BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(70), 2012, Pages 53–62
ISSN 1024–7696

Solving the games generated by the informational extended strategies of the players

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Abstract. In this article we study the non-informational extended games which are generated by the two-directional informational flow extended strategies of the players. The theorem on the existence of the Nash equilibrium profiles in this type of games is also proved.

Mathematics subject classification: 91A10, 47H04, 47H10.
Keywords and phrases: Noncooperative game, payoff function, set of strategies, informational extended games, normal form game, Nash equilibrium, best response mapping, point-to-set mapping, fixed point theorem.

Let

$$\Gamma = \langle I; X_p, p \in I; H_p : X \to \mathbb{R} \rangle \tag{1}$$

be the strategic form or normal form of the static noncooperative games with complete and imperfect information[1] where $I = \{1, 2, ..., n\}$ is the set of players, X_p is a set of available alternatives of the player $p \in I$, $H_p : X_p \to R$ is the payoff function of the player $p \in I$ and $X = \prod_{p \in I} X_i$ is the set of strategy profiles for the game. In [1] the author studied informational extensions of the games (1), generated by a one-way directional informational flow, denoted by $j \overset{\text{inf}}{\to} i$, which means: the player i, and only he, knows exactly what value of the strategy will be chosen by the player j, and two-directional informational flow, denoted by $i \overset{\text{inf}}{\leftrightarrows} j$, which means[2]: at any time simultaneously player i knows exactly what value of the strategy will be chosen by the player j and player j knows exactly what value of the strategy will be chosen by the player i. We mention that the game is static, in other words, the order of the chosen strategies is not significant. The players do not known the informational type of the other players, so the player i (respectively j) does not know that the player j (respectively i) knows what value of the strategies will be chosen. In the general case [2, 3] the set of the informational extended strategies of the player i (respectively j) is the set of the functions $\Theta_i = \{\theta_i : X_j \to X_i\}$ (respectively $\Theta_j = \{\theta_j : X_i \to X_j\}$)

[1] So the players know exactly their and of the other players payoff functions and they know the sets of strategies. Players do not know what kind of the strategy will be chosen by the players.

[2] Notation $i \overset{\text{inf}}{\leftrightarrows} j$ means the following: "the information about the concrete chosen value of the strategies of player i will be transmitted to the player j" and vice versa "the information about the concrete chosen value of the strategies of player j will be transmitted to the player i".

such that $\forall x_j \in X_j$, $\theta_i(x_j) \in X_i$ (respectively $\forall x_i \in X_i$, $\theta_j(x_i) \in X_j$). Following [1], if in the game Γ the sets of strategies $X_i = \left\{x_i^1, ..., x_i^l, ..., x_i^{|X_i|}\right\}$ and $X_j = \left\{x_j^1, ..., x_j^l, ..., x_j^{|X_j|}\right\}$ of the players i and j are at most countable, H_p is the discrete payoff function of the player $p \in I$, the sets of the informational extended strategies can be represented as $\Theta_i = \{\theta_i^\alpha : X_j \to X_i^\alpha, \alpha = 1, ..., \varkappa_i\}$ and respectively $\Theta_j = \left\{\theta_j^\beta : X_i \to X_j^\beta, \beta = 1, ..., \varkappa_j\right\}$, where

$$X_i^\alpha = \left\{\left(x_i^{\alpha_1}, x_i^{\alpha_2}, ..., x_i^{\alpha_l}, ..., x_i^{\alpha|X_j|}\right) : x_i^{\alpha_l} \in X_i, \forall l = \overline{1, |X_j|}\right\} \subseteq X_i,$$

$$X_j^\beta = \left\{\left(x_j^{\beta_1}, x_j^{\beta_2}, ..., x_j^{\beta_k}, ..., x_j^{\beta|X_j|}\right) : x_j^{\beta_k} \in X_i, \forall k = \overline{1, |X_j|}\right\} \subseteq X_j$$

