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Asymptotic Stability of Infinite-Dimensional Nonautonomous Dynamical Systems

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Abstract. This paper is dedicated to the study of the problem of asymptotic stability for general non-autonomous dynamical systems (both with continuous and discrete time). We study the relation between different types of attractions and asymptotic stability in the framework of general non-autonomous dynamical systems. Specially we investigate the case of almost periodic systems, i.e., when the base (driving system) is almost periodic. We apply the obtained results we apply to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Functional Differential Equations (both with finite retard and neutral type) and Semi-Linear Parabolic Equations.

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1 Introduction

The aim of this paper is the study the problem of asymptotic stability (both local and global) for non-autonomous differential systems. We study this problem in the framework of general *non-autonomous dynamical systems* (NDS). We formulate and prove our results for general (abstract) non-autonomous dynamical systems. We apply the obtained results to the study the problem of asymptotic stability for ordinary differential equations (ODEs), functional-differential equations (FDEs) and semi-linear parabolic equations (SLPEs).

Let $\mathbb{R} := (-\infty, +\infty)$, E be a Banach space with the norm $|\cdot|$, W be an open subset of E containing the origin, $C(\mathbb{R} \times W, E)$ be the space of all continuous functions $f : \mathbb{R} \times W \mapsto E$ equipped with compact open topology.

Consider a differential equation

$$u' = f(t, u), \tag{1}$$

where $f \in C(\mathbb{R} \times W, E)$. Denote by $(C(\mathbb{R} \times W, E), \mathbb{R}, \sigma)$ the *shift dynamical system* [7, 14] on the space $C(\mathbb{R} \times W, \mathbb{R}^n)$ (*dynamical system of translations* or *Bebutov's dynamical system*), i.e. $\sigma(\tau, f) := f_\tau$ for any $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R} \times W, E)$, where $f_\tau(t, x) := f(t + \tau, x)$ for any $(t, x) \in \mathbb{R} \times W$.

Below we will use the following conditions:

(A): for any $(t_0, x_0) \in \mathbb{R}_+ \times W$ equation (1) admits a unique solution $x(t; t_0, x_0)$ with initial data (t_0, x_0) defined on $\mathbb{R}_+ := [0, +\infty)$, i.e. $x(t_0; t_0, x_0) = x_0$;

(B): the right hand side f *is positively compact* if the set $\Sigma_f^+ := \{f_\tau : \tau \in \mathbb{R}_+\}$ is a relatively compact subset of $C(\mathbb{R} \times W, E)$;

(C): the equation

$$v' = g(t, v), \quad g \in \Omega_f \tag{2}$$

is called a *limiting equation* for (1), where Ω_f is the ω-limit set of f with respect to the shift dynamical system $(C(\mathbb{R} \times W, E), \mathbb{R}, \sigma)$, i.e. $\Omega_f := \{g :$ there exists a sequence $\{\tau_k\} \to +\infty$ such that $f_{\tau_k} \to g$ as $k \to \infty\}$;

(D): *equation* (1) (or its right hand side f) *is regular* if for all $p \in H^+(f)$ the equation

$$x' = p(t, x)$$

admits a unique solution $\varphi(t, x_0, p)$ defined on $\mathbb{R}_+$ with initial condition $\varphi(0, x_0, p) = x_0$, where $H^+(f) := \overline{\{f_\tau : \tau \in \mathbb{R}_+\}}$ and by bar the closure in the space $C(\mathbb{R} \times W, E)$ is denoted;

(E): equation (1) admits a null (trivial) solution, i.e. $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$.

The null solution of equation (1) is said to be:

1. *uniformly stable* if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t, u, f_\tau)| < \varepsilon$ for any $t, \tau \in \mathbb{R}_+$;
2. *uniformly attracting*, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, u, f_\tau)| = 0$$

   uniformly with respect to $|u| \le a$ and $\tau \in \mathbb{R}_+$;
3. *uniformly asymptotically stable* if it is uniformly stable and uniformly attracting;
4. *globally asymptotically stable* if it is asymptotically stable and

$$\lim_{t \to +\infty} |\varphi(t, v, g)| = 0$$

   for any $(v, g) \in E \times H^+(f)$, where $\varphi(t, v, g)$ is a unique solution of equation (2) with initial data $\varphi(0, v, g) = v$.

The main results are contained in the following three theorems. The firs two (Theorems 1 and 2) are related to equation (1) and the third (Theorem 3) to equation (1) with almost periodic right hand side f.

Let E be a Banach space with the norm $|\cdot|$.

Theorem 1. *Let $f \in C(\mathbb{R}\times E, E)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
4. *the cocycle φ generated by equation (1) is locally compact, i.e. for every point $u \in E$ there exists a neighborhood U of the point u and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact.*

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:

1. $$\lim_{t\to+\infty} \sup_{v\in K, g\in\Omega_f} |\varphi(t, v, g)| = 0$$
   for every compact subset K of E;
2. *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (2) is relatively compact on $\mathbb{R}_+$.*

Theorem 1 generalizes a statement (Theorem 2.6) established in the work [2] for finite-dimensional equation (1) (see also [13, Ch.I] and the bibliography therein).

Theorem 2. *Let $f \in C(\mathbb{R}\times E, E)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t,0) = 0$ for all $t \in \mathbb{R}_+$;*
4. *the cocycle φ generated by equation (1) is completely continuous, i.e., for every bounded subset $M \subseteq E$ there exists a positive number l such that the set $\varphi(l, M, H^+(f))$ is relatively compact.*

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:

a) for every $g \in \Omega_f$ limiting equation (2) does not admit nontrivial bounded on $\mathbb{R}$ solutions;

b) for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (2) is bounded on $\mathbb{R}_+$.

Recall that a function $f \in C(\mathbb{R} \times W, E)$ is called *almost periodic* (respectively, *almost recurrent*) in $t \in \mathbb{R}$ uniformly in u on every compact subset K of W if for an arbitrary number $\varepsilon > 0$ and compact subset $K \subseteq W$ there exists a positive number $L = L(K, \varepsilon)$ such that on every segment $[a, a+L]$ $(a \in \mathbb{R})$ of the length L there exists at least one number τ such that

$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+s+\tau, u) - f(t+s)| < \varepsilon$$

(respectively,

$$\max_{u \in K,\ |t| \le 1/\varepsilon} |f(t+\tau, u) - f(t, u)| < \varepsilon)$$

for all $s \in \mathbb{R}$. If the function $f \in C(\mathbb{R} \times W, E)$ is almost recurrent and $H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$ is compact, then f is called *recurrent* (in $t \in \mathbb{R}$ uniformly in u on every compact subset K of W).

Theorem 3. *Suppose that the following conditions are fulfilled:*

1. *the function $f \in C(\mathbb{R} \times W, E)$ is recurrent in $t \in \mathbb{R}$ uniformly in u on every compact subset of W;*
2. *$f(t, 0) = 0$ for all $t \in \mathbb{R}_+$;*
3. *the function f is regular;*
4. *the cocycle φ associated by equation (1) is asymptotically compact;*
5. *the null solution of equation (1) is uniformly stable;*
6. *there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t, u, f)| = 0$$

for any $|u| \le a$.

Then the null solution of equation (1) is asymptotically stable.

Remark 1. For finite-dimensional equation (1) with almost periodic hand right side f Theorem 3 was established by Z. Artstein [3] (see also [1, 12] and [13, Ch.I]).

We establish also analogical results for the functional-differential equations and for semi-linear parabolic equations.

The paper is organized as follows.

In Section 2 we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, recurrence, shift dynamical systems, cocycles, non-autonomous dynamical systems, etc) and facts from the theory of dynamical systems which will be needed in this paper.

Section 3 is devoted to the analysis of different types of stabilities for non-autonomous dynamical systems (NDSs). We prove that from the uniform attractivity the uniform asymptotic stability follows. It is proved that for an asymptotically compact dynamical system the asymptotic stability and the uniform asymptotic stability are equivalent. We formulate and prove some tests of asymptotic stability (global asymptotic stability) for infinite-dimensional NDSs (Theorem 6, Theorem 7 and Theorem 8).

In Section 4 we present some results about NDSs with minimal base (driving system). The main result of this Section (Theorem 11) gives a sufficient condition of global asymptotic stability for this type of systems.

Finally, Section 5 contains a series of applications of our general results from Sections 3-4 for Ordinary Differential Equations (Theorem 12, Theorem 13 and Theorem 14), Functional-Differential Equations (both Functional-Differential Equations with finite delay (Theorem 17, Theorem 18 and Theorem 19) and Neutral Functional-Differential Equations (Theorem 20)) and Semi-Linear Parabolic Equations (Theorem 21, Theorem 22 and Theorem 23).

2 Some Notions and Facts from Dynamical Systems

2.1 Stable and asymptotically stable sets. Global attractors and Levinson center

Let (X,ρ) be a complete metric space with the metric ρ, $\mathbb{R}$ ($\mathbb{Z}$) be the group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be the semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T}\subseteq\mathbb{S}$ be one of the sub-semigroups $\mathbb{R}_+$ (respectively, $\mathbb{Z}_+$) or $\mathbb{R}$ (respectively, $\mathbb{Z}$).

A triplet $(X,\mathbb{T},\pi)$, where $\pi:\mathbb{T}\times X\to X$ is a continuous mapping satisfying the following conditions: $\pi(0,x)=x$ and $\pi(s,\pi(t,x))=\pi(s+t,x)$ is called a *dynamical system.* If $\mathbb{T}=\mathbb{R}$ ($\mathbb{R}_+$) or $\mathbb{Z}$ ($\mathbb{Z}_+$), then $(X,\mathbb{T},\pi)$ is called a *group (semi-group) dynamical system.* In the case when $\mathbb{T}=\mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X,\mathbb{T},\pi)$ is called a *flow*, but if $\mathbb{T}\subseteq\mathbb{Z}$, then $(X,\mathbb{T},\pi)$ is called a *cascade (discrete flow).*

The function $\pi(\cdot,x):\mathbb{T}\to X$ is called a *motion* passing through the point x at moment $t=0$ and the set $\Sigma_x:=\pi(\mathbb{T},x)$ is called a *trajectory* of this motion.

A nonempty set $M\subseteq X$ is called *positively invariant* (respectively, *negatively invariant*, *invariant*) with respect to dynamical system $(X,\mathbb{T},\pi)$ or, simply, positively invariant (respectively, negatively invariant, invariant) if $\pi(t,M)\subseteq M$ ($M\subseteq\pi(t,M),\pi(t,M)=M$) for every $t\in\mathbb{T}_+:=\{t\in\mathbb{T}:\ t\geq 0\}$.

