Convex function.

## 1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : \qquad |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f$ which are univalent in $\mathbb{U}$.
A function $f \in \mathcal{A}$ is the starlike function of order $\alpha$, $0 \le \alpha < 1$ if $f$ satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in \mathbb{U}.$$

We denote this class by $\mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is a convex function of order $\alpha$, $0 \le \alpha < 1$, if $f$ satisfies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \qquad z \in \mathbb{U}.$$

We denote this class by $\mathcal{K}(\alpha)$.

In [4], J. Stankiewicz and A. Wisniowska introduced the class of univalent functions $\mathcal{SH}(\beta)$, $\beta > 0$, defined by

$$\left|\frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2}-1\right)\right| < \mathrm{Re}\left\{\sqrt{2}\frac{zf'(z)}{f(z)}\right\} + 2\beta\left(\sqrt{2}-1\right) \tag{1}$$

for all $z \in \mathbb{U}$.

Also, in [3], F. Ronning introduced the class of univalent functions $\mathcal{SP}$, defined by

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \tag{2}$$

for all $z \in \mathbb{U}$.

The geometric interpretation of the relation (2) is that the class $\mathcal{SP}$ is the class of all functions $f \in \mathcal{S}$ for which the expression $zf'(z)/f(z)$, $z \in \mathbb{U}$, takes all values in the parabolic region

$$\begin{aligned}\Omega &= \{\omega : \quad |\omega - 1| \leq \mathrm{Re}\omega\}\\ &= \{\omega = u + iv : \quad v^2 \leq 2u - 1\}.\end{aligned}$$

In [2], F. Ronning introduced the class of univalent functions $\mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$, as the class of all functions $f \in \mathcal{S}$ which have the property

$$\left|\frac{zf'(z)}{f(z)} - (\alpha+\beta)\right| \leq \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) + \alpha - \beta, \tag{3}$$

for all $z \in \mathbb{U}$.

Geometric interpretation: $f \in \mathcal{SP}(\alpha, \beta)$ if and only if $zf'(z)/f(z)$, $z \in \mathbb{U}$, takes all values in the parabolic region

$$\begin{aligned}\Omega_{\alpha,\beta} &= \{\omega : \quad |\omega - (\alpha+\beta)| \leq \mathrm{Re}\omega + \alpha - \beta\}\\ &= \{\omega = u + iv : \quad v^2 \leq 4\alpha\,(u - \beta)\}.\end{aligned}$$

In the present paper, we will obtain the order of convexity of the following integral operator:

$$F(z) = \int_0^z \left(te^{f(t)}\right)^{\gamma} dt \tag{4}$$

where the function $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$.

*Remark* 1. The integral operator defined by (4) was introduced by Frasin and Ahmad in [1].

## 2 Main results

**Theorem 1.** *Let $f \in \mathcal{A}$ be in the class $\mathcal{SH}(\beta)$, $\beta > 0$ and $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1$ for all $z \in \mathbb{U}$. If* $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1$, *$z \in \mathbb{U}$, then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma| \left[(4\beta(\sqrt{2}-1) + \sqrt{2})M + 1\right]$$

*and*

$$|\gamma| \left[(4\beta(\sqrt{2}-1) + \sqrt{2})M + 1\right] < 1, \qquad \gamma \in \mathbb{C}.$$

*Proof.* We calculate for $F(z)$ the derivatives of the first and second order. From (4) we obtain

$$F'(z) = \left(ze^{f(z)}\right)^{\gamma}$$

and

$$F''(z) = \gamma \left(ze^{f(z)}\right)^{\gamma-1} \left(e^{f(z)} + zf'(z)e^{f(z)}\right).$$

After the calculus, we obtain that

$$\begin{aligned} \frac{zF''(z)}{F'(z)} &= \gamma\left(1 + zf'(z)\right) \\ &= \gamma\left(\frac{zf'(z)}{f(z)} f(z) + 1\right). \end{aligned} \tag{5}$$

It follows from (5) that

$$\begin{aligned} \left|\frac{zF''(z)}{F'(z)}\right| &\leq |\gamma| \left(\left|\frac{zf'(z)}{f(z)}\right| |f(z)| + 1\right) \\ &\leq \gamma\left(\left(\left|\frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2}-1\right)\right| + 2\beta\left(\sqrt{2}-1\right)\right)|f(z)| + 1\right). \end{aligned} \tag{6}$$