for any $\alpha = 1, ..., \varkappa_i = |X_i|^{|X_j|}$, $\beta = 1, ..., \varkappa_j = |X_j|^{|X_i|}$. Thereby, the informational extended strategies of the player i are functions $\theta_i^\alpha : X_j \to X_i^\alpha$ such that for all $x_j^l \in X_j$ there is $x_i^{\alpha_l} \in X_i$ such that $\theta_i^\alpha\left(x_j^l\right) = x_i^{\alpha_l}$ and it means the following: the player i will choose the non-informational extended strategy $x_i^{\alpha_l} \in X_i^\alpha$ in case the player j will choose the non-informational extended strategy $x_j^l \in X_j$. Accordingly the informational extended strategies of the player j are functions $\theta_j^\beta : X_i \to X_j^\beta$ such that for all $x_i^k \in X_i$ there is $x_j^{\beta_k} \in X_j$ such that $\theta_j^\beta\left(x_i^k\right) = x_j^{\beta_k}$ and it means the following: the player j will choose the non-informational extended strategy $x_j^{\beta_k} \in X_j^\beta$ in case the player i will choose the non-informational extended strategy $x_i^k \in X_i$. Under the assumption that the players want maximize their payoffs we define the payoff functions of the player as follows:

$$\mathcal{H}_p\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) = \begin{cases} \max\limits_{(x_i,x_j)\in\left[gr\theta_i^\alpha \bigcap gr\theta_j^\beta\right]} H_p\left(x_i, x_j, x_{[-ij]}\right) & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) \neq \varnothing, \\ -\infty & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) = \varnothing. \end{cases}$$

Here $X\left(\theta_i^\alpha, \theta_j^\beta\right) \subseteq X$ is the set of the strategy profiles of the players in the game (1) "generated" by the informational extended strategies θ_i^α and θ_j^β, $gr\theta_i^\alpha$, $gr\theta_j^\beta$ denotes the graph of the function θ_i^α and θ_j^β, $x_{[-ij]} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, x_{j+1}, ..., x_n)$. And finally, the normal form of the two-directional $i \overset{\text{inf}}{\leftrightarrows} j$ informational extended game will be $\Gamma\left(i \overset{\text{inf}}{\leftrightarrows} j\right) = \left\langle I, \Theta_i, \Theta_j, \{X\}_{p\in I\setminus\{i,j\}}, \{\mathcal{H}_p\}_{p\in I}\right\rangle$. Also by [1] for the bimatricial game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ the normal form of the $1 \overset{\text{inf}}{\leftrightarrows} 2$ informational extended game will be the bimatricial game with the following payoff matrices for the player 1

$$\mathcal{H}_1 = \begin{pmatrix} 3 & 5 & 4 & 3 & 5 & 3 & 4 & 5 & 4 \\ 6 & 7 & 2 & 7 & 6 & 2 & 6 & 2 & 7 \\ 3 & 5 & 2 & 3 & 5 & 3 & -\infty & 3 & -\infty \\ 3 & 7 & 4 & 7 & -\infty & 3 & 4 & -\infty & 7 \\ 6 & 5 & 4 & -\infty & 7 & -\infty & 3 & 5 & 4 \\ 3 & 7 & 2 & 7 & -\infty & 3 & -\infty & 2 & 7 \\ 6 & 7 & 4 & -\infty & 6 & 2 & 6 & 5 & -\infty \\ 6 & 7 & 4 & 7 & 6 & -\infty & 6 & -\infty & 7 \end{pmatrix}$$

and for the player 2 correspondingly

$$\mathcal{H}_2 = \begin{pmatrix} 0 & 5 & 1 & 0 & 5 & 0 & 1 & 5 & 1 \\ 4 & 3 & 2 & 3 & 4 & 2 & 4 & 2 & 3 \\ 0 & 5 & 2 & 0 & 5 & 2 & -\infty & 5 & -\infty \\ 0 & 3 & 1 & 3 & -\infty & 0 & 1 & -\infty & 3 \\ 4 & 5 & 1 & -\infty & 5 & -\infty & 0 & 5 & 1 \\ 0 & 3 & 2 & 3 & -\infty & 2 & -\infty & 2 & 3 \\ 4 & 5 & 2 & -\infty & 5 & 2 & 4 & 5 & -\infty \\ 4 & 3 & 1 & 3 & 4 & -\infty & 4 & -\infty & 3 \end{pmatrix}.$$