A closed positively invariant set (respectively, invariant set) which does not contain own closed positively invariant (respectively, invariant) subset is called *minimal.*

Let $M\subseteq X$. The set

$$\Omega(M):=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi(\tau,M)}$$

is called ω-limit for M. If the set M consists of single point x, i.e. $M=\{x\}$, then $\Omega(\{x\}):=\omega_x$ is called the ω-limits set of the point x.

The set $W^s(\Lambda)$, defined by the equality

$$W^s(\Lambda):=\{x\in X|\lim_{t\to+\infty}\rho(\pi(t,x),\Lambda)=0\}$$

is called a *stable manifold* (or *domain of attraction*) of the set $\Lambda \subseteq X$.

The set M is called:

- *orbitally stable*, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \geq 0$;
- *attracting* if there exists $\gamma > 0$ such that $B(M, \gamma) \subset W^s(M)$, where $B(M, \gamma) := \{x \in X : \ \rho(x, M) < \gamma\}$;
- *asymptotically stable* if it is orbital stable and attracting;
- *global asymptotic stable*, if it is asymptotically stable and $W^s(M) = X$;
- *uniformly attracting* if there exists $\gamma > 0$ such that

$$\lim_{t \to +\infty} \sup_{x \in B(M, \gamma)} \rho(\pi(t, x), M) = 0.$$

The system $(X, \mathbb{T}, \pi)$ is called:

– *point dissipative* if there exists a nonempty compact subset $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(\pi(t, x), K) = 0; \tag{3}$$

– *compactly dissipative* if equality (3) takes place uniformly in x on the compact subsets from X;

– *locally dissipative* if for any point $p \in X$ there exists $\delta_p > 0$ such that equality (3) takes place uniformly in $x \in B(p, \delta_p)$;

– *bounded dissipative* if equality (3) holds uniformly in x on every bounded subset of X;

– *locally completely continuous* (*compact*) if for any point $p \in X$ there are two positive numbers δ_p and l_p such that the set $\pi(l_p, B(p, \delta_p))$ is relatively compact.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a compact set attracting every compact subset of X. Let us set

$$J = \Omega(K). \tag{4}$$

It can be shown [7, Ch.I] that the set J defined by equality (4) does not depend on the choice of the attractor K, but it is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called the *Levinson center* of the compactly dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Lemma 1 (see [8])**.** *Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $x \in X$ be a point with relatively compact semi-trajectory $\Sigma_x^+ := \{\pi(t, x) : \ t \geq 0\}$. Then the following statements hold:*

1. *the dynamical system* $(X, \mathbb{T}, \pi)$ *induces on* $H^+(x) := \overline{\Sigma_x^+}$ *a dynamical system* $(H^+(x), \mathbb{T}_+, \pi)$*;*

2. *the dynamical system* $(H^+(x), \mathbb{T}_+, \pi)$ *is compactly dissipative;*

3. *the Levinson center* $J_{H^+(x)}$ *of* $(H^+(x), \mathbb{T}_+, \pi)$ *coincides with the* ω*-limit set* ω_x *of the point* x*.*

2.2 Almost periodic and recurrent points (motions)

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*.

Remark 2. Suppose that the phase space X of dynamical system $(X, \mathbb{T}, \pi)$ is not a metric-space, but it is a pseudo metric space with the family of pseudo-metrics $\mathcal{P}$. For $\varepsilon > 0$ and $\rho \in \mathcal{P}$ a number τ is called (ε, ρ)-shift (respectively, (ε, ρ)-almost period) of $x \in X$, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$). Now it is easy to modify the notion of almost recurrence (respectively, almost periodicity, recurrence) for a pseudo-metric space.

2.3 Bebutov's dynamical system

Let X, W be two metric spaces. Denote by $C(\mathbb{T} \times W, X)$ the space of all continuous mappings $f : \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and by σ the mapping from $\mathbb{T} \times C(\mathbb{T} \times W, X)$ into $C(\mathbb{T} \times W, X)$ defined by the equality $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where f_τ is the τ-translation (shift) of f with respect to variable t, i.e. $f_\tau(t, x) = f(t + \tau, x)$ for all $(t, x) \in \mathbb{T} \times W$. Then [7, Ch.I],[15, Ch.I] the triplet $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$ is a dynamical system on $C(\mathbb{T} \times W, X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

A function $f \in C(\mathbb{T} \times W, X)$ is said to be *almost periodic* (respectively, *recurrent* in $t \in \mathbb{T}$ uniformly in $x \in W$ on every compact subset of W) if $f \in C(\mathbb{T} \times W, X)$ is an almost periodic (respectively, recurrent) point of the Bebutov's dynamical system $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$.

2.4 Cocycles

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on the metric space Y. Recall that a triplet $\langle W, \varphi, (Y, T_2, \sigma) \rangle$ (or shortly φ), where W is a metric space and φ is a mapping from $\mathbb{T}_1 \times W \times Y$ into W, is said to be a *cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W if the following conditions are fulfilled:

1. $\varphi(0,u,y) = u$ for all $u \in W$ and $y \in Y$;

2. $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau \in \mathbb{T}_1$, $u \in W$ and $y \in Y$;

3. the mapping $\varphi : \mathbb{T}_1 \times W \times Y \mapsto W$ is continuous.

Example 1. *Consider differential equation (1) with regular right hand side $f \in C(\mathbb{R}\times W,\mathbb{R}^n)$, where $W \subseteq \mathbb{R}^n$. Denote by $(H^+(f),\mathbb{R}_+,\sigma)$ a semi-group shift dynamical system on $H^+(f)$ induced by Bebutov's dynamical system $(C(\mathbb{R}\times W,\mathbb{R}^n),\mathbb{R},\sigma)$, where $H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}_+\}}$. Let $\varphi(t,u,g)$ be a unique solution of the equation*

$$y' = g(t,y), \quad (g \in H^+(f)),$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

1. *$\varphi(0,u,g) = u$ for all $u \in W$ and $g \in H^+(f)$;*

2. *$\varphi(t+\tau,u,g) = \varphi(t,\varphi(\tau,u,g),g_\tau)$ for all $t,\tau \in \mathbb{R}_+$, $u \in W$ and $g \in H^+(f)$;*

3. *the mapping $\varphi : \mathbb{R}_+ \times W \times H^+(f) \mapsto W$ is continuous.*

From above it follows that the triplet $\langle W,\varphi,(H^+(f),\mathbb{R}_+,\sigma)\rangle$ is a cocycle over $(H^+(f),\ \mathbb{R}_+,\ \sigma)$ with the fiber $W \subseteq \mathbb{R}^n$. Thus, every non-autonomous equation (1) with regular f naturally generates a cocycle which plays a very important role in the qualitative study of equation (1).

Suppose that $W \subseteq E$, where E is a Banach space with the norm $|\cdot|$, $0 \in W$ (0 is the null element of E) and the cocycle $\langle W,\varphi,(Y,T_2,\sigma)\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t,0,y) = 0$ for all $t \in \mathbb{T}_1$ and $y \in Y$.

The trivial motion/solution of cocycle φ is said to be:

1. *uniformly stable*, if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0,\varepsilon)$) such that $|u| < \delta$ implies $|\varphi(t,u,y)| < \varepsilon$ for all $t \geq 0$ and $y \in Y$;

2. *uniformly attracting* if there exists a positive number a such that

$$\lim_{t\to+\infty} |\varphi(t,u,y)| = 0$$

uniformly with respect to $|u| \leq a$ and $y \in Y$;

3. *uniformly asymptotically stable* if it is uniformly stable and uniformly attracting.

2.5 Nonautonomous Dynamical Systems (NDS)

Recall [7] that a triplet $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be a *nonautonomous dynamical system* (NDS), where $(X,\mathbb{T}_1,\pi)$ (respectively, $(Y,\mathbb{T}_2,\sigma)$) is a dynamical system on X (respectively, Y) and h is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$.

Below we will give some examples of nonautonomous dynamical systems which play a very important role in the study of nonautonomous differential equations.

Example 2. (***NDS generated by cocycle***.) *Note that every cocycle $\langle W,\ \varphi,\ (Y,\ \mathbb{T}_2,\ \sigma)\rangle$ naturally generates a NDS. In fact, let $X := W\times Y$ and $(X,\mathbb{T}_1,\pi)$ be a skew-product dynamical system on X (i.e. $\pi(t,x) := (\varphi(t,u,y),\sigma(t,y))$ for all $t\in\mathbb{T}_1$ and $x := (u,y)\in X$). Then the triplet $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$, where $h := pr_2 : X\mapsto Y$ is the second projection (i.e. $h(u,y) = y$ for all $u\in W$ and $y\in Y$), is a NDS.*

Remark 3. There are Examples of NDS which are not generated by cocycles (see, for instance, [8]).

Let (X,h,Y) be a vector bundle [11]. Denote by θ_y the null element of the vectorial space $X_y := \{x\in X :\ h(x) = y\}$ and $\Theta := \{\theta_y :\ y\in Y\}$ the null section of (X,h,Y).

A vectorial bundle (X,h,Y) is said to be *locally trivial* with fiber F if for every point $y\in Y$ there exists a neighborhood U of the point y (U is an open subset of Y containing y) such that $h^{-1}(U)$ and $U\times F$ are homeomorphic, i.e. there exists a homeomorphism $\alpha : h^{-1}(U)\mapsto U\times F$ (trivialization).

Lemma 2 (see [8])**.** *Let (X,h,Y) be a vector bundle and Θ be its null section. Suppose that the following conditions hold:*

1. *the space Y is compact;*
2. *the vectorial bundle (X,h,Y) is locally trivial.*

Then the trivial section Θ is compact.

Consider a NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ on the vector bundle (X,h,Y). Everywhere in this paper we suppose that the null section Θ of (X,h,Y) is a positively invariant set, i.e. $\pi(t,\theta)\in\Theta$ for all $\theta\in\Theta$ and $t\geq 0$ ($t\in\mathbb{T}_1$).

The *null (trivial) section* Θ of NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be:

1. *uniformly stable* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(t,x)| < \varepsilon$ for all $t\geq 0$ ($t\in\mathbb{T}_1$);
2. *attracting* if there exists a number $\nu > 0$ such that $B(\Theta,\nu)\subseteq W^s(\Theta)$, where $B(\Theta,\nu) := \{x\in X :\ |x| < \nu\}$;
3. *uniformly attracting* if there exists a number $\nu > 0$ such that

$$\lim_{t\to+\infty}\sup\{|\pi(t,x)| :\ |x|\leq\nu\} = 0;$$

4. *asymptotically stable* (respectively, *uniformly asymptotically stable*) if Θ is uniformly stable and attracting (respectively, uniformly attracting);

5. globally asymptotically (respectively, uniformly asymptotically) stable if Θ is asymptotically (respectively, uniformly asymptotically) stable and $W^s(\Theta) = X$.

