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Invariant transformations of loop transversals. 2. The case of isotopy

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Abstract. The investigation of special transformations of loop transversals is continued. These transformations correspond to arbitrary isotopies of loop transversal operations (witch correspond to the considered loop transversals). Isotopies of loop transversal operations with the same unit are investigated.

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1 Introduction

This article is a continuation of the research of some special class of loop transversal transformations, begun in [5]. Transformations from the studied class correspond to arbitrary isotopies of transversal operations (which correspond to the considered loop transversals). We find a new class of loop transversal transformations which preserve the property to be a loop transversal. This investigation (as it was mentioned in [5]) is important for solving some other problems – for example, it can be used in the classification of G-loops.

2 Necessary definitions and statements

All necessary definitions and statements can be found in [5], §2. We remind the most important ones.

Definition 1. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H,$$

Definition 2. If a system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then such left transversal $T = \{t_x\}_{x\in E}$ is called a **loop transversal**.

Further we are going to use the following permutation representation $\widehat{G}$ of a group G by the left cosets of its subgroup H (see [2,3]):

$$\widehat{g}(x) = y \quad \stackrel{def}{\Longleftrightarrow} \quad gt_xH = t_yH.$$

For simplicity we assume that

$$Core_G(H) = \bigcap_{g \in G} gHg^{-1} = \{e\},$$

then this representation is exact (see Lemma 6 in [3]), and we have $\widehat{G} \cong G$. Notice that $\widehat{H} = St_1(\widehat{G})$.

Lemma 1 (see [3], Lemma 4)**.** *Let $T = \{t_x\}_{x \in E}$ be a left transversal in G to H. Then the following statements are true:*

1. $\widehat{h}(1) = 1 \ \forall h \epsilon H$;
2. $\forall x, y \in E$:

   $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y = \widehat{L}_x(y), \qquad \widehat{t_1}(x) = \widehat{t_x}(1) = x,$

   $\widehat{t_x^{-1}}(y) = x \overset{(T)}{\diagdown} y = \widehat{L}_x^{-1}(y), \qquad \widehat{t_x^{-1}}(1) = x \overset{(T)}{\diagdown} 1, \qquad \widehat{t_x^{-1}}(x) = 1,$

   where "$\overset{(T)}{\diagdown}$" *is a left division for the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *(i.e.* $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$*).*

Lemma 2 (see [3], Lemma 7)**.** *Let $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$ be left transversals in G to H. Then there is a set of elements $\{h_{(x)}\}_{x \in E}$ from H such that:*

1. $p_x = t_x h_{(x)} \ \forall x \in E$;
2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)$.

This set $\{h_{(x)}\}_{x \in E}$ is called (see [4]) a **derivation set** for the transversal T (and for the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$).

Definition 3 (see [1])**.** A triple of permutations $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) is called an **isotopy** of the operation $\langle E, \cdot\rangle$ on the operation $\langle E, \circ\rangle$ if

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\alpha, \beta, id)$, then such an isotopy is called a **principal isotopy**.

According to Lemma 1.2 from [1] we have

Lemma 3. *If a loop $\langle E, \cdot, e_1\rangle$ is isotopic to a loop $\langle E, \circ, e_2\rangle$, then it is isomorphic to some principal isotope of a loop $\langle E, \circ\rangle$ (and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id), a \cdot b = e_2$).*

Remark 1. If a loop $\langle E, \cdot, 1\rangle$ is principally isotopic to a loop $\langle E, \circ, 1\rangle$, then this principal isotopy has the form $T_0 = (R^{-1}_{a\diagdown 1}, L_a^{-1}, id)$ for some $a \in E$ ($a^{-1} = a\diagdown 1$ is the right inverse element to a in the loop $\langle E, \cdot, 1\rangle$).

3 Transformations of loop transversals which correspond to an isotopy of their transversal operations

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a group G to its subgroup H, and $\langle E, \overset{(T)}{\cdot}, 1\rangle$, $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be their transversal operations. Fix one of these loop transversals, for example $T = \{t_x\}_{x\in E}$.

