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Stability analysis of Pareto optimal portfolio of multicriteria investment maximin problem in the Hölder metric

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Abstract. We analyzed the stability of a Pareto-optimal portfolio of the multicriteria discrete variant of Markowitz's investment problem with Wald's maximin efficiency criteria. We obtained lower and upper bounds for the stability radius of such portfolio in the case of the Hölder metric l_p, $1 \leq p \leq \infty$, in the three-dimensional space of problem parameters. We also show the attainability of bounds in particular cases.

Mathematics subject classification: 90C09, 90C29, 90C31, 90C47.
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In paper [1] we obtained lower and upper attainable bounds for the stability radius of a Pareto-optimal portfolio of the multicriteria Boolean investment problem with Savage's minimax risk criteria in the case of the Chebyshev metric l_1 in the three-dimensional space of problem parameters. In the present paper we obtained the results of similar nature for the stability radius of the multicriteria investment problem with Wald's maximin efficiency criteria and any Hölder metric l_p, $1 \leq p \leq \infty$, in the spaces of criteria, portfolio and market states.

1 Problem statement and definitions

We consider the multicriteria discrete variant of Markowitz's investment managing problem [2]. To this end, we introduce the following notations:

$N_n = \{1, 2, \ldots, n\}$ be a set of investment alternative projects (assets);

N_m be a set of market states (conditions, scenarios);

N_s be a set of project efficiency measures;

$x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n$ be an investment portfolio, where $|X| \geq 2$, $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1 & \text{if the project } j \text{ is implemented,} \\ 0 & \text{otherwise;} \end{cases}$$

e_{ijk} be an assessment of efficiency of measure $k \in N_s$ of investment project $j \in N_n$ in the situation when the market is in state $i \in N_m$;

$E = [e_{ijk}]$ be a three-dimensional $m \times n \times s$ matrix with elements from $\mathbf{R}$.

Note that there are several approaches to evaluate efficiency of investment projects (NPV, NFV, PI et al.), which take into account risk and uncertainty in different ways (see e.g. [3–5]). That way it is worth to consider a decision making problem with multiple criteria (several measures of project efficiency).

Let the following vector objective function

$$f(x, E) = (f_1(x, E_1), f_2(x, E_2), \dots, f_s(x, E_s)),$$

be given on a set of investment portfolios X whose components are Wald's maximin criteria [6]

$$f_k(x, E_k) = \min_{i \in N_m} E_{ik}x = \min_{i \in N_m} \sum_{j \in N_n} e_{ijk}x_j \to \max_{x \in X}, \qquad k \in N_s,$$

where $E_k \in \mathbf{R}^{m \times n}$ is the k-th cut of matrix $E = [e_{ijk}] \in \mathbf{R}^{m \times n \times s}$, $E_{ik} = (e_{i1k}, e_{i2k}, \dots, e_{ink})$ is the i-th row of that cut. Thus, the investor, following Wald's criteria, takes extreme caution and optimizes portfolio efficiency $E_{ik}x$ (for the k-th criteria), assuming that the market was in the worst state, namely the efficiency is minimal. Obviously such pessimistic approach in the market state estimation is justified when we are talking about the guaranteed result.

A multicriteria *investment Boolean problem* $Z^s(E)$, $s \in \mathbf{N}$, with Wald's criteria means the problem of searching the set of *Pareto-optimal investment portfolios* (the Pareto set)

$$P^s(E) = \{x \in X : \nexists x' \in X \ \ (g(x', x, E) \geq 0_{(s)} \ \ \& \ \ g(x', x, E) \neq 0_{(s)})\},$$

where

$$g(x', x, E) = (g_1(x', x, E_1), g_2(x', x, E_2), \ \dots, g_s(x', x, E_s)),$$

$$g_k(x', x, E_k) = f_k(x', E_k) - f_k(x, E_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x), \ k \in N_s,$$