3 Some Tests of Global Asymptotical Stability of NDS

Let $(Y, \mathbb{T}_2, \sigma)$ be a compactly dissipative dynamical system, J_Y its Levinson center and $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h\rangle$ be a NDS. Denote by $\tilde{X} := h^{-1}(J_Y) = \{x \in X :\ h(x) = y \in J_Y\}$, then evidently the following statements are fulfilled:

1. $\tilde{X}$ is closed;

2. $\pi(t, \tilde{X}) \subseteq \tilde{X}$ for all $t \in \mathbb{T}_1$ and, consequently, on the set $\tilde{X}$ a dynamical system $(\tilde{X}, \mathbb{T}_1, \pi))$ is induced by $(X, \mathbb{T}_1, \pi)$;

3. the triplet $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h\rangle$ is a NDS.

A dynamical system $(X, \mathbb{T}_1, \pi)$ is said to be:

1. *completely continuous (compact)* if for every bounded subset $B \subseteq X$ there exists a number $l = l(B) > 0$ such that the set $\pi(l, M)$ is relatively compact, where $\pi(l, M) := \{\pi(l, x) :\ x \in M\}$;

2. *locally completely continuous* (*locally compact*) if for every point $p \in X$ there exit positive numbers $l = l(p)$ and $\delta = \delta(p)$ such that the set $\pi(l, B(p, \delta))$ is relatively compact, where $B(p, \delta) := \{x \in X :\ \rho(x, p) < \delta\}$;

3. *asymptotically compact* if for any positively invariant subset $M \subseteq X$ there exists a compact subset $K \subseteq X$ such that $\lim\limits_{t\to+\infty} \beta(\pi(t, M), K) = 0$, where $\beta(A, B) := \sup\limits_{a\in A} \rho(a, B)$ and $\rho(a, B) := \inf\limits_{b\in B} \rho(a, b)$.

Remark 4. 1. The dynamical system $(X, \mathbb{T}_1, \pi)$ is completely continuous if one of the following conditions is fulfilled:

1. the space X possesses the property of Heine-Borel, i.e. every bounded set $B \subseteq X$ is relatively compact;

2. for some $t_0 \in T_1$ the mapping $\pi^{t_0} : X \mapsto X$, defined by the equality $\pi^{t_0}(x) := \pi(t_0, x)$ ($\forall\ x \in X$) is completely continuous, i.e. for any bounded subset B of X the set $\pi^{t_0}(B)$ is relatively compact.

2. Every completely continuous dynamical system $(X, \mathbb{T}_1, \pi)$ is locally completely continuous and asymptotically compact.

3. Let $(X, \mathbb{T}, \pi)$ be a dynamical system associated by cocycle $\langle (W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ and Y be a compact space. Then $(X, \mathbb{T}, \pi)$ is asymptotically compact if and only

if for every bounded sequence $\{u_n\} \subseteq W$, $\{y_n\} \subseteq Y$ and $t_n \to +\infty$ the sequence $\{\varphi(t_n, u_n, y_n)\}$ is relatively compact if it is bounded. In this case the cocycle φ is called asymptotically compact.

Theorem 4 (see [8]). *Let $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h\rangle$ be a NDS. Suppose that the following conditions are fulfilled:*

1. *Y is compact;*
2. *the dynamical system $(X, \mathbb{T}_1, \pi)$ is locally compact;*
3. *the trivial section Θ of (X, h, Y) is positively invariant;*
4. *the trivial section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{T}_1, \pi),\ (J_Y, \mathbb{T}_2, \sigma),\ h\rangle$ is uniformly attracting.*

Then the trivial section Θ of non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi),$ $(Y, \mathbb{T}_2, \sigma),\ h\rangle$ is uniformly stable.

Remark 5. Theorem 4 remains true:

1. if we replace the condition of uniform attraction of Θ by the following one: there exists a positive number $\tilde{\alpha}$ such that for any compact subset $K \subseteq B[\tilde{\Theta}, \tilde{\alpha}]$ we have
$$\lim_{t\to+\infty} \sup\{|\pi(t, x)| :\ x \in K\} = 0,$$
where $B[M, r] := \{x \in X :\ \rho(x, M) \le r\}$;
2. if we replace the condition of local compactness for $(X, \mathbb{T}_1, \pi)$ by the following: there are positive numbers α and l such that the set $\pi(l, B(\Theta, \alpha))$ is relatively compact, where $B(M, r) := \{x \in X :\ \rho(x, M) < r\}$.

Corollary 1 (see [8]). *Under the conditions of Theorem 4 the trivial section Θ of NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is uniformly asymptotically stable.*

Theorem 5. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h)\rangle$ be a NDS and the following conditions hold:*

1. *the trivial section Θ of (X, h, Y) is positively invariant;*
2. *Y is compact.*

Then the following statements are equivalent:

a) *$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h)\rangle$ is compactly dissipative and its Levinson center J_X is included in Θ;*

b) *the trivial section Θ is globally asymptotically stable;*

c) *the equality*
$$\lim_{t\to+\infty} |\pi(t, x)| = 0$$
holds for all $x \in X$ uniformly in x on every compact subset M of X.

Proof. Suppose that condition a. is fulfilled. We will show that Θ is globally asymptotically stable. Under condition a. it is sufficient to show that Θ is stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $0 < \delta_n \to 0$, $|x_n| < \delta_n$ and $t_n \to +\infty$ such that

$$|\pi(t_n, x_n)| \geq \varepsilon_0. \tag{5}$$

By Lemma 2 the set Θ is compact, then the sequence $\{x_n\}$ is relatively compact. Since $(X, \mathbb{T}_1, \pi)$ is compactly dissipative, then the sequence $\{\pi(t_n, x_n)\}$ is relatively compact. Thus, without loss of generality, we can suppose that the sequence $\{\pi(t_n, x_n)\}$ is convergent. Denote by $\bar{x} := \lim_{n\to\infty} \pi(t_n, x_n)$. Then $\bar{x} \in J_X \subseteq \Theta$ and, consequently, $|\bar{x}| = 0$. On the other hand, passing to limit in (5) as $n \to \infty$ we obtain $0 = |\bar{x}| \geq \varepsilon_0$. The obtained contradiction proves our statement.

Now we will prove that condition b) implies a). Indeed, according to Theorem 3.6 [6] the set Θ is orbitally stable. By Theorem 1.13 [7, Ch.I] the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is included in Θ.

Suppose that condition c). is fulfilled. We will show that c. implies a). Let M be an arbitrary compact subset of X, then by condition c). we have the following equality

$$\lim_{t\to+\infty} \sup_{x\in M} \rho(\pi(t,x), \Theta) = 0. \tag{6}$$

In fact

$$\rho(\pi(t,x), \Theta) \leq \rho(\pi(t,x), \theta_{h(\pi(t,x))}) = |\pi(t,x)| \leq \max_{x\in M} |\pi(t,x)| \to 0$$

as $t \to +\infty$. Since the sets M and Θ are compact, then by Lemma 1.3 [7, Ch.I] we have:

1. the set $\Sigma_M^+ := \bigcup\{\pi(t,x) : \ t \geq 0,\ x \in M\}$ is relatively compact;
2. the set $\Omega(M)$ is nonempty, compact and invariant;
3. $$\lim_{t\to+\infty} \sup_{x\in M} \rho(\pi(t,x), \Omega(M)) = 0. \tag{7}$$

From (6) and (7) we obtain $\Omega(M) \subseteq \Theta$ for any compact subset M of X, i.e. the compact subset Θ attracts every compact subset M of X. This means that the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and, evidently, its Levinson center J_X is included in Θ, i.e. c) implies a).

Finally we will establish the implication a) $\Rightarrow$ c). Suppose that it is not true, then there are a compact subset $M_0 \subseteq X$, a sequence $\{x_n\} \subseteq M_0$, $t_n \to +\infty$ and $\varepsilon_0 > 0$ such that

$$|\pi(t_n, x_n)| \geq \varepsilon_0. \tag{8}$$

Since $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and Y is compact, then without loss of generality, we can consider that the sequences $\{\pi(t_n, x_n)\}$ and $\{\sigma(t_n, y_n)\}$ are convergent, where $y_n := h(x_n)$. Denote by $\bar{y} = \lim_{n\to\infty} \sigma(t_n, y_n)$ and $\bar{x} = \lim_{n\to\infty} \pi(t_n, x_n)$, then $\bar{x} \in J_X$ and $h(\bar{x}) = \bar{y}$. Since $J_X \subseteq \Theta$, then $|\bar{x}| = 0$. Taking into account the last equality and passing to limit in (8) as $n \to \infty$ we will have $\varepsilon_0 \le 0$. The obtained contradiction proves our statement. Theorem is proved. □

A continuous mapping $\gamma : \mathbb{S} \mapsto X$ is called *an entire motion (trajectory)* of the semi-group dynamical system $(X, \mathbb{T}, \pi)$ passing through the point x if $\gamma(0) = x$ and $\pi(t, \gamma(s)) = \gamma(t+s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

Denote by $\mathcal{F}_x(\pi)$ the set of all entire trajectories of $(X, \mathbb{T}, \pi)$ passing through the point x and $\mathcal{F}(\pi) := \bigcup_{x \in X} \mathcal{F}_x(\pi)$.

Theorem 6. *Let Y be a compact metric space and $(X, \mathbb{T}_1, \pi)$ be asymptotically compact. The following statements hold:*

1. *if the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is globally asymptotically stable, then:*
   - a) *every motion of $(X, \mathbb{T}_1, \pi)$ is bounded on $\mathbb{T}_1^+$, i.e. $\sup_{t \in \mathbb{T}_1^+} |\pi(t,x)| < +\infty$ for all $x \in X$, where $\mathbb{T}_1^+ := \{t \in \mathbb{T}_1 : \ t \ge 0\}$;*
   - b) *the dynamical system $(X, \mathbb{T}_1, \pi)$ does not have nontrivial entire bounded on $\mathbb{S}$ motions;*
2. *if $(X, \mathbb{T}_1, \pi)$ is locally compact, then under conditions a) and b) the trivial section Θ of NDS $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$ is globally asymptotically stable.*

Proof. Let Y be compact, $(X, \mathbb{T}_1, \pi)$ be asymptotically compact and the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be globally asymptotically stable. According to Theorem 5 the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is included in Θ. Hence, every positive semi-trajectory $\Sigma_x^+ := \{\pi(t,x) : \ t \ge 0\}$ is relatively compact and, in particular, it is bounded. Let now $\gamma \in \mathcal{F}(\pi)$ be an arbitrary entire trajectory of dynamical system $(X, \mathbb{T}_1, \pi)$ bounded on $\mathbb{S}$. Since the dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact, then $\gamma(\mathbb{S})$ is relatively compact. Taking into account that the Levinson center J_X is a maximal compact invariant set of dynamical system $(X, \mathbb{T}_1, \pi)$, then $\gamma(\mathbb{S}) \subseteq J_X \subseteq \Theta$. Thus the first statement of the theorem is proved.