As follows from Lemma 3, to investigate loop transversals transformations which correspond to an isotopy of operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ it is enough to study the case of principal isotopy $T_0 = (R^{-1}_{a\diagdown 1}, L_a^{-1}, id)$ (because the transformations which corresponds to an isomorphism of transversal operations were studied earlier in [5]).

Theorem 1. *Let loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be principally isotopic and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id)$ for some $a \in E$ (note that $a, b \in E$, $a \overset{(T)}{\cdot} b = 1$). Then*

$$\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}.$$

Proof. Let the conditions of Theorem hold. Then

$$x \overset{(P)}{\cdot} y = R_b^{-1}(x) \overset{(T)}{\cdot} L_a^{-1}(y)$$

for some $a, b \in E$, $a \overset{(T)}{\cdot} b = 1$, and L_a, R_b are left and right translations in the loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Then the left translation $\mathbf{L}_x$ in the loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ has the form:

$$\mathbf{L}_x(y) = x \overset{(P)}{\cdot} y = R_b^{-1}(x) \overset{(T)}{\cdot} L_a^{-1}(y) = L_{R_b^{-1}(x)} L_a^{-1}(y), \quad \forall x, y \in E,$$

that is

$$\mathbf{L}_x = L_{R_b^{-1}(x)} L_a^{-1} \quad \forall x \in E. \tag{1}$$

By Lemma 1 (item 2) we have

$$\{\mathbf{L}_x\}_{x\in E} \equiv \{\widehat{p}_x\}_{x\in E} = \widehat{P}$$

and

$$\{L_x\}_{x\in E} \equiv \left\{\widehat{t}_x\right\}_{x\in E} = \widehat{T}.$$

Since R_b^{-1} is a permutation on the set E for every $b \in E$, then it follows from (1): $\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}$ for some $a \in E$. □

Lemma 4. *Let loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be isotopic. Then the following statement holds:*

$$\widehat{P} = \widehat{h}_0 \widehat{T} \widehat{t}_a^{-1} \widehat{h}_0^{-1}$$

for some $h_0 \in \widehat{H}$ and some $a \in E$.

Proof. Let loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be isotopic. Then according to Lemma 3, their isotopy can be represented in the form of composition of a principal isotopy and an isomorphism:

$$(\alpha, \beta, \gamma) = (R_b^{-1}, L_a^{-1}, id) \circ (\gamma, \gamma, \gamma),$$

where $\gamma(1) = 1$, $\quad a \overset{(T)}{\cdot} b = 1$. Now our statement is a simple corollary from Theorem 1 and Lemma 7 of [5]. □

Theorem 2. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in G to H, and $a \in E$ be an arbitrary element of the set E. Define the following set $P = \{p_{x'}\}_{x'\in E}$ of permutations:*

$$\widehat{p}_{x'} \overset{def}{=} \widehat{t}_x \widehat{t}_a^{-1} \quad \forall x \in E.$$

Then

1. *$P = \{p_{x'}\}_{x'\in E}$ is a left transversal in G to H;*
2. *A transversal operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is principally isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and the principal isotopy S has the following form: $S = (R_{a\diagdown 1}^{-1}, L_a^{-1}, id)$;*
3. *P is a loop transversal in G to H.*

Proof. 1. We have

$$x' = \widehat{p}_{x'}(1) = \widehat{t}_x \widehat{t}_a^{-1}(1) = \widehat{t}_x(a\diagdown 1) = x \overset{(T)}{\cdot} (a\diagdown 1) = R_{a\diagdown 1}(x).$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then $R_{a\diagdown 1}$ is a permutation on the set E for every $a \in E$. Therefore the element x' runs over all the set E. So there is at least one element of P (element $p_{x'}$) in each left coset $H_{x'}$. It means that P is a left transversal in G to H. Moreover, $\quad e = t_a t_a^{-1} \in P$.