$$0_{(s)} = (0, 0, \dots, 0) \in \mathbf{R}^s.$$

It is easy to see, in the particular case for $m = 1$ our multicriteria investment problem $Z^s(E)$ becomes the multicriteria problem of linear Boolean programming

$$Z^s_B(E): \quad Ex \to \max_{x \in X}, \tag{1}$$

where $X \subseteq \mathbf{E}^n$, $E = [e_{1jk}] \in \mathbf{R}^{1 \times n \times s}$ is the matrix with rows $E_k = (e_{11k}, \ e_{12k}, \dots$ $\dots, e_{1nk}) \in \mathbf{R}^n$, $k \in N_s$. Such case can be interpreted as the situation when the investor has not got another alternative market state.

For any positive integer $d \geq 2$ in the real space $\mathbf{R}^d$ we introduce *the Hölder metric* l_p, $1 \leq p \leq \infty$, where the norm of $a = (a_1, a_2, \dots, a_d) \in \mathbf{R}^d$ is defined by the formula

$$\|a\|_p = \begin{cases} \left(\sum_{j \in N_d} |a_j|^p\right)^{1/p} & \text{if } \ 1 \leq p < \infty, \\ \max\{|a_j| : \ j \in N_d\} & \text{if } \ p = \infty, \end{cases}$$

and by the norm of a matrix means the norm of the vector composed of all matrix elements. Hence for matrix $E \in \mathbf{R}^{m\times n\times s}$ and any metric l_p, $1 \le p \le \infty$, we get the equalities

$$\|E\|_p = \|(\|E_1\|_p, \|E_2\|_p, \dots, \|E_s\|_p)\|_p, \tag{2}$$

$$\|E_k\|_p = \|(\|E_{1k}\|_p, \|E_{2k}\|_p, \dots, \|E_{mk}\|_p)\|_p, \quad k \in N_s. \tag{3}$$

Thus for $p < \infty$ the equations

$$\|E\|_p = (\sum_{k\in N_s} \|E_k\|_p^p)^{1/p}, \tag{4}$$

$$\|z\|_p = \|z\|_1^{1/p} \ \text{ for } \ z \in \{-1, 0, 1\}^n \tag{5}$$

hold. In addition, from (2) and (3) it follows that

$$\|E_{ik}\|_p \le \|E_k\|_p \le \|E\|_p, \quad i \in N_m,\ k \in N_s. \tag{6}$$

It is known, that the metric l_p defined in the space $\mathbf{R}^d$ includes the metric l_q in the dual space $(\mathbf{R}^d)^*$, and p, q, as it is well known, are related by the formula

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 < p < \infty. \tag{7}$$

In addition, as usual, we set $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. Thus, in what follows, we assume that the domain of variation of p and q is the interval $[1,\infty]$, while p, q obey the above conditions, moreover, we assume $1/p = 0$ for $p = \infty$.

Using (6) and the Hölder inequality

$$ab \le \|a\|_p \|b\|_q,$$

where $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$, $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$, it is easy to see that for x^0, $x \in X$ and $1 < p \le \infty$ the following inequalities hold:

$$E_{i'k}x^0 - E_{ik}x \ge -\|E_k\|_p(\|x^0\|_q + \|x\|_q), \qquad i,\ i' \in N_m, \quad k \in N_s, \tag{8}$$

and for $p = 1$:

$$E_{i'k}x^0 - E_{ik}x \ge -\|E_k\|_1, \qquad i,\ i' \in N_m, \quad k \in N_s. \tag{9}$$

In addition, for any $p \in [1,\infty]$ the following equality is obvious:

$$\|a\|_p = m^{1/p}\alpha \tag{10}$$

if any component of $a \in \mathbf{R}^m$ is the number $\alpha > 0$.