Now we will establish the second statement of the theorem. From condition a) and asymptotical compactness of $(X, \mathbb{T}_1, \pi)$ it follows that every semi-trajectory Σ_x^+ is relatively compact and, consequently, every ω–limit set ω_x $(x \in X)$ is non-empty, compact and invariant. Note that $\omega_x \subseteq \Theta$. In fact, let $x \in X$ and $p \in \omega_x$ be an arbitrary point from ω_x. Since the set ω_x is compact and invariant, then there exists an entire trajectory $\gamma \in \mathcal{F}_x$ such that $\gamma(\mathbb{S}) \subseteq \omega_x$. According to condition b. we have $\gamma(0) = p \in \gamma(\mathbb{S}) \subseteq \Theta$. Thus we established the inclusion $\Omega_X := \overline{\bigcup \{\omega_x : \ x \in X\}} \subseteq \Theta$. This means that the dynamical system $(X, \mathbb{T}_1, \pi)$ is point dissipative. By Theorem 1.10 [7, Ch.I] it is also compactly dissipative. Let J_X be its Levinson center and $x \in J_X$. Since J_X is a compact invariant set of dynamical system $(X, \mathbb{T}_1, \pi)$, then there exists an entire motion $\gamma \in \mathcal{F}_x$ such that $\gamma(\mathbb{S}) \subseteq J_X$. According to condition b. we obtain $x \in \gamma(\mathbb{S}) \subseteq \Theta$ and, consequently, $J_X \subseteq \Theta$. Now to finish the proof of Theorem it is sufficient to apply Theorem 5. □

Remark 6. 1. Under the conditions of Theorem 6 condition a) is equivalent to the following one: $\lim_{t\to+\infty} |\pi(t,x)| = 0$ for all $x \in X$.-

2. It is not difficult to check that Theorem 6 remains true if we replace condition b) by the following one:

$b')$ the dynamical system $(\tilde{X}, \mathbb{T}_1, \pi)$ does not have nontrivial entire bounded on $\mathbb{S}$ motions.

The second statement of Remark 6 directly follows from Theorem 6. In fact if $\gamma \in \mathcal{F}(\pi)$ is a bounded on $\mathbb{S}$ motion of $(X, \mathbb{T}_1, \pi)$, then under the conditions of Theorem 6 the set $\gamma(S)$ is relatively compact and, consequently, $\nu := h \circ \gamma$; (i.e. $\nu(s) := h(\gamma(s))\ \forall\ s \in \mathbb{S}$) is an entire trajectory with relatively compact rank $\nu(\mathbb{S})$. This means that $\nu(\mathbb{S}) \subseteq J_Y$ and, consequently, $\gamma(\mathbb{S}) \subseteq \tilde{X}$.

From Theorem 6 and Remark 4 follows the following statement follows immediately.

Corollary 2. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a NDS and the following conditions hold:*

1. *Y is compact;*
2. *the dynamical system $(X, \mathbb{T}_1, \pi)$ is completely continuous.*

Then the trivial section Θ is globally asymptotically stable if and only if conditions a) and b) of Theorem 6 hold.

Remark 7. Corollary 2 was established in [4] in the particular case when (X, h, Y) is finite-dimensional and Y is a compact and invariant set.

Theorem 7. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a NDS and Y be compact. The trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section $\tilde{\Theta}$ of $\langle (\tilde{X}, \mathbb{T}_1, \pi), (J_Y, \mathbb{T}_2, \sigma), h\rangle$ is globally asymptotically stable;*
2. *for any compact subset $K \subseteq X$ the set Σ^+_K is relatively compact.*

Proof. Necessity. Suppose that the trivial section Θ of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is globally asymptotically stable, then by Theorem 5 the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is contained in Θ. Since the Levinson center J_Y of $(Y, \mathbb{T}_2, \sigma)$ is its maximal compact invariant set, then the set $\tilde{\Theta}$ is also invariant and, consequently, $J_X = \tilde{\Theta}$. Taking into account that $\Theta \supseteq \tilde{\Theta} = J_X$, then it is easy to check that $\tilde{\Theta}$ is a globally asymptotically stable set of NDS

$\langle(\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$. To finish the proof of the first statement it is sufficient to note that since the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative, then by Theorem 1.5 [7, Ch.I] for every compact subset $K\subseteq X$ the set Σ_K^+ is relatively compact.

Sufficiency. Let the trivial section $\tilde{\Theta}$ of NDS $\langle(\tilde{X},\mathbb{T}_1,\pi)$, $(J_Y,\mathbb{T}_2,\sigma),h\rangle$ be globally asymptotically stable. By Theorem 5 the dynamical system $(\tilde{X},\mathbb{T}_1,\pi)$ is compactly dissipative and its Levinson center $J_{\tilde{X}}$ is included in $\tilde{\Theta}$. Reasoning as in the proof of the first statement of Theorem and taking into account the invariance of the set J_Y we conclude that $J_{\tilde{X}}=\tilde{\Theta}$. Now we will establish that the dynamical system $(X,\mathbb{T}_1,\pi)$ is also compactly dissipative. To prove this statement, according to Theorem 1.15 [7, Ch.I], it is sufficient to establish that $(X,\mathbb{T}_1,\pi)$ is point dissipative. Let x be an arbitrary point of X, since the positive semi-trajectory Σ_x^+ of x is relatively compact, then its ω–limit set ω_x is a non-empty, compact, invariant set, and

$$\lim_{t\to+\infty}\rho(\pi(t,x),\omega_x)=0.$$

Note that $h(\omega_x)\subseteq J_Y$, since J_Y is a maximal compact invariant set of $(Y,\mathbb{T}_2,\sigma)$, and, consequently, $\omega_x\subseteq\tilde{X}$. On the other hand $\tilde{\Theta}$ is a maximal compact invariant set of $(\tilde{X},\mathbb{T}_2,\sigma)$, hence $\omega_x\subseteq\tilde{\Theta}$. Thus $\Omega_X:=\overline{\{\omega_x:\ x\in X\}}$ is a compact set, i.e. the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative and, consequently, it is compactly dissipative, too. Let now J_X be its Levinson center, then $h(J_X)\subseteq J_Y$ and, consequently, $J_X\subseteq\tilde{X}$. On the other hand, $J_{\tilde{X}}=\tilde{\Theta}$ is a maximal compact set of $(\tilde{X},\mathbb{T}_1,\pi)$ and, consequently, $J_X\subseteq\tilde{\Theta}$. Now we will prove that the set Θ is uniformly stable. Suppose that it is not true, then there are $\delta_n\to0$ $(\delta_n>0)$, $\{x_n\}\subseteq X$ and $t_n\to+\infty$ such that

$$|x_n|<\delta_n\ \text{ and }\ |\pi(t_n,x_n)|\geq\varepsilon_0\tag{9}$$

for any $n\in\mathbb{N}$. By Lemma 2 Θ is a compact set and the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative, then without loss of generality, we can suppose that the sequences $\{x_n\}$ and $\{\pi(t_n,x_n)\}$ are convergent. Denote by x_0 (respectively, by $\bar{x}_0$) the limit of $\{x_n\}$ (respectively, $\{\pi(t_n,x_n)\}$). Then by (9) we have $x_0\in\Theta$ and $|\bar{x}|\geq\varepsilon_0>0$. On the other hand $\bar{x}\in J_X\subseteq\tilde{\Theta}$ and, consequently, $|\bar{x}|=0$. The obtained contradiction proves our statement. Let now x be an arbitrary point from X, then $\lim\limits_{t\to+\infty}|\pi(t,x)|=0$. In fact if we suppose the contrary, then there exist $x_0\in X$, $\varepsilon_0>0$, and $t_n\to+\infty$ such that

$$|\pi(t_n,x_0)|\geq\varepsilon_0\tag{10}$$

for any $n\in\mathbb{N}$. Since the semi-trajectory $\Sigma_{x_0}^+$ of x_0 is relatively compact, then we can suppose that the sequence $\{\pi(t_n,x_0)\}$ is convergent. Let $\bar{x}_0$ be its limit, then from (10) we have $|\bar{x}_0|\geq\varepsilon_0>0$. On the other hand, $\bar{x}_0\in\omega_{x_0}\subseteq J_X\subseteq\tilde{\Theta}$ and, consequently, $|\bar{x}_0|=0$. The obtained contradiction completes the proof of the global asymptotic stability of trivial section Θ. Theorem is proved. □

Theorem 8. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS, Y be compact and $(X,\mathbb{T}_1,\pi)$ be locally compact. The trivial section Θ of $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section $\tilde{\Theta}$ of $\langle (\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable;*
2. *for any $x\in X$ the set Σ_x^+ is relatively compact.*

Proof. The necessity of Theorem follows from Theorem 7. To prove the sufficiency, according to Theorem 7, it is enough to show that the set Σ_K^+ is relatively compact for any compact subset $K\subseteq X$. To this end we note (reasoning as in the proof of Theorem 7) that the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative. Since dynamical system $(X,\mathbb{T}_1,\pi)$ is locally compact, then by Theorem 1.10 [7, Ch.I] this system is also compactly dissipative. Due to Theorem 1.15 [7, Ch.I] for any compact subset $K\subseteq X$ the set Σ_K^+ is relatively compact. □

Corollary 3. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a NDS, Y be compact and $(X,\mathbb{T}_1,\pi)$ be completely continuous. The trivial section Θ of $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable if and only if the following conditions hold:*

1. *the trivial section $\tilde{\Theta}$ of $\langle (\tilde{X},\mathbb{T}_1,\pi),(J_Y,\mathbb{T}_2,\sigma),h\rangle$ is globally asymptotically stable;*
2. *for any $x\in X$ the set Σ_x^+ is bounded.*

Proof. This statement follows directly from Theorem 8. To this end it is sufficient to note that every completely continuous dynamical system is locally compact and every bounded semi-trajectory Σ_x^+ is relatively compact if $(X,\mathbb{T}_1,\pi)$ is completely continuous. □

Lemma 3. *Suppose that the following conditions hold:*

1. *$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is a NDS;*
2. *Y is compact;*
3. *the trivial section Θ of (X,h,Y) is positively invariant.*

Then the following two statements are equivalent:

1. *Θ is uniformly stable;*
2. *Θ is orbitally stable with respect to $(X,\mathbb{T}_1,\pi)$.*