Below the correspondence between Nash equilibrium profiles in the $\Gamma\left(1 \overset{\text{inf}}{\leftrightarrows} 2\right)$ game and profiles in the Γ game is shown:

$$\left(\theta_1^1,\theta_2^8\right) \Rightarrow (1,2)\,;\ \left(\theta_1^2,\theta_2^1\right) \Rightarrow (2,1)\,;\ \left(\theta_1^2,\theta_2^7\right) \Rightarrow (2,1)\,;\ \left(\theta_1^4,\theta_2^2\right) \Rightarrow (2,2)\,;$$
$$\left(\theta_1^4,\theta_2^4\right) \Rightarrow (2,2);\ \left(\theta_1^4,\theta_2^9\right) \Rightarrow (2,2);\ \left(\theta_1^5,\theta_2^5\right) \Rightarrow (2,1);\ \left(\theta_1^5,\theta_2^8\right) \Rightarrow (1,2);$$
$$\left(\theta_1^6,\theta_2^2\right) \Rightarrow (2,2)\,;\ \left(\theta_1^6,\theta_2^4\right) \Rightarrow (2,2);\ \left(\theta_1^6,\theta_2^9\right) \Rightarrow (2,2);\ \left(\theta_1^7,\theta_2^2\right) \Rightarrow (1,2)\,;$$
$$\left(\theta_1^7,\theta_2^8\right) \Rightarrow (1,2);\ \left(\theta_1^8,\theta_2^1\right) \Rightarrow (2,1)\,;\ \left(\theta_1^8,\theta_2^7\right) \Rightarrow (2,1).$$

Here the informational extended strategy of the player 1 is the function with the following values:

$\theta_1^1(j) = 1\ \forall j = 1,2,3;\ \theta_1^2(j) = 2\ \forall j = 1,2,3;\ \theta_1^3(1) = \theta_1^3(2) = 1,\ \theta_1^3(3) = 2;$
$\theta_1^4(1) = \theta_1^4(3) = 1,\ \theta_1^4(2) = 2;\ \theta_1^5(2) = \theta_1^5(3) = 1,\ \theta_1^5(1) = 2;\ \theta_1^6(2) =$
$\theta_1^6(3) = 2,\ \theta_1^6(1) = 1;\ \theta_1^7(1) = \theta_1^7(3) = 2,\ \theta_1^7(2) = 1;\ \theta_1^8(1) = \theta_1^8(2) = 2,$
$\theta_1^8(3) = 1$

and correspondingly for the player 2:

$\theta_2^1(i) = 1\ \forall i = 1,2;\ \theta_2^2(i) = 2\ \forall i = 1,2;\ \theta_2^3(i) = 3\ \forall i = 1,2;\ \theta_2^4(1) = 1,$
$\theta_2^4(2) = 2;\ \theta_2^5(2) = 1,\ \theta_2^5(1) = 2;\ \theta_2^6(1) = 1,\ \theta_2^6(2) = 3;\ \theta_2^7(2) = 1,$
$\theta_2^7(1) = 3;\ \theta_2^8(1) = 2,\ \theta_2^8(2) = 3;\ \theta_2^9(1) = 3,\ \theta_2^9(2) = 2.$

Remark 1. We have to mention the following: the informational extended game (considered in this article) is not a dynamic game (in terms of the choice of the strategy, but not in terms of the strategies structure) because the strategies are chosen simultaneously.

In this article we study the case when the informational strategies of the players have already been chosen and so appears the necessity to study the informational non-extended game generated by the chosen informational extended strategies. These games differ in: a) the sets of the strategies that are the subsets of the sets of strategies in the initial non-extended informational game; b) how the payoff functions of the players will be constructed.

Let the payoff functions of the players be defined as $\widetilde{H}_p : \prod_{p\in I} X_p \rightarrow R$ where for all $x_i \in X_i$, $x_j \in X_j$, $x_{[-ij]} \in X_{[-ij]}$ we have $\widetilde{H}_p\left(x_i, x_j, x_{[-ij]}\right) \equiv H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right)$.