2. Let us study the following set of elements:

$$\widehat{p}_{x'} = \widehat{t}_x \widehat{t}_a^{-1}, \quad x \in E,$$

where a is an arbitrary fixed element of the set E. As we have seen,

$$x' = x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1). \tag{2}$$

For the transversal operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ we have (by the definition):

$$p_{x'}p_{y'} = p_{x' \overset{(P)}{\cdot} y'} h^*, \quad h^* \in H.$$

Then by Lemma 1 and the definition of transversal operation we have

$$\begin{aligned} x' \overset{(P)}{\cdot} y' &= \widehat{p}_{x' \overset{(P)}{\cdot} y'} \widehat{h}^*(1) = \widehat{p}_{x'}\widehat{p}_{y'}(1) = \\ &= \widehat{t}_x \widehat{t}_a^{-1} \widehat{t}_y \widehat{t}_a^{-1}(1) = x \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} (y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1)) \right]. \end{aligned} \tag{3}$$

Using (2) in (3), we obtain

$$\left[x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) \right] \overset{(P)}{\cdot} \left[y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) \right] = x \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} (y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1)) \right]. \tag{4}$$

We replace:

$$\begin{cases} x = u \overset{(T)}{/} (a \overset{(T)}{\diagdown} 1) \\ y = v \overset{(T)}{/} (a \overset{(T)}{\diagdown} 1) \end{cases} \iff \begin{cases} u = x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) = R_{a \diagdown 1}(x) \\ v = y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) = R_{a \diagdown 1}(y). \end{cases}$$

Since $R_{a \diagdown 1}$ is a permutation for every $a \in E$ in the loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$, then u and v run over all the set E. Then we have from (4):

$$\begin{aligned} u \overset{(P)}{\cdot} v &= (u \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} ((v \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown 1)) \right] = \\ &= (u \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown v) = R_{a \diagdown 1}^{-1}(u) \overset{(T)}{\cdot} L_a^{-1}(v). \end{aligned}$$

From the last equality it follows that the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is principally isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and this principal isotopy has the following form: $S = (R_{a \diagdown 1}^{-1}, L_a^{-1}, id)$.

3. According to item 2 the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a principal isotope of the loop operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and this principal isotopy has the form $S = (R_{a \diagdown 1}^{-1}, L_a^{-1}, id)$. It is well known that any isotope of a loop is a quasigroup, so the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a quasigroup. Moreover, the element 1 is a unit element of this quasigroup, that is the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a loop. It means that the transversal $P = \{p_x\}_{x \in E}$ is a loop transversal. $\square$

Lemma 5. *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two transversals in G to H which correspond to principally isotopic transversal operations. Let $p_x = t_x h_{(x)}$ and $\left\{h_{(x)}\right\}_{x\in E}$ be a derivation set. Then*

$$h_{(x)} = t_x^{-1} t_{x \overset{(T)}{/} (a\diagdown 1)} t_a^{-1}$$

for some $a \in E$.

Proof. According to Theorem 2 (item 2) we have for every $x \in E$:

$$\widehat{p}_{x \overset{(T)}{\cdot} (a\diagdown 1)} = \widehat{t}_x \widehat{t}_a^{-1}$$

for some element $a \in E$. Let us replace $u = x \overset{(T)}{\cdot} (a\diagdown 1)$, so $x = u \overset{(T)}{/} (a\diagdown 1)$. Then

$$p_u = t_{u \overset{(T)}{/} (a\diagdown 1)} t_a^{-1}, \quad \forall u \in E$$

On the other hand,

$$p_u = t_u h_{(u)}, \quad \forall u \in E.$$

So

$$t_u h_{(u)} = t_{u/(a\diagdown 1)} t_a^{-1},$$

and our Lemma is proved. □

Lemma 6. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in G to H, and $a \in E$ be some element of the set E. Define the following set $S = \{s_{x'}\}_{x'\in E}$ of elements:*

$$s_{x'} \overset{def}{=} t_a t_x t_a^{-1} \quad \forall x \in E.$$

Then:

1. *$S = \{s_{x'}\}_{x'\in E}$ is a left transversal in G to H;*
2. *A transversal operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and the isotopy S has the following form: $S = (\beta\alpha, \alpha, \beta^{-1})$, where $\alpha = L_a^{-1}$, $\beta = R_{a\diagdown 1}^{-1}$;*
3. *S is a loop transversal in G to H.*

Proof. 1. We have:

$$x' = \widehat{s}_{x'}(1) = \widehat{t}_a \widehat{t}_x \widehat{t}_a^{-1}(1) = \widehat{t}_a \widehat{t}_x (a\diagdown 1) = a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (a\diagdown 1)) = L_a R_{a\diagdown 1}(x).$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then L_a and $R_{a\diagdown 1}$ are permutations on the set E for every $a \in E$. Therefore an element x' runs over all the set E. So every left coset $H_{x'}$

contains an element of S (element $s_{x'}$). So $S=\{s_{x'}\}_{x'\in E}$ is a left transversal in G to H. Moreover, $e=t_a e t_a^{-1}=t_a t_1 t_a^{-1}\in E$.