As usual [1, 7–9], *the stability radius of the investment portfolio* $x^0 \in P^s(E)$ in the Hölder metric l_p is defined as follows:

$$\rho^s(x^0, p, m) = \begin{cases} \sup\ \Xi_p & \text{if } \ \Xi_p \ne \emptyset, \\ 0 & \text{if } \ \Xi_p = \emptyset, \end{cases}$$

where

$$\Xi_p = \{\varepsilon > 0 : \ \forall E' \in \Omega_p(\varepsilon) \quad (x^0 \in P^s(E + E'))\},$$
$$\Omega_p(\varepsilon) = \{E' \in \mathbf{R}^{m\times n\times s} : \ \|E'\|_p < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is the set of perturbing matrixes, and $P^s(E + E')$ is the Pareto set of the perturbed problem $Z^s(E + E')$.

Thus, the stability radius defines an extreme level of problem initial data perturbations (elements of matrix E) preserving Pareto-optimality of the portfolio.

2 Lemmas

For the vector $a = (a_1, a_2, \dots, a_s) \in \mathbf{R}^s$ we introduce the positive cutoff function:

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_s^+),$$

where $a_k^+ = [a_k]^+ = \max\{0, a_k\}$, $k \in N_s$.

Lemma 1. *Let $\varphi_1 > 0$, $x^0 \neq x$,*

$$\|g^+(x^0, x, E)\|_1 \geq \varphi_1. \tag{11}$$

Then

$$\forall E' \in \Omega_1(\varphi_1) \quad \exists l \in N_s \qquad (g_l(x^0, x, E_l + E'_l) > 0). \tag{12}$$

Proof. Suppose, to the contrary, that there exists the perturbing matrix $E^0 \in \Omega_1(\varphi_1)$ such that the inequalities

$$g_k(x^0, x, E_k + E_k^0) \leq 0, \quad k \in N_s \tag{13}$$

hold.

Then, involving (9), we derive

$$0 \geq g_k(x^0, x, E_k + E_k^0) = \max_{i\in N_m} \min_{i'\in N_m} (E_{i'k}x^0 - E_{ik}x + E^0_{i'k}x^0 - E^0_{ik}x) \geq$$
$$\geq g_k(x^0, x, E_k) - \|E_k^0\|_1,$$

i. e. $g_k^+(x^0, x, E_k) \leq \|E_k^0\|_1$, $k \in N_s$. Hence, taking into account $E^0 \in \Omega_1(\varphi_1)$ it follows that the inequality

$$\|g^+(x^0, x, E)\|_1 = \sum_{k\in N_s} g_k^+(x^0, x, E_k) \leq \sum_{k\in N_s} \|E_k^0\|_1 = \|E^0\|_1 < \varphi_1$$

holds.

This inequality contradicts the condition (11) of Lemma 1. □

Lemma 2. *Let $1 < p \leq \infty$, $\varphi_2 > 0$, $x^0 \neq x$,*

$$\|g^+(x^0, x, E)\|_p \geq \varphi_2(\|x^0\|_q + \|x\|_q). \tag{14}$$

Then

$$\forall E' \in \Omega_p(\varphi_2) \quad \exists l \in N_s \qquad (g_l(x^0, x, E_l + E'_l) > 0). \tag{15}$$

Proof. We again suppose, to the contrary, that there exists the perturbing matrix $E^0 \in \Omega_p(\varphi_2)$ with the conditions (13) and for any index $k \in N_s$ in view of (8) we find

$$0 \geq g_k(x^0, x, E_k + E^0_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x + E^0_{i'k}x^0 - E^0_{ik}x) \geq$$

$$\geq g_k(x^0, x, E_k) - \|E^0_k\|_p(\|x^0\|_q + \|x\|_q),$$

i.e.