Proof. Let Θ be uniformly stable, then it is orbitally stable with respect to $(X,\mathbb{T}_1,\pi)$. If we suppose that it is not true, then there are $\varepsilon_0>0$, $0<\delta_n\to 0$, $\{x_n\}$ and $t_n\to+\infty$ such that

$$\rho(x_n,\Theta)<\delta_n \ \text{ and } \ \rho(\pi(t_n,x_n),\Theta)\ge\varepsilon_0. \tag{11}$$

Since Θ is compact then, without loss of generality, we can suppose that the sequence $\{x_n\}$ is convergent. Denote its limit by x_0, then $y_0 = \lim_{n\to\infty} y_n$, where $y_n := h(x_n)$. Denote by $\delta_0 = \delta(\varepsilon_0/2)$ a positive number chosen for $\varepsilon_0/2$ from the uniform stability of Θ, i.e. $|x| < \delta_0$ implies $|\pi(t,x)| < \varepsilon_0/2$ for all $t \geq 0$ ($t \in \mathbb{T}_1$). Since $|x_n| = \rho(x_n, \theta_{y_n}) \leq \rho(x_n, \theta_{y_0}) + \rho(\theta_{y_0}, \theta_{y_n}) \to 0$ as $n \to \infty$. Thus, there exists a number $n_0 \in \mathbb{N}$ such that $|x_n| < \delta_0$ for all $n \geq n_0$ and, consequently, we obtain

$$|\pi(t_n, x_n)| < \varepsilon_0/2. \tag{12}$$

On the other hand from (11) we receive

$$|\pi(t_n, x_n)| \geq \rho(\pi(t_n, x_n), \Theta) \geq \varepsilon_0. \tag{13}$$

The inequalities (12) and (13) are contradictory. The obtained contradiction proves our statement.

Now we will show that from the orbital stability of Θ it follows that it is uniformly stable. This statement may be proved using the same reasoning as in the proof of Theorem 5. □

Let $M \subset X$. Denote by $D^+(M) := \bigcap_{\varepsilon>0} \overline{\bigcup\{\pi(t, B(M,\varepsilon)) | t \geq 0\}}$.

Theorem 9. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system, Y be a compact metric space, (X, h, Y) be a finite-dimensional vector bundle and Θ be its null section. If Θ is uniformly stable, then the following properties are equivalent:*

1. *for every $\varepsilon > 0$ and $x \in X$ there exists a number $\tau = \tau(\varepsilon, x) > 0$ such that $|\pi(\tau, x)| < \varepsilon$;*
2. *for every $\varepsilon > 0$ and $x \in X$ there exists a number $l = l(\varepsilon, x) > 0$ such that $|\pi(t, x)| < \varepsilon$ for any $t \geq l$;*
3. *the dynamical system $(X, \mathbb{T}_1, \pi)$ is point dissipative and $\Omega_X \subseteq \Theta$;*
4. *$\omega_x \bigcap \Theta \neq \emptyset$ for any $x \in X$;*
5. *for any $\varepsilon > 0$ and $r > 0$ there exists $L = L(\varepsilon, r) > 0$ such that*

$$|\pi(t,x)| < \varepsilon \text{ for any } t \geq L(\varepsilon, r) \quad \text{and} \quad |x| \leq r. \tag{14}$$

Proof. It is easy to check that, under the conditions of Theorem, the following implications 2. $\Longleftrightarrow$ 3. $\Rightarrow$ 4. $\Longleftrightarrow$ 1. hold. Now we will establish the implication 4. $\Rightarrow$ 3. To this end we note that by Lemma 3 the set Θ is orbitally stable and, consequently, $D^+(\Theta) = \Theta$. According to Theorem 1.13 [7, Ch.I] the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and its Levinson center J_X is included in $D^+(\Theta)$. Thus we obtain $J_X \subseteq \Theta$. Since $(X, \mathbb{T}_1, \pi)$ is point dissipative and $\Omega_X \subseteq J_X$ we obtain the necessary statement.

To finish the proof of Theorem it is sufficient, for example, to show that 3. $\Longleftrightarrow$ 5. The implication 5. $\Rightarrow$ 3. is evident. According to condition 3. the dynamical system $(X, \mathbb{T}_1, \pi)$ is point dissipative and $\Omega_X \subseteq \Theta$. By Theorem 1.10 [7, Ch.I] the dynamical system $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and $J_X = D^+(\Omega_X) \subseteq \Theta$, since the set Θ is uniformly stable. Since the Levinson center J_X attracts every compact subset of J_X we have (14). Indeed if we suppose that it is not true, then there are $\varepsilon_0 > 0$, $r_0 > 0$, $\{x_n\}$ and $t_n \to +\infty$ such that

$$|x_n| \le r_0 \quad \text{and} \quad |\pi(t_n, x_n)| \ge \varepsilon_0 \tag{15}$$

for any $n \in \mathbb{N}$. Since Y is compact, (X, h, Y) is finite-dimensional and $(X, \mathbb{T}_1, \pi)$ is compact dissipative, then we can suppose that the sequence $\{\pi(t_n, x_n)\}$ is convergent. Denote by $\bar{x}$ its limit, then passing to limit in (15) we obtain $|\bar{x}| \ge \varepsilon_0 > 0$. On the other hand $\bar{x} \in J_X \subseteq \Theta$ and, consequently, $|\bar{x}| = 0$. The obtained contradiction completes the proof of Theorem. □

Remark 8. 1. Note that Theorem 9 remains true also for the infinite-dimensional case too (i.e. (X, h, Y) is infinite-dimensional) if we suppose that the dynamical system $(X, \mathbb{T}_1, \pi)$ is completely continuous.

2. Theorem 9 remains true if we replace the uniform stability of the set Θ by the uniform stability of $\tilde{\Theta} = h^{-1}(J_Y) \bigcap \Theta$.

4 Asymptotic Stability of NDS with Minimal Base

In this section we suppose that the complete metric space Y is compact and the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal, i.e. every trajectory $\Sigma_y := \{\sigma(t, y) : t \in \mathbb{T}_2\}$ is dense in Y (this means that $H(y) = Y$ for any $y \in Y$, where $H(y) := \overline{\Sigma}_y$).

Theorem 10. *Suppose that the following conditions are fulfilled:*

1. *the trivial section Θ is uniformly stable with respect to NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$;*
2. *$L^+(X) = X$, where $L^+(X) := \{x \in X : \Sigma_x^+$ is relatively compact $\}$;*
3. *there exists a point $y_0 \in Y$ such that $X_{y_0}^s = X_{y_0}$, where $X_y := \{x \in X : h(x) = y\}$ and $X_y^s := \{x \in X_y : \lim_{t \to +\infty} |\pi(t, x)| = 0\}$.*

Then $X_y^s = X_y$ for any $y \in Y$.

Proof. Suppose that there exists $\tilde{y} \in Y$ such that $X_{\tilde{y}}^s \neq X_{\tilde{y}}$ and let $\tilde{x} \in X_{\tilde{y}} \setminus X_{\tilde{y}}^s$. Since $\Sigma_{\tilde{x}}^+$ is relatively compact, then the ω-limit set $\omega_{\tilde{x}}$ of the point $\tilde{x}$ is a nonempty compact and invariant set. According to the choice of the point $\tilde{x}$ there exists at least one point $\bar{x} \in \omega_{\tilde{x}}$ such that $|\bar{x}| \neq 0$. Let $\gamma \in \mathcal{F}_{\bar{x}}(\pi)$ be an entire trajectory of $(X, \mathbb{T}_1, \pi)$ passing through the point $\bar{x}$ at initial moment with the condition $\gamma(\mathbb{S}) \subseteq \omega_{\tilde{x}}$. We will show that

$$\alpha := \inf_{s \le 0} |\gamma(s)| > 0. \tag{16}$$

If we suppose that (16) is not true, then there exists a sequence $s_n \to -\infty$ such that $|\gamma(s_n)| \to 0$ as $n \to \infty$. Since Θ is uniformly stable then for all $0 < \varepsilon < |\bar{x}|/2$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $|x| < \delta$ implies the inequality $|\pi(t,x)| < \varepsilon$ for all $t \geq 0$. Let $n_0 \in \mathbb{N}$ be a sufficiently large number (such that $|\gamma(s_n)| < \delta$ for all $n \geq n_0$), then we have $|\bar{x}| = |\pi(-s_{n_0}, \gamma(s_{n_0}))| < \varepsilon < |\bar{x}|/2$. The obtained contradiction proves our statement. Denote by ν the entire trajectory of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ defined by the equality $\nu := h \circ \gamma$, i.e. $\nu(s) = h(\gamma(s))$ for all $s \in \mathbb{S}$, then $\nu \in \mathcal{F}_{\bar{y}}(\sigma)$, where $\bar{y} := h(\bar{x})$. Since Y is minimal, then there exists a sequence $\{\tau_n\}$ from $\mathbb{S}$ such that $\tau_n \to -\infty$ and $\nu(\tau_n) \to y_0$. Under the conditions of Theorem, without loss of generality, we may suppose that the functional sequences $\{\gamma(t+\tau_n)\}_{t\in\mathbb{S}}$ and $\{\nu(t+\tau_n)\}_{t\in\mathbb{S}}$ are convergent (uniformly with respect to t on every compact subset of $\mathbb{S}$). Let $\tilde{\gamma}$ (respectively, $\tilde{\nu}$) be the limit of the sequence $\{\gamma(t+\tau_n)\}_{t\in\mathbb{S}}$ (respectively, $\{\nu(t+\tau_n)\}_{t\in\mathbb{S}}$). Then it is clear that $\tilde{\gamma} \in \mathcal{F}_{\tilde{\gamma}(0)}(\pi)$, $\tilde{\gamma}(\mathbb{S}) \subseteq \alpha_\gamma := \{z :$ there exists a sequence $s_n \to -\infty$ such that $\gamma(s_n) \to z\}$ and $|\tilde{\gamma}(s)| \geq \alpha$ for all $s \in \mathbb{S}$. On the other hand $\tilde{\gamma}(t) = \pi(t, \tilde{\gamma}(0))$ for any $t \geq 0$, $\tilde{\gamma}(0) \in X_{y_0}$ and, consequently, $\lim\limits_{t\to+\infty} |\pi(t, \tilde{\gamma}(0))| = 0$. The obtained contradiction completes the proof of Theorem. □

Lemma 4. *Suppose that the trivial section Θ is uniformly stable with respect to NDS $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$. Let $y_0 \in Y$ be an arbitrary point, then the following conditions are equivalent:*

1. $X^s_{y_0} = X_{y_0}$;
2. *for every $x \in X_{y_0}$ the semi-trajectory Σ^+_x is relatively compact and $\omega_x \subseteq \Theta$;*
3. $\omega_x \bigcap \Theta \neq \emptyset$ *for any* $x \in X_{y_0}$;
4. *for arbitrary $\varepsilon > 0$ and $x \in X_{y_0}$ there exists a positive number $\tau = \tau(x, \varepsilon)$ such that $|\pi(\tau, x)| < \varepsilon$.*

Proof. Note that the implications 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 4. are evident. To finish the proof of Lemma it is sufficient to show that 4. implies 1.. Indeed, let $\varepsilon > 0$ be an arbitrary positive number, $x \in X$, $\varepsilon_k := 1/k$ $(k \in \mathbb{N})$, and τ_k be a positive number such that $|\pi(\tau_k, x)| < 1/k$. Denote by $\delta(\varepsilon)$ the positive number from the uniform stability of Θ for ε (i.e. $|x| < \delta$ implies $|\pi(t,x)| < \varepsilon$ for any $t \geq 0$), then for the sufficiently large k $(1/k < \delta)$ we have $|\pi(t+\tau_k, x)| < \varepsilon$ for any $t \geq 0$. Thus for $\varepsilon > 0$ there exists $l(\varepsilon, x) > 0$ such that $|\pi(t,x)| < \varepsilon$ for any $t \geq l(\varepsilon, x)$, i.e. $x \in X^s_{y_0}$. □

Remark 9. 1. The implications 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 4. are true without assumption of uniform stability of Θ.