Definition 1. The game with the following normal form

$$\Gamma\left(\theta_i, \theta_j\right) = \left\langle I, \left\{X_p\right\}_{p\in I}, \left\{\widetilde{H}_p\right\}_{p\in I}\right\rangle \tag{2}$$

will be called informational non-extended game generated by the informational extended strategies θ_i and θ_j.

The game $\Gamma\left(\theta_i, \theta_j\right)$ is played as follows: independently and simultaneously each player $p \in I$ chooses the informational non-extended strategy $x_p \in X_p$, after that the players i and j calculate the value of the informational extended strategies $\theta_i(x_j)$ and $\theta_j\left(x_j\right)$, after that each player calculates the payoff values $H_p\left(\theta_i(x_j), \theta_j\left(x_j\right), x_{[-ij]}\right)$, and with this the game is finished. To all strategy profiles $\left(x_i, x_j, x_{[-ij]}\right)$ in the game (2) the following realization $\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right)$ in terms of the informational extended strategies will correspond.

We introduce the following definition of the Nash equilibrium profile for normal form game $\Gamma\left(\theta_i, \theta_j\right)$.

Definition 2. The strategy profile $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in X$ is called the Nash equilibrium of the game $\Gamma\left(\theta_i, \theta_j\right)$ if and only if the following conditions are satisfied:

$$\begin{cases} \widetilde{H}_i\left(x_i^*, x_j^*, x_{[-ij]}^*\right) \geqslant \widetilde{H}_i\left(x_i, x_j^*, x_{[-ij]}^*\right) \text{ for all } x_i \in X_i, \\ \widetilde{H}_j\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \geqslant \widetilde{H}_j\left(x_i^*, x_j, x_{[-ij]}^*\right) \text{ for all } x_j \in X_j, \\ \widetilde{H}_p\left(x_i^*, x_i^*, x_p^*\right) \geqslant \widetilde{H}_p\left(x_i^*, x_j^*, x_p\right) \text{ for all } x_p \in X_p \text{ and for all } p \in I\backslash\{i, j\}. \end{cases}$$

We denote by $NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ the set of Nash equilibrium profiles of the game $\Gamma\left(\theta_i, \theta_j\right)$. According to Definition 1 we have that $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ if and only if

$$\begin{cases} H_i\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_i\left(\theta_i(x_j^*), \theta_j\left(x_i\right), x_{[-ij]}^*\right) \text{ for all } x_i \in X_i, \\ H_j\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_j\left(\theta_i(x_j), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \text{ for all } x_j \in X_j, \\ H_p\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_p\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_p\right) \text{ for all } x_p \in X_p\ p \in I\backslash\{i,j\}. \end{cases}$$

Another, and some times more convenient way of defining Nash equilibrium is via the best response correspondences $Br_p : \prod\limits_{k\in I\backslash\{p\}} X_k \to 2^{X_p}$ such that:

- for player i :

$$\begin{aligned} Br_i\left(x_{[-i]}\right) &= \{x_i \in X_i : H_i\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right) \geq H_i\left(\theta_i(x_j), \theta_j\left(x_i'\right),\right. \\ &\quad \left. x_{[-ij]}\right) \text{ for all } x_i' \in X_i\}; \end{aligned}$$

- for player j :

$$\begin{aligned} Br_j\left(x_{[-j]}\right) &= \{x_j \in X_j : H_j\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right) \geq H_j\left(\theta_i(x_j'), \theta_j\left(x_i\right),\right. \\ &\quad \left. x_{[-ij]}\right) \text{ for all } x_j' \in X_j\}; \end{aligned}$$

- for player $p \neq i, j$:

$$\begin{aligned} Br_p\left(x_{[-p]}\right) &= \{x_p \in X_p : H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_p, x_{[-ijp]}\right) \geq H_p\left(\theta_i(x_j),\right. \\ &\quad \left. \theta_j\left(x_i\right), x_p', x_{[-ijp]}\right) \text{ for all } x_p' \in X_p\}. \end{aligned}$$