$$g^+_k(x^0, x, E_k) \leq \|E^0_k\|_p(\|x^0\|_q + \|x\|_q), \quad k \in N_s.$$

Thus, taking into account (4) and $E^0 \in \Omega_p(\varphi_2)$ for $p < \infty$ we have

$$\|g^+(x^0, x, E)\|_p = \Big(\sum_{k \in N_s} (g^+_k(x^0, x, E_k))^p\Big)^{1/p} \leq$$

$$\leq \Big(\sum_{k \in N_s} \|E^0_k\|^p_p\Big)^{1/p}(\|x^0\|_q + \|x\|_q) = \|E^0\|_p(\|x^0\|_q + \|x\|_q) < \varphi_2(\|x^0\|_q + \|x\|_q),$$

and for $p = \infty$ we derive

$$\|g^+(x^0, x, E)\|_\infty = \max_{k \in N_s} g^+_k(x^0, x, E_k) \leq \max_{k \in N_s} \|E^0_k\|_\infty(\|x^0\|_1 + \|x\|_1) =$$

$$= \|E^0\|_\infty(\|x^0\|_1 + \|x\|_1) < \varphi_2(\|x^0\|_1 + \|x\|_1).$$

This inequality is contrary to the condition (14). □

By contradiction we can easily prove the following lemma.

Lemma 3. *Let $x^0 \in P^s(E)$, $\gamma > 0$ and $1 \leq p \leq \infty$. If for any portfolio $x \in X \setminus \{x^0\}$ and any perturbing matrix $E' \in \Omega_p(\gamma)$ there exists $l \in N_s$ such that the inequality $g_l(x^0, x, E_l + E'_l) > 0$ is true, then the portfolio x^0 is a Pareto-optimal portfolio of the perturbing problem $Z^s(E + E')$, i.e. $x^0 \in P^s(E + E')$ for $E' \in \Omega_p(\gamma)$.*

Lemma 4. *Let $1 \leq p \leq \infty$, $x^0 \neq x$, $\delta = (\delta_1, \delta_2, \ldots, \delta_s)$, $\delta_k > 0$, $k \in N_s$,*

$$\delta_k\|x^0 - x\|_q > g^+_k(x^0, x, E_k), \qquad k \in N_s. \tag{16}$$

Then for any number $\varepsilon > m^{1/p}\|\delta\|_p$ there exists a matrix $E^0 \in \Omega_p(\varepsilon)$ such that $x^0 \notin P^s(E + E^0)$.

Proof. Using components of δ (see (16)), we define elements of the perturbing matrix $E^0 = [e^0_{ijk}] \in \mathbf{R}^{m \times n \times s}$ as follows:

$$e^0_{ijk} = \delta_k \, \frac{x_j - x^0_j}{\|x^0 - x\|_p}, \quad i \in N_m, \;\; j \in N_n, \;\; k \in N_s.$$

Because all rows E^0_{ik}, $i \in N_m$, of the cut $E^0_k \in \mathbf{R}^{m\times n}$ are equal, then denoting such rows as A_k, we have

$$A_k = \delta_k \frac{(x - x^0)^T}{\|x^0 - x\|_p}, \quad k \in N_s. \tag{17}$$

Thus $\|E^0_{ik}\|_p = \|A_k\|_p = \delta_k$, $i \in N_m$, $k \in N_s$. Hence, according to (2), (3) and (10) we find

$$\|E^0_k\|_p = m^{1/p}\delta_k, \quad k \in N_s,$$

$$\|E^0\|_p = m^{1/p}\|\delta\|_p,$$

and, therefore, $E^0 \in \Omega_p(\varepsilon)$ for any $\varepsilon > m^{1/p}\|\delta\|$. Here $1/p = 0$ is for $p = \infty$.

Further we prove that for any $p \in [1, \infty]$ and $k \in N_s$ the equality

$$A_k(x^0 - x) = -\delta_k\|x^0 - x\|_q \tag{18}$$

holds. Actually, for $p = \infty$ we have (in view of (17))

$$A_k(x^0 - x) = -\delta_k\|x^0 - x\|_1, \quad k \in N_s,$$

and for $1 \leq p < \infty$, considering (5), (7) and (17), we get the following chain of equalities

$$A_k(x^0 - x) = -\delta_k \frac{\|x^0 - x\|_1}{\|x^0 - x\|_p} =$$

$$= -\delta_k \frac{\|x^0 - x\|_1}{\|x^0 - x\|_1^{1/p}} = -\delta_k\|x^0 - x\|_1^{1/q} = -\delta_k\|x^0 - x\|_q, \quad k \in N_s.$$

At last, using (16) and (18), we conclude that for any index $k \in N_s$ the relations

$$g_k(x^0, x, E_k + E^0_k) = \min_{i\in N_m}(E_{ik} + A_k)x^0 - \min_{i\in N_m}(E_{ik} + A_k)x =$$

$$= g_k(x^0, x, E_k) + A_k(x^0 - x) \leq g^+_k(x^0, x, E_k) - \delta_k\|x^0 - x\|_q < 0$$

hold.