2. Lemma 4 remains true without compactness and minimality of Y.

From Theorem 10 and Lemma 4 we have the following statement.

Corollary 4. *Suppose that the following conditions are fulfilled:*

1. *the trivial section* Θ *is uniformly stable with respect to NDS* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$;

2. $L^+(X) = X$;

3. *there exists a point* $y_0 \in Y$ *such that one of the conditions 1.–4. of Lemma 4 is fulfilled.*

Then $X^s_y = X_y$ *for all* $y \in Y$.

Below we give a local version of Theorem 10.

Theorem 11. *Suppose that the following conditions are fulfilled:*

1. *the dynamical system* $(X,\mathbb{T}_1,\pi)$ *is asymptotically compact;*

2. *the trivial section* Θ *is uniformly stable with respect to NDS* $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$;

3. *there exist positive number* δ_0 *and point* $y_0 \in Y$ *such that* $B(\theta_{y_0},\delta_0) \subset X^s_{y_0}$, *where* $B(\theta_y,r) := \{x \in X_y : \ |x| < r\}$.

Then the trivial section Θ *is asymptotically stable, i.e. there exists a positive number* β *such that* $B(\Theta,\beta) \subset X^s$, *where* $B(\Theta,\beta) := \bigcup\{B(\theta_y,\beta) : \ y \in Y\}$ *and* $X^s := \bigcup\{X^s_y : \ y \in Y\}$.

Proof. Since Θ is uniformly stable, then there exists a positive number δ_1 such that $|\pi(t,x)| \le \delta_0$ for any $t \ge 0$ and $x \in X$ with $|x| \le \delta_1$. Let now $\beta := \min\{\delta_0,\delta_1\}$. We will show that $B(\Theta,\beta) \subset X^s$. If we suppose that it is not so, then using the same reasoning as in the proof of Theorem 10 we obtain a contradiction which proves our statement. □

Remark 10. All results of Sections 3–4 remain true if:

1. we replace the positive invariance of the trivial section Θ by the following condition: there exists a compact positively invariant set $M \subseteq X$ such that $M_y := \{x \in M : \ h(x) = y\}$ consists of a single point for any $y \in Y$;

2. we the compact metric space Y by an arbitrary compact regular topological space.

5 Some Applications

5.1 Ordinary differential equations

Consider a differential equation

$$u' = f(t,u), \tag{17}$$

where $f \in C(\mathbb{R} \times W, E)$.

Applying general results from Sections 3-4 we will obtain a series of results for equation (17). Below we formulate some of them.

Denote by $\Omega_f := \{g \in H^+(f)$: there exists a sequence $\tau_n \to +\infty$ such that $g = \lim_{n\to\infty} f_{\tau_n}\}$ the ω-limit set of f.

A trivial solution of equation (17) is called *uniformly attracting* (respectively, *eventually uniformly attracting* [2]) if for every compact subset $K \subset E$ and for every $\varepsilon > 0$ there exists $L = L(K,\varepsilon) > 0$ (respectively, there exist $\gamma = \gamma(K) > 0$ and $L = L(K,\varepsilon) > 0$) such that

$$x_0 \in K,\ t \geq t_0 + L \ \text{ implies } \ |x_f(t; t_0, x_0)| < \varepsilon$$

(respectively,

$$x_0 \in K,\ t_0 \geq \gamma,\ \ t \geq t_0 + L \ \text{ implies } \ |x_f(t; t_0, x_0)| < \varepsilon),$$

where by $x_f(t; t_0, x_0)$ a unique solution $x(t)$ of equation (17) with initial data $x(t_0) = x_0$ is denoted.

The solutions of equation (17) are said to be *uniformly bounded* [2] if for any $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that

$$|x_0| \leq \alpha,\ \ t_0 \in \mathbb{R}_+,\ t \geq t_0 \ \Rightarrow \ |x_f(t; t_0, x_0)| \leq \beta.$$

Lemma 5. *Suppose that the following conditions are fulfilled:*

1. $f \in C(\mathbb{R} \times E, E)$*;*
2. *the function* f *is regular;*
3. *the set* $H^+(f)$ *is compact;*
4. $f(t,0) = 0$ *for any* $t \in \mathbb{R}_+$*.*

Let φ *be a cocycle, generated by equation (17) (see Example 1), then the following statements hold:*

1. *if the trivial solution of equation (17) is uniformly attracting, then the trivial solution/motion of the cocycle* φ *is uniformly attracting;*
2. *if the trivial solution of equation (17) is eventually uniformly attracting, then the trivial solution/motion of the cocycle* φ *possesses the following property:*

$$\lim_{t\to+\infty} \max_{x\in K, g\in\Omega_f} |\varphi(t,x,g)| = 0 \tag{18}$$

   for any compact subset K *of* E*;*

3. *if the solutions of equation (17) are uniformly bounded, then the solutions/motions of the cocycle* φ *are uniformly bounded, i.e. for any* $\alpha > 0$ *there exists* $\beta = \beta(\alpha) > 0$ *such that* $|x| \leq \alpha$ *implies* $|\varphi(t,x,g)| \leq \beta$ *for any* $t \in \mathbb{R}_+$ *and* $g \in H^+(f)$*.*

Proof. The first statement of Lemma is well known [14, Ch.VIII].

To prove the second statement we note that $\varphi(t, x, f_{t_0}) = x(t + t_0; t_0, x)$ for any $t, t_0 \in \mathbb{R}_+$ and $x \in E$. Let now K be an arbitrary compact subset of E and $\varepsilon > 0$ be an arbitrary positive number. Denote by $\gamma = \gamma(K)$ and $L = L(K, \varepsilon)$ positive numbers from eventually uniform attractivity of null solution for equation (17). Let now $x \in K$ and $g \in \Omega_f$, then there exists a sequence $t_n \to +\infty$ such that $f_{t_n} \to g$ (in the space $C(\mathbb{R} \times E, E)$) and, consequently, $t_n \geq \gamma$ for sufficiently large n. Note that

$$|\varphi(t, x, g)| = \lim_{n \to +\infty} |\varphi(t, x, f_{t_n})| = \lim_{n \to +\infty} |x_f(t + t_n; t_n, x)| \leq \varepsilon \tag{19}$$
$$\text{for all} \quad t \geq L(K, \varepsilon).$$

From (19) evidently follows (18).

Finally we will prove the third statement. Let $\alpha > 0$ and $\beta = \beta(\alpha) > 0$ is taken from the uniform boundedness of the solutions of (17). Let $|x| \leq \alpha$, $g \in H^+(f)$ and $t \in \mathbb{R}_+$, then there exists a sequence $\{t_n\} \subseteq \mathbb{R}_+$ such that $g = \lim_{t \to +\infty} f_{t_n}$. Note that

$$|\varphi(t, x, g)| = \lim_{n \to \infty} |\varphi(t, x, f_{t_n})| = \lim_{n \to \infty} |x_f(t + t_n; t_n, x)| \leq \beta(\alpha).$$

Lemma is completely proved. □

Theorem 12. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t, 0) = 0$ for any $t \in \mathbb{R}_+$;*
4. *the cocycle φ generated by equation (17) is locally compact, i.e. for every point $u \in E$ there exists a neighborhood U of the point u and a positive number l such that the set $\varphi(l, U, H^+(f))$ is relatively compact.*

Then the null solution of equation (17) is globally asymptotically stable if and only if the following conditions hold:

1. $$\lim_{t \to +\infty} \sup_{v \in K, g \in \Omega_f} |\varphi(t, v, g)| = 0 \tag{20}$$
   for every compact subset K of E;
2. *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (2) is relatively compact on $\mathbb{R}_+$.*

Proof. Consider the dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center (maximal compact invariant set) $J_{H^+(f)}$ by Lemma 1 coincides with ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote by $X := W \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ, i.e. $\pi(t,(v,g)) := (\varphi(t,v,g), \sigma(t,g))$ for all $t \in \mathbb{R}_+$ and $(v,g) \in X$. Now consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h \rangle$ ($h := pr_2$) associated with equation (17). It is easy to check that under the conditions of Theorem 12 this NDS possesses the following properties:

1. by Lemma 1 the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compactly dissipative and its Levinson center J_Y coincides with Ω_f;
2. the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h \rangle$ coincides with $\{0\} \times Y$;
3. Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
4. according to (20) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h \rangle$ is uniformly attracting because $|\pi(t,x)| = |\varphi(t,v,g)|$ for any $t \in \mathbb{R}_+$ and $x := (v,g) \in X$;
5. every trajectory $\Sigma^+_{(u,g)}$ ($(u,g) \in E \times H^+(f)$) of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$, generated by equation (17), is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 5 and Theorem 8. □

Corollary 5. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t,0) = 0$ for any $t \in \mathbb{R}_+$;*
4. *the cocycle φ generated by equation (17) is locally compact;*
5. *the null solution of equation (17) is eventually uniformly attracting;*
6. *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t,v,g)$ of equation (2) is relatively compact on $\mathbb{R}_+$.*

Then the null solution of equation (17) is globally asymptotically stable.

Proof. This statement follows from Theorem 12. Indeed, according to Lemma 5 from the uniform eventual attraction of the null solution of equation (17) follows condition (20). Now to finish the proof of this statement it is sufficient to apply Theorem 12. □

Remark 11. 1. For finite-dimensional equation (17) Corollary 5 generalizes a statement (Theorem 2.6) established in the work [2] (see also [13, Ch.I] and the bibliography therein).