Because $f \in \mathcal{SH}(\beta)$, $\beta > 0$ and $|f(z)| \leq M$, $M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (6) the inequality (1) and we obtain

$$\begin{aligned} \left|\frac{zF''(z)}{F'(z)}\right| &\leq |\gamma| \left(\left(\mathrm{Re}\left\{\sqrt{2}\frac{zf'(z)}{f(z)}\right\} + 4\beta\left(\sqrt{2}-1\right)\right) M + 1\right) \\ &\leq |\gamma| \left(\left(\sqrt{2}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) + 4\beta\left(\sqrt{2}-1\right)\right) M + 1\right) \end{aligned}$$

From the hypothesis of Theorem 1 we have $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1$ and we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq |\gamma| \left[(4\beta(\sqrt{2}-1) + \sqrt{2})M + 1\right] = 1 - \delta$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$. $\square$

**Theorem 2.** *Let the function $f \in \mathcal{SP}$, where $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1$, $z \in \mathbb{U}$. If* $\operatorname{Re}\left(\frac{f'(z)}{f(z)}\right) \leq 1$, *$z \in \mathbb{U}$, then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma|(2M+1)$$

*and*

$$|\gamma|(2M+1) < 1, \qquad \gamma \in \mathbb{C}.$$

*Proof.* Following the same steps as in Theorem 1, we have

$$\frac{zF''(z)}{F'(z)} = \gamma\left(\frac{zf'(z)}{f(z)}f(z) + 1\right). \tag{7}$$

It follows from (7) that

$$\begin{aligned} \left|\frac{zF''(z)}{F'(z)}\right| &\leq |\gamma|\left(\left|\frac{zf'(z)}{f(z)}\right||f(z)| + 1\right) \\ &\leq \gamma\left(\left(\left|\frac{zf'(z)}{f(z)} - 1\right| + 1\right)|f(z)| + 1\right). \end{aligned} \tag{8}$$

Because $f \in \mathcal{SP}$ and $|f(z)| \leq M$, $M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (8) the inequality (2) and we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq |\gamma|\left(\left(\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) + 1\right)M + 1\right).$$

Because $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1$, we obtain that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq |\gamma|(2M+1) = 1 - \delta$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$. □

**Theorem 3.** *Let the function $f \in \mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$, where $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1$, $z \in \mathbb{U}$. If* $\operatorname{Re}\left(\frac{f'(z)}{f(z)}\right) \leq 1$, *$z \in \mathbb{U}$ then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma|[(1+2\alpha)M + 1)$$

*and*

$$|\gamma|[(1+2\alpha)M + 1) < 1, \qquad \gamma \in \mathbb{C}.$$

*Proof.* From the proof of Theorem 1, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq |\gamma|\left(\left|\frac{zf'(z)}{f(z)}\right||f(z)| + 1\right)$$

$$\leq |\gamma| \left( \left( \left| \frac{zf'(z)}{f(z)} - (\alpha+\beta) \right| + (\alpha+\beta) \right) |f(z)| + 1 \right). \tag{9}$$

Because $f \in \mathcal{SP}(\alpha,\beta)$, $\alpha > 0$, $\beta \in [0,1)$ and $|f(z)| \leq M$, $M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (9) the inequality (3) and we obtain

$$\begin{aligned} \left| \frac{zF''(z)}{F'(z)} \right| &\leq |\gamma| \left( \left( \operatorname{Re}\left( \frac{zf'(z)}{f(z)} \right) + \alpha - \beta + \alpha + \beta \right) M + 1 \right) \\ &\leq |\gamma| \left( \left( \operatorname{Re}\left( \frac{zf'(z)}{f(z)} \right) + 2\alpha \right) M + 1 \right). \end{aligned}$$

Because $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1$, $z \in \mathbb{U}$, we obtain that

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq |\gamma| \left[ (1+2\alpha)M + 1 \right] = 1 - \delta$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$. $\square$

**Acknowledgements.** This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

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*Received February 17, 2012*

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