Hence, $x^0 \notin P^s(E + E^0)$. □

3 Stability radius bounds

For a Pareto-optimal portfolio x^0 of the problem $Z^s(E)$ denote

$$\varphi_1 = \varphi_1(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \|g^+(x^0, x, E)\|_p,$$

$$\varphi_2 = \varphi_2(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \frac{\|g^+(x^0, x, E)\|_p}{\|x^0\|_q + \|x\|_q},$$

$$\psi = \psi(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \frac{\|g^+(x^0, x, E)\|_p}{\|x^0 - x\|_q}.$$

Evidently, $\psi \geq 0$, $\varphi_i \geq 0$, $i \in N_2$, herewith $\varphi_1(x^0, 1, m) = \psi(x^0, 1, m)$ and $\varphi_2(x^0, p, m) \leq \psi(x^0, p, m)$ for $1 < p \leq \infty$.

Theorem. *For any* m, $s \in \mathbf{N}$ *and* $1 \leq p \leq \infty$ *the stability radius* $\rho^s(x^0, p, m)$ *of the investment portfolio* $x^0 \in P^s(E)$ *in the Hölder metric* l_p *has the following lower and upper bounds*

$$m^{1/p}\psi(x^0, p, m) \geq \rho^s(x^0, p, m) \geq \begin{cases} \varphi_1(x^0, p, m), & \text{if } p = 1, \\ \varphi_2(x^0, p, m), & \text{if } 1 < p \leq \infty. \end{cases} \tag{19}$$

Proof. Let $x^0 \in P^s(E)$. First we will prove the validity of lower bounds (19). Without loss of generality we assume that $\varphi_i > 0$, $i \in N_2$ (otherwise, the inequalities $\rho \geq \varphi_i$, $i \in N_2$, are obvious). We shall consider separately the two possible cases.

Case 1: $p = 1$. According to the definition of $\varphi_1 = \varphi_1(x^0, 1, m)$ for any portfolio $x \neq x^0$ the inequality

$$\|g^+(x^0, x, E)\|_1 \geq \varphi_1,$$

holds. Therefore, due to Lemma 1 the formula (12) is valid. Then, according to Lemma 3 the portfolio $x^0 \in P^s(E + E')$ for any perturbing matrix $E' \in \Omega_1(\varphi_1)$. Thus, $\rho^s(x^0, 1, m) \geq \varphi_1(x^0, 1, m)$.

Case 2: $1 < p \leq \infty$. According to the definition of $\varphi_2 = \varphi_2(x^0, p, m)$ the inequalities hold

$$\|g^+(x^0, x, E)\|_p \geq \varphi_2(\|x^0\|_q + \|x\|_q), \quad x \in X \setminus \{x^0\}.$$

Applying Lemma 2 yields the conclusion that for any portfolio $x \neq x^0$ the formula (15) holds. Hence from Lemma 3 it follows that the portfolio $x^0 \in P^s(E + E')$ for $E' \in \Omega_p(\varphi_2)$. Therefore, $\rho^s(x^0, p, m) \geq \varphi_2(x^0, p, m)$.