Here 2^{X_p} denotes the set of all subsets of the set X_p and $x_{[-ijp]}$ denotes the strategies profiles without the strategies of the players i, j and p. If the payoff functions $H_p(\cdot), \left(p = \overline{1,n}\right)$ are continuous on the compact $\prod\limits_{p\in I} X_p$ and the functions $\theta_i : X_j \to X_i$, $\theta_j : X_i \to X_j$ are continuous on the compact X_j (correspondingly X_i) then the functions $\widetilde{H}_p, p = \overline{1,n}$ are continuous on the compact $\prod\limits_{p\in I} X_p$ as composite functions. Then according to the Weierstrass theorem we can write

$$\begin{aligned} Br_i\left(x_{[-i]}\right) &= Arg \max_{x_i \in X_i} H_i\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right), \\ Br_j\left(x_{[-j]}\right) &= Arg \max_{x_j \in X_j} H_j\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right), \\ Br_p\left(x_{[-p]}\right) &= Arg \max_{x_p \in X_p} H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_p, x_{[-ijp]}\right) \end{aligned}$$

for all $p \in I$, $p \neq i, j$. In this case $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ if and only if

$$\begin{cases} x_i^* \in Br_i\left(x_{[-i]}^*\right), \\ x_j^* \in Br_j\left(x_{[-j]}^*\right), \\ x_p^* \in Br_p\left(x_{[-p]}^*\right) \ \forall p \neq i, j. \end{cases}$$

Construct the point to set mapping $Br : X \to 2^X$ so that for all $x \in X$,

$$Br(x) = \left(Br_1\left(x_{[-1]}\right), ..., Br_i\left(x_{[-i]}\right), ..., Br_n\left(x_{[-n]}\right)\right) \subseteq X.$$

Then

$$\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$$

if and only if

$$\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \in Br\left(x_i^*, x_i^*, x_{[-ij]}^*\right),$$

that is $\left(x_i^*, x_i^*, x_{[-ij]}^*\right)$ is the fixed point of the mapping Br.

We shall analyze in more details the case of the bimatricial game. So we consider the following normal form game

$$\Gamma = \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, H_1 = \|a_{ij}\|_{i\in I}^{j\in J}, H_2 = \|b_{ij}\|_{i\in I}^{j\in J} \right\rangle.$$

For this game we construct the game according to Definition 1. The informational extended strategies are $\theta_1 : J \to I$ and $\theta_2 : I \to J$, the payoff matrices are $\widetilde{H}_1 = \|\widetilde{a}_{ij}\|_{i\in I}^{j\in J}$, $\widetilde{H}_2 = \left\|\widetilde{b}_{ij}\right\|_{i\in I}^{j\in J}$ where $\widetilde{a}_{ij} = a_{\theta_1(j)\theta_2(i)}$ and $\widetilde{b}_{ij} = b_{\theta_1(j)\theta_2(i)}$ for all $i \in I$, $j \in J$. So we will obtain the following normal form game

$$\Gamma\left(\theta_1, \theta_2\right) = \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, \widetilde{H}_1 = \|\widetilde{a}_{ij}\|_{i\in I}^{j\in J}, \ \widetilde{H}_2 = \left\|\widetilde{b}_{ij}\right\|_{i\in I}^{j\in J} \right\rangle \equiv$$
$$\equiv \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, \widetilde{H}_1 = \left\|a_{\theta_1(j)\theta_2(i)}\right\|_{i\in I}^{j\in J}, \ \widetilde{H}_2 = \left\|b_{\theta_1(j)\theta_2(i)}\right\|_{i\in I}^{j\in J} \right\rangle.$$

The strategy profile $(i^e, j^e) \in NE\left(\Gamma\left(\theta_1, \theta_2\right)\right)$ if and only if

$$\begin{cases} \widetilde{a}_{i^e j^e} \geqslant \widetilde{a}_{i j^e} \text{ for all } i \in I, \\ \widetilde{b}_{i^e j^e} \geqslant \widetilde{b}_{i^e j} \text{ for all } j \in J, \end{cases}$$

and according to Definition 1 we have that

$$\begin{cases} a_{\theta_1(j^e)\theta_2(i^e)} \geqslant a_{\theta_1(j^e)\theta_2(i)} \text{ for all } i \in I, \\ b_{\theta_1(j^e)\theta_2(i^e)} \geqslant b_{\theta_1(j)\theta_2(i^e)} \text{ for all } j \in J. \end{cases}$$