2. Let us examine the following set of elements

$$s_{x'}=t_a t_x t_a^{-1}, \quad x\in E,$$

where a is an element of the set E. As we have seen,

$$x'=a\overset{(T)}{\cdot}(x\overset{(T)}{\cdot}(a\diagdown 1)). \tag{5}$$

For the transversal operation $\langle E,\overset{(S)}{\cdot},1\rangle$ we have

$$s_{x'}s_{y'}=s_{x'\overset{(S)}{\cdot}y'}h^*, \quad h^*\in H.$$

Then

$$\begin{aligned} x'\overset{(S)}{\cdot}y' &= \widehat{s}_{x'\overset{(S)}{\cdot}y'}\widehat{h}^*(1)=\widehat{s}_{x'}\widehat{s}_{y'}(1)=(\widehat{t}_a\widehat{t}_x\widehat{t}_a^{-1})(\widehat{t}_a\widehat{t}_y\widehat{t}_a^{-1})(1)= \\ &= \widehat{t}_a\widehat{t}_x\widehat{t}_y\widehat{t}_a^{-1}(1)=\widehat{t}_a\widehat{t}_x\widehat{t}_y(a\diagdown 1)=a\overset{(T)}{\cdot}(x\overset{(T)}{\cdot}(y\overset{(T)}{\cdot}(a\diagdown 1))). \end{aligned}$$

By (5) from the last equality we obtain:

$$\left[a\overset{(T)}{\cdot}(x\overset{(T)}{\cdot}(a\diagdown 1))\right]\overset{(S)}{\cdot}\left[a\overset{(T)}{\cdot}(y\overset{(T)}{\cdot}(a\diagdown 1))\right]=a\overset{(T)}{\cdot}(x\overset{(T)}{\cdot}(y\overset{(T)}{\cdot}(a\diagdown 1))). \tag{6}$$

We replace:

$$\left\{\begin{array}{l} a\overset{(T)}{\cdot}(x\overset{(T)}{\cdot}(a\diagdown 1))=u \\ a\overset{(T)}{\cdot}(y\overset{(T)}{\cdot}(a\diagdown 1))=v \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} x=(a\diagdown u)/(a\diagdown 1) \\ y=(a\diagdown v)/(a\diagdown 1) \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} u=L_aR_{a\diagdown 1}(x) \\ v=L_aR_{a\diagdown 1}(y), \end{array}\right.$$

that is the elements u,v run over all the set E. Then from (6) we obtain:

$$\begin{aligned} u\overset{(S)}{\cdot}v &= a\overset{(T)}{\cdot}\left[((a\diagdown u)/(a\diagdown 1))\overset{(T)}{\cdot}\left[((a\diagdown v)/(a\diagdown 1))\overset{(T)}{\cdot}(a\diagdown 1)\right]\right]= \\ &= a\overset{(T)}{\cdot}\left[((a\diagdown u)/(a\diagdown 1))\overset{(T)}{\cdot}(a\diagdown v)\right]=L_a\left[(R_{a\diagdown 1}^{-1}L_a^{-1}(u))\overset{(T)}{\cdot}(L_a^{-1}(v))\right] \end{aligned}$$

and

$$L_a^{-1}(u\overset{(S)}{\cdot}v)=R_{a\diagdown 1}^{-1}L_a^{-1}(u)\overset{(T)}{\cdot}L_a^{-1}(v). \tag{7}$$

It means that the operations $\langle E,\overset{(S)}{\cdot},1\rangle$ and $\langle E,\overset{(T)}{\cdot},1\rangle$ are isotopic and the isotopy S has the form $S=(\beta\alpha,\alpha,\alpha)$, where $\alpha=L_a^{-1}$, $\beta=R_{a\diagdown 1}^{-1}$.