Further we will prove the validity of the upper bound (19) for any number $p \in [1, \infty]$. Let $\varepsilon > m^{1/p}\psi > 0$, and a portfolio $x^* \neq x^0$ be such that

$$\|g^+(x^0, x^*, E)\|_p = \psi\|x^0 - x^*\|_q.$$

Then, taking into account the continuous dependence of the norm of a vector on its coordinates we find a vector $\delta \in \mathbf{R}^s$ with positive components, which satisfy inequalities (16) shach that $\varepsilon/m^{1/p} > \|\delta\|_p > \psi$. Hence, due to Lemma 4 there exists a perturbing matrix $E^0 \in \Omega_p(\varepsilon)$ such that the portfolio $x^0 \in P^s(E)$ is not a Pareto-optimal portfolio of the perturbed problem $Z^s(E + E^0)$. Thus, we proved that for any number $\varepsilon > m^{1/p}\psi$ the inequality $\rho^s(x^0, p, m) < \varepsilon$ holds, i.e. the inequality $\rho^s(x^0, p, m) \leq m^{1/p}\psi(x^0, p, m)$ is true for any number $p \in [1, \infty]$. □

4 Corollary

All of the following corollaries from Theorem are obvious and are valid for any number of criteria $s \in \mathbf{N}$.

Corollary 1. *For any* $m \in \mathbf{N}$ *the following bounds are true:*

$$m\varphi_1(x^0, 1, m) \geq \rho^s(x^0, 1, m) \geq \varphi_1(x^0, 1, m).$$

Hence we get the following well-known result, which shows that lower and upper bounds (19) are attainable for $p = m = 1$.

Corollary 2 [7, 10]. *The following formula holds:*

$$\rho^s(x^0, 1, 1) = \varphi_1(x^0, 1, 1) = \min_{x \in X \setminus \{x^0\}} \|[E(x^0 - x)]^+\|_1.$$

Corollary 3. *For any $m \in \mathbf{N}$ the following bounds are true:*

$$\psi(x^0, \infty, m) = \min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} \frac{E_{i'k} x^0 - E_{ik} x}{\|x^0 - x\|_1} \geq \rho^s(x^0, \infty, m) \geq$$
$$\geq \min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} \frac{E_{i'k} x^0 - E_{ik} x}{\|x^0\|_1 + \|x\|_1} = \varphi_2(x^0, \infty, m). \quad (20)$$

In paper [1] we proved the attainability of such bounds for the stability radius of the Pareto-optimal portfolio of the multicriteria investment problem with Savage's minimax criteria in the metric l_∞. Using the developed there techniques it is easy to prove, that lower and upper bounds (20), obtained here, are also attainable. In addition, the next statement follows from Corollary 3 and shows that lower and upper bound are attainable for $p = \infty$.

Corollary 4. *If for any portfolio $x \in X \setminus \{x^0\}$ the inequality $\|x^0\|_1 + \|x\|_1 = \|x^0 - x\|_1$ holds, then for index $m \in \mathbf{N}$ the following formula is true:*

$$\rho^s(x^0, \infty, m) = \varphi_2(x^0, \infty, m) = \psi(x^0, \infty, m).$$

Note that earlier in paper [7] (see also [8, 9]) the formula of the stability radius of the Pareto-optimal solution x^0 of the multicriteria linear Boolean programming problem $Z^s_B(E)$ (see (1)) in the Hölder metric was obtained:

$$\rho^s(x^0, p, 1) = \psi(x^0, p, 1) = \min_{x \in X \setminus \{x^0\}} \frac{\|[E(x^0 - x)]^+\|_p}{\|x^0 - x\|_q}, \qquad 1 \leq p \leq \infty.$$

This result shows that upper bound (19) is attainable in the linear case ($m = 1$).

Corollary 5. *For any parameters $m \in \mathbf{N}$ and $p \in [1, \infty]$ the stability radius $\rho^s(x^0, p, m) > 0$ if and only if*

$$\min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} g_k^+(x^0, x, E_k) > 0.$$

Remark. Due to equivalence of any two metrics in finite dimensional linear spaces (see e. g. [11]), Corollary 5 is also valid not only for the Hölder metric l_p, but for another metrics in the space $\mathbf{R}^{m \times n \times s}$ of perturbing parameters of $Z^s(E)$.

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