From the set of all informational extended strategies of the players i and j we will highlight the following class of "best responses" strategies

$$\widetilde{\Theta}_i = \left\{ \widetilde{\theta}_i : X_j \to X_i \mid \forall x_j \in X_j, \ \widetilde{\theta}_i(x_j) = \arg\max_{x_i \in X_i} H_i\left(x_i, x_j, x_{[-ij]}\right) \right\}, \quad (3)$$

$$\widetilde{\Theta}_j = \left\{ \widetilde{\theta}_j : X_i \to X_j \mid \forall x_i \in X_i, \ \widetilde{\theta}_j(x_i) = \arg\max_{x_j \in X_j} H_j\left(x_i, x_j, x_{[-ij]}\right) \right\}. \quad (4)$$

We consider now the following examples of the informational non-extended game $\Gamma\left(\theta_i, \theta_j\right)$ generated by the strategies type (3)-(4) of the players.

Example 1. Let us consider the two person game for which $X = [0,1]$, $Y = [0,1]$, $H_1(x,y) = \frac{3}{2}xy - x^2$, $H_2(x,y) = \frac{3}{2}xy - y^2$ are the sets of strategies and the payoff functions of the players. We construct the normal form game $\Gamma(\theta_i, \theta_j)$ generated by the informational extended strategies type (3)-(4) and determine the Nash equilibrium solution.

Solution. We determine the equilibrium profile using the "best response" mapping. We derive the best response (reaction) function for each player given the other players strategy. Because the problem is symmetric, first we will show only for player 1 and then apply the result to the case for player 2. First order condition will give us that $Br_1(y) = Arg \max\limits_{x\in[0,1]} H_1(x,y) = \left\{x \in [0,1]/x = \frac{3}{4}y\right\}$ where $Br_1(y)$ is the best response correspondence for player 1. Similarly, the best response corresponding to player 2 is $Br_2(x) = Arg \max\limits_{y\in[0,1]} H_2(x,y) = \left\{y \in [0,1]/y = \frac{3}{4}x\right\}$. So the solution of the problem $\left\{\begin{array}{l} x^* \in Br_1(y^*) \\ y^* \in Br_2(x^*) \end{array}\right.$ is $x^* = y^* = 0$. Consider the informational non-extended game generated by the informational extended strategies of the two-directional informational flow type $1 \overset{\inf}{\leftrightarrows} 2$. As informational extended strategies we will use the functions $\theta_1 : Y \to X$ where $\forall y \in Y$, $\theta_1(y) = \arg\max\limits_{x\in X} H_1(x,y)$, respectively $\theta_2 : X \to Y$ where $\forall x \in X$, $\theta_2(x) = \arg\max\limits_{y\in Y} H_2(x,y)$. Using the necessary condition of optimality we obtain that $\theta_1(y) = \frac{3}{4}y$ $\forall y \in [0,1]$ and $\theta_2(x) = \frac{3}{4}x$ $\forall x \in [0,1]$. Thus $\widetilde{H}_1(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_1(y))^2 = \frac{3}{2}\left(\frac{3}{4}y\right)\left(\frac{3}{4}x\right) - \left(\frac{3}{4}y\right)^2 = \frac{27}{32}xy - \frac{9}{16}y^2$ and $\widetilde{H}_2(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_2(x))^2 = \frac{3}{2}\left(\frac{3}{4}y\right)\left(\frac{3}{4}x\right) - \left(\frac{3}{4}x\right)^2 = \frac{27}{32}xy - \frac{9}{16}x^2$. So we obtain the following normal form game $\Gamma(\theta_1,\theta_2) = \left\langle X, Y, \widetilde{H}_1, \widetilde{H}_2 \right\rangle$. Determine the equilibrium profile of the game $\Gamma(\