2. If the cocycle φ associated with equation (17) is asymptotically compact (in particular if it is completely continuous), then Theorem 12 remains true if we replace condition (ii) by the following one: for any $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ is bounded on $\mathbb{R}_+$.

Theorem 13. *Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t, 0) = 0$ for any $t \in \mathbb{R}_+$;*
4. *the cocycle φ generated by equation (17) is completely continuous, i.e. for every bounded subset $M \subseteq E$ there exists a positive number l such that the set $\varphi(l, M, H^+(f))$ is relatively compact.*

Then the null solution of equation (17) is globally asymptotically stable if and only if the following conditions hold:

a) *for every $g \in \Omega_f$ limiting equation (2) does not have nontrivial bounded on $\mathbb{R}$ solutions;*

b) *for every $v \in E$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (2) is bounded on $\mathbb{R}_+$.*

Proof. This statement follows from Corollary 2 and can be proved using the same arguments as in the proof of Theorem 12. □

Remark 12. Theorem 13 remains true if we replace the completely continuity by the following two conditions:

1. the cocycle φ is asymptotically compact:
2. the cocycle φ is locally completely continuous.

Theorem 14. *Suppose that the following conditions are fulfilled:*

1. *the function $f \in C(\mathbb{R} \times W, E)$ is recurrent in $t \in \mathbb{R}$ uniformly in u on every compact subset from W;*
2. *$f(t, 0) = 0$ for any $t \in \mathbb{R}_+$;*
3. *the function f is regular;*
4. *the cocycle φ associated with equation (17) is asymptotically compact;*

5. *the null solution of equation (17) is uniformly stable;*

6. *there exists a positive number a such that*

$$\lim_{t\to+\infty} |\varphi(t,u,f)| = 0$$

for any $|u| \le a$.

Then the null solution of equation (17) is asymptotically stable.

Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 12. □

Remark 13. For finite-dimensional equation (17) with almost periodic right hand side f Theorem 14 was established by Z. Artstein [3] (see also [1,12] and [13, Ch.I]).

5.2 Difference equations

Consider a difference equation

$$u(t+1) = f(t,u(t)), \tag{21}$$

where $f \in C(\mathbb{Z} \times W, E)$.

Along with equation (21) we consider the family of equations

$$v(t+1) = g(t,v(t)), \tag{22}$$

where $g \in H^+(f) := \overline{\{f_\tau :\ \tau \in \mathbb{Z}_+\}}$. Let $\varphi(t,v,g)$ be a unique solution of equation (22) with initial data $\varphi(0,v,g) = v$. Denote by $(H^+(f), \mathbb{Z}_+, \sigma)$ the shift dynamical system on $H^+(f)$, then the triplet $\langle W, \varphi, (H^+(f), \mathbb{Z}_+, \sigma)\rangle$ is a cocycle (with discrete time) over $(H^+(f), \mathbb{Z}_+, \sigma)$ with the fibre W.

Applying the results from Sections 3-4 we will obtain a series of results for difference equation (21). Below we formulate two of them.

Theorem 15. *Let $f \in C(\mathbb{Z} \times W, E)$. Assume that the following conditions are fulfilled:*

1. *the set $H^+(f)$ is compact;*

2. *$f(t,0) = 0$ for any $t \in \mathbb{Z}_+$;*

3. *there exists a neighborhood U of 0 and a positive number l such that $\varphi(l,U,H^+(f))$ is relatively compact;*

4. *there exists a positive number a such that*

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0.$$

Then the null solution of equation (21) is uniformly asymptotically stable.

Theorem 16. *Let $f \in C(\mathbb{Z} \times W, E)$. Assume that the following conditions are fulfilled:*

1. *the function $f \in C(\mathbb{Z}_+ \times W, E)$ is recurrent in $t \in \mathbb{Z}_+$ uniformly in u on every compact subset of W;*
2. *$f(t, 0) = 0$ for any $t \in \mathbb{Z}_+$;*
3. *the null solution of equation (21) is uniformly stable;*
4. *there exists a positive number a such that*

$$\lim_{t\to+\infty} |\varphi(t, u, f)| = 0$$

for any $|u| \leq a$.

Then the null solution of equation (21) is uniformly asymptotically stable.

5.3 Functional-differential equations

We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

5.3.1 Functional-differential equations (FDEs) with finite delay

Let us first recall some notions and notations from [9]. Let $r > 0$, $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a, b] = [-r, 0]$, then we set $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for any $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t + \theta)$, $-r \leq \theta \leq 0$. Consider a functional differential equation

$$\dot{u} = f(t, u_t), \tag{23}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^n$ equipped with the compact open topology. On the space $C(\mathbb{R}\times\mathcal{C}, \mathbb{R}^n)$ is defined (see, for example,[7, ChI] and [15, ChI]) a shift dynamical system $(C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, where $\sigma(\tau, f) := f_\tau$ for any $f \in C(\mathbb{R}\times\mathcal{C}, \mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f_τ is τ-translation of f, i.e. $f_\tau(t, \phi) := f(t + \tau, \phi)$ for any $(t, \phi) \in \mathbb{R} \times \mathcal{C}$. Let us set $H^+(f) := \overline{\{f_s : s \in \mathbb{R}_+\}}$.

Along with equation (23) let us consider the family of equations

$$\dot{v} = g(t, v_t), \tag{24}$$

where $g \in H^+(f)$.

Below, in this subsection, we suppose that equation (23) is regular.

Remark 14. *1. Denote by $\tilde{\varphi}(t,u,f)$ the solution of equation (23) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0,u,f) = u \in \mathcal{C}$, i.e. $\varphi(s,u,f) = u(s)$ for any $s \in [-r,0]$. By $\varphi(t,u,f)$ we will denote below the trajectory of equation (23), corresponding to the solution $\tilde{\varphi}(t,u,f)$, i.e. the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t,u,f)(s) := \tilde{\varphi}(t+s,u,f)$ for any $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r,0]$.*

2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (23) as synonymous concepts.

It is well known [5, 14] that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H^+(f) \mapsto \mathbb{R}^n$ possesses the following properties:

1. $\varphi(0,v,g) = v$ for any $v \in \mathcal{C}$ and $g \in H^+(f)$;
2. $\varphi(t+\tau,v,g) = \varphi(t,\varphi(\tau,v,g),\sigma(\tau,g))$ for any $t,\tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H^+(f)$;
3. the mapping φ is continuous.

Thus, a triplet $\langle \mathcal{C}, \varphi, (H^+(f), \mathbb{R}_+, \sigma)\rangle$ is a cocycle which is associated to equation (23). Applying the results from Sections 3-4 we will obtain a series of results for functional differential equation (23). Below we formulate some of them.

Lemma 6 (see [8]). *Suppose that the following conditions hold:*

1. *the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is regular;*
2. *the set $H^+(f)$ is compact;*
3. *the function f is completely continuous, i.e. the set $f(\mathbb{R}_+ \times A)$ is bounded for any bounded subset $A \subseteq \mathcal{C}$.*

Then the cocycle φ associated with (23) is completely continuous, i.e. for any bounded subset $A \subseteq W$ there exists a positive number $l = l(A)$ such that the set $\varphi(l, A, H^+(f))$ is relatively compact in $\mathcal{C}$.

Theorem 17. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and completely continuous;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t,0) = 0$ for any $t \in \mathbb{R}_+$.*

Then the null solution of equation (23) is globally asymptotically stable if and only if the following conditions hold:

a) for every $a > 0$

$$\lim_{t\to+\infty} \sup_{|v|\le a, g\in\Omega_f} |\varphi(t,v,g)| = 0; \tag{25}$$

b) for every $v \in \mathcal{C}$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (24) is bounded on $\mathbb{R}_+$.

Proof. Consider the dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$. Since the space $H^+(f)$ is compact, then $(H^+(f), \mathbb{R}_+, \sigma)$ is compactly dissipative and by Lemma 1 its Levinson center $J_{H^+(f)}$ coincides with the ω-limit set Ω_f of f. Let $Y := H^+(f)$ and $(Y, \mathbb{R}_+, \sigma)$ be the shift dynamical system on Y. Denote $X := \mathcal{C} \times Y$ and $(X, \mathbb{R}_+, \pi)$ the skew-product dynamical system generates by $(Y, \mathbb{R}_+, \sigma)$ and cocycle φ. Now consider a NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h \rangle$ $(h := pr_2)$ associated with equation (23). It is easy to verify that this NDS has the following properties:

1. the dynamical system $(Y, \mathbb{R}_+, \sigma)$ is compact dissipative and its Levinson center J_Y coincides with Ω_f;
2. the null section Θ of $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \pi), h \rangle$ coincides with $\{0\} \times Y$;
3. Θ is a positively invariant subset of $(X, \mathbb{R}_+, \pi)$;
4. according to (25) the null section $\tilde{\Theta}$ of NDS $\langle (\tilde{X}, \mathbb{R}_+, \pi), (J_Y, \mathbb{R}_+, \sigma), h \rangle$ is uniformly attracting because $|\pi(t, x)| = |\varphi(t, v, g)|$ for any $t \in \mathbb{R}_+$ and $x := (v, g) \in X$;
5. according to Lemma 6 the dynamical system $(X, \mathbb{R}_+, \pi)$ is completely continuous;
6. every positive semi-trajectory Σ_x^+ of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is relatively compact.

Now to finish the proof it is sufficient to apply Corollary 3. □

Theorem 18. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and completely continuous;*
2. *the set $H^+(f)$ is compact;*
3. *$f(t, 0) = 0$ for any $t \in \mathbb{R}_+$.*

Then the null solution of equation (23) is globally asymptotically stable if and only if the following conditions hold:

a. for every $g \in \Omega_f$ limiting equation (24) does not have nontrivial bounded on $\mathbb{R}$ solutions;

b. for every $v \in \mathcal{C}$ and $g \in H^+(f)$ the solution $\varphi(t, v, g)$ of equation (24) is bounded on $\mathbb{R}_+$.

Proof. This statement follows from Corollary 2 and can be proved using the same arguments as in the proof of Theorem 17. □

Theorem 19. *Suppose that the following conditions are fulfilled:*

1. *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is recurrent in* $t \in \mathbb{R}$ *uniformly in* u *on every compact subset of* $\mathcal{C}$*;*
2. $f(t, 0) = 0$ *for any* $t \in \mathbb{R}_+$*;*
3. *the function* f *is regular and completely continuous;*
4. *the null solution of equation (23) is uniformly stable;*
5. *there exists a positive number* a *such that*

$$\lim_{t \to +\infty} \sup_{|u| \le a} |\varphi(t, u, f)| = 0.$$

Then the null solution of equation (23) is asymptotically stable.

Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 17. □

5.3.2 Neutral functional-differential equations

Now consider the neutral functional-differential equation

$$\frac{d}{dt} D u_t = f(t, u_t), \tag{26}$$

where $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is a regular function and the operator $D : \mathcal{C} \mapsto \mathbb{R}^n$ is atomic at zero [9, p.67]. Like (23), equation (26) generates a NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$, where $X := \mathcal{C} \times Y,\ Y := H^+(f)$, and $\pi := (\varphi, \sigma)$.

An operator D is said to be stable if the zero solution of difference equation $Dy_t = 0$ is uniformly asymptotically stable (see, for example,[9, p.337]).

Lemma 7. *Let* $H^+(f)$ *be compact. If the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ *is completely continuous, then the NDS* $(X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ *generated by equation (26) is asymptotically compact.*

Proof. This statement can be proved by slight modification of the proof of Theorem 12.6.3 and Lemma 12.6.1 from [9, Ch.XII] and taking into account that $Y = H^+(A)$ is compact. □

Theorem 20. *Suppose that the following conditions are fulfilled:*

1. *the function* $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ *is recurrent in* $t \in \mathbb{R}$ *uniformly in* u *on every compact subset of* $\mathcal{C}$*;*
2. $f(t, 0) = 0$ *for any* $t \in \mathbb{R}_+$*;*
3. *the function* f *is regular and completely continuous;*

4. *the null solution of equation (26) is uniformly stable;*

5. *there exists a positive number a such that*

$$\lim_{t\to+\infty} |\varphi(t,u,f)| = 0 \tag{27}$$

for any $|u| \le a$.

Then the null solution of equation (26) is asymptotically stable, i.e., there exists a positive number δ *such that* $\lim\limits_{t\to+\infty} |\varphi(t,v,g)| = 0$ *for any* $|v| < \delta$ *and* $g \in H^+(f)$.

Proof. Let $(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a NDS generated by equation (26). It is easy to check that under the conditions of Theorem 20 the following statements hold:

1. the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center J_Y coincides with $Y = H^+(f) = \Omega_f$;

2. the null section Θ of $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;

3. Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;

4. according to (27) we have $B(0_f,a) \subset X^s_f$, where $0_f := (0,f)$, 0 is the null element of $\mathcal{C}$ and $B(x_0,a) := \{x \in \mathcal{C} :\ |x - x_0| < a\}$;

5. according to Lemma 7 the dynamical system $(X,\mathbb{R}_+,\pi)$ is asymptotically compact.

Now to finish the proof of Theorem it is sufficient to apply Theorem 11. □

5.4 Semi-linear parabolic equations

Let E be a Banach space, and let $A : D(A) \to E$ be a linear closed operator with the dense domain $D(A) \subseteq E$.

An operator A is called [10] sectorial if for some $\phi \in (0,\pi/2)$, some $M \ge 1$, and some real a, the sector

$$S_{a,\phi} := \{\lambda : \phi \le |\arg(\lambda - a)| \le \pi, \lambda \ne a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \le M|\lambda - a|^{-1}$ for all $\lambda \in S_{a,\varphi}$.

If A is a sectorial operator, then there exists an $a_1 \ge 0$ such that $Re\,\sigma(A + a_1 I) > 0$ ($\sigma(A) := \mathbb{C} \setminus \rho(A)$). Let $A_1 = A + a_1 I$. For $0 < \alpha < 1$, one defines the operator [10]

$$A_1^{-\alpha} := \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha}(\lambda I + A_1)^{-1} d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha := D(A_1^\alpha)$, and let us equip the space E^α with the graph norm $|x|_\alpha := |A_1^\alpha x|$ ($x \in E$), $E^0 := E$, and $E^1 := D(A)$.

Then E^α is a Banach space with the norm $|\cdot|_\alpha$ and is densely and continuously embedded in E.

Consider the differential equation

$$x' + Ax = F(t,x), \tag{28}$$

where $F \in C(\mathbb{R} \times E^\alpha, E)$ and $C(\mathbb{R} \times E^\alpha, E)$ is the space of all the continuous functions equipped with compact open topology.

Along with equation (28), consider the family of equations

$$y' + Ay = G(t,y), \tag{29}$$

where $G \in H^+(F) := \overline{\{F_\tau : \tau \in \mathbb{R}_+\}}$.

Recall that a function F is said to be regular if for every $(v,G) \in E^\alpha \times H^+(F)$ equation (29) admits a unique solution [10, Ch.III] $\varphi(t,v,G)$ with initial data $\varphi(0,v,G) = v$ and the mapping $\varphi : \mathbb{R}_+ \times E^\alpha \times H^+(F) \mapsto E^\alpha$ is continuous.

Regularity conditions for F are given in Theorems 3.3.3, 3.3.4, 3.3.6, and 3.4.1 in [10, Ch.III].

Assuming that F is regular, a non-autonomous dynamical system can be associated in a natural way with equation (28). Namely, we set $Y := H^+(F)$ and by $(Y, \mathbb{R}_+, \sigma)$ denote the dynamical system of translations on Y. Further, let $X := E^\alpha \times Y$, and let $(X, \mathbb{R}_+, \pi)$ be the dynamical system on X defined by the relation $\pi^\tau(v,G) = \langle \varphi(\tau,v,G), G_\tau \rangle$. Finally, by setting $h = pr_2 : X \to Y$, we obtain the non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$ determined by equation (28).

Applying results from Sections 3-4 we obtain a series of results for evolution equation (28). Now we will formulate some of them.

Recall that a function $F \in C(\mathbb{R} \times E^\alpha, E)$ is said to be locally Hölder continuous in t and locally Lipschitz in x if for every $(t_0, x_0) \in \mathbb{R} \times E^\alpha$ there exists a neighborhood V $((t_0,x_0) \in V)$ and positive numbers L and θ such that

$$|F(t_1,x_1) - F(t_2,x_2)| \leq L(|t_1 - t_2|^\theta + |x_1 - x_2|_\alpha)$$

for any $(t_i, x_i) \in V$ $(i = 1,2)$.

Lemma 8. *Suppose that the following conditions are fulfilled:*

1. *A is a sectorial operator;*
2. *the resolvent of operator A is compact;*
3. *$0 \leq \alpha < 1$ and $F \in C(\mathbb{R} \times E^\alpha, E)$;*
4. *the function F is locally Hölder continuous in t and locally Lipschitz in x.*

Under the conditions listed above if the function F is regular and the set $H^+(F)$ is compact, then the cocycle φ associated with equation (28) is completely continuous.

Proof. This statement can be proved with the slight modification of the proof of Theorem 3.3.6 [10, Ch.III]. □

Theorem 21. *Assume that the following conditions are fulfilled:*

1. *the function F is regular;*
2. *the set $H^+(F)$ is compact;*
3. *$F(t,0)=0$ for any $t\in\mathbb{R}_+$;*
4. *$0\le\alpha<1$ and $F\in C(\mathbb{R}\times E^\alpha,E)$;*
5. *the function F is locally Hölder continuous in t and locally Lipschitz in x.*

Then the null solution of equation (28) is globally asymptotically stable if and only if the following conditions hold:

1. $$\lim_{t\to+\infty}\sup_{|v|\le a, g\in\Omega_f}|\varphi(t,v,G)|=0$$
   for every $a>0$;
2. *for every $v\in E^\alpha$ and $G\in H^+(F)$ the solution $\varphi(t,v,G)$ of equation (28) is bounded on $\mathbb{R}_+$.*

Proof. Consider the dynamical system $(H^+(F),\mathbb{R}_+,\sigma)$. Since the space $H^+(F)$ is compact, then $(H^+(f),\mathbb{R}_+,\sigma)$ is compactly dissipative and its Levinson center $J_{H^+(F)}$ coincides with the ω-limit set Ω_F of F. Let $Y:=H^+(F)$ and $(Y,\mathbb{R}_+,\sigma)$ be the shift dynamical system on Y. Denote by $X:=E^\alpha\times Y$ and $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system generated by $(Y,\mathbb{R}_+,\sigma)$ and cocycle φ. Consider a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ ($h:=pr_2$) associated with equation (28). It is easy to verify that for this NDS the following properties hold:

1. the dynamical system $(Y,\mathbb{R}_+,\sigma)$ is compactly dissipative and by Lemma 1 its Levinson center J_Y coincides with Ω_F;
2. the null section Θ of $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\pi),h\rangle$ coincides with $\{0\}\times Y$;
3. Θ is a positively invariant subset of $(X,\mathbb{R}_+,\pi)$;
4. according to (30) the null section $\tilde\Theta$ of NDS $\langle(\tilde X,\mathbb{R}_+,\pi),(J_Y,\mathbb{R}_+,\sigma),h\rangle$ is uniformly attracting because $|\pi(t,x)|=|\varphi(t,v,g)|$ for any $t\in\mathbb{R}_+$ and $x:=(v,g)\in X$;
5. by Lemma 8 the cocycle φ (and, consequently, the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ too) is completely continuous;
6. every positive semi-trajectory Σ_x^+ of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 8. □

Theorem 22. *Assume that the following conditions are fulfilled:*

1. *the function F is regular;*
2. *the set $H^+(F)$ is compact;*
3. *$F(t,0) = 0$ for any $t \in \mathbb{R}_+$;*
4. *$0 \le \alpha < 1$ and $F \in C(\mathbb{R} \times E^\alpha, E)$;*
5. *the function F is locally Hölder continuous in t and locally Lipschitz in x.*

Then the null solution of equation (28) is globally asymptotically stable if and only if the following conditions hold:

a. *for every $G \in \Omega_F$ limiting equation (29) does not have nontrivial bounded on $\mathbb{R}$ solutions;*

b. *for every $v \in \mathcal{C}$ and $G \in H^+(F)$ the solution $\varphi(t,v,g)$ of equation (29) is bounded on $\mathbb{R}_+$.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 21 plus applying Corollary 2. □

Theorem 23. *Suppose that the following conditions are fulfilled:*

1. *$0 \le \alpha < 1$ and $F \in C(\mathbb{R} \times E^\alpha, X)$;*
2. *the function F is locally Hölder continuous in t and locally Lipschitz in x;*
3. *the function F is recurrent in $t \in \mathbb{R}$ uniformly in u on every compact subset of $W \subseteq E^\alpha$;*
4. *$F(t,0) = 0$ for any $t \in \mathbb{R}_+$;*
5. *the function F is regular;*
6. *the null solution of equation (28) is uniformly stable;*
7. *there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t,u,F)| = 0$$

   for any $|u| \le a$.

Then the null solution of equation (28) is asymptotically stable.

Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 21. □

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