3. By item 2 the operation $\langle E,\overset{(S)}{\cdot},1\rangle$ is an isotope of the loop operation and this isotopy has the form $S=(\beta\alpha,\alpha,\alpha)$, where $\alpha=L_a^{-1}$, $\beta=R_{a\diagdown 1}^{-1}$. It is well known

that any isotope of a loop is a quasigroup, so the operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is a quasigroup. Moreover,

$$s_{1'} = t_a t_1 t_a^{-1} = t_a \cdot e \cdot t_a^{-1} = e = t_1,$$

that is the element 1 is a unit element of this quasigroup. So the operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is a loop and $S' = \{s_x\}_{x\in E}$ is a loop transversal. □

Lemma 7. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in G to H and $a \in E$ be an arbitrary element in E. Define the following set $M = \{m_{x'}\}_{x'\in E}$ of elements:*

$$m_{x'} \overset{def}{=} t_a^{-1} t_x, \quad \forall x \in E.$$

Then:

1. *$M = \{m_{x'}\}_{x'\in E}$ is a left transversal in G to H.*
2. *The transversal operation $\langle E, \overset{(M)}{\cdot}, 1\rangle$ is isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and the isotopy Q has the following form: $Q = (L_a, id, L_a)$.*
3. *M is a loop transversal in G to H.*

Proof. 1. We have

$$x' = \widehat{m}_{x'}(1) = \widehat{t}_a^{-1}\widehat{t}_x(1) = a\diagdown x = L_a^{-1}(x). \tag{8}$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop then L_a^{-1} is a permutation on the set E for every $a \in E$. So the element x' runs over all the set E, and M is a loop transversal in G to H.

2. Let us examine the following set of elements:

$$m_{x'} \overset{def}{=} t_a^{-1} t_x, \quad x \in E$$

where a is some element in E. As we have seen above, $x' = a\diagdown x$. For the transversal operation $\langle E, \overset{(M)}{\cdot}, 1\rangle$ we have

$$m_{x'} m_{y'} = m_{x' \overset{(M)}{\cdot} y'} h^*, \quad h^* \in H.$$

Then

$$\begin{aligned} x' \overset{(M)}{\cdot} y' &= \widehat{m}_{x' \overset{(M)}{\cdot} y'} \widehat{h}^*(1) = \widehat{m}_{x'}\widehat{m}_{y'}(1) = (\widehat{t}_a^{-1}\widehat{t}_x)(\widehat{t}_a^{-1}\widehat{t}_y)(1) = \\ &= \widehat{t}_a^{-1}\hat{t}_x(a\diagdown y) = a\diagdown \left[x \overset{(T)}{\cdot} (a\diagdown y)\right]. \end{aligned}$$

By (8) we obtain:

$$(a\diagdown x) \overset{(M)}{\cdot} (a\diagdown y) = a\diagdown \left[x \overset{(T)}{\cdot} (a\diagdown y)\right]. \tag{9}$$

We use the change of variables:

$$\begin{cases} a\diagdown x = u \\ a\diagdown y = v \end{cases} \Longleftrightarrow \begin{cases} x = a \overset{(T)}{\cdot} u \\ y = a \overset{(T)}{\cdot} v \end{cases} \Longleftrightarrow \begin{cases} u = L_a^{-1}(x) \\ v = L_a^{-1}(y) \end{cases}$$

So elements u, v run over all the set E. Then we have

$$u \overset{(M)}{\cdot} v = a\diagdown \left[(a \overset{(T)}{\cdot} u) \overset{(T)}{\cdot} (a\diagdown(a \overset{(T)}{\cdot} v)) \right] = a\diagdown \left[(a \overset{(T)}{\cdot} u) \overset{(T)}{\cdot} v \right],$$

that is

$$L_a(u \overset{(M)}{\cdot} v) = L_a(u) \overset{(T)}{\cdot} v.$$

It is an isotopy of the type (L_a, id, L_a).

3. Similar to the item 3 of Lemma 5 and Lemma 6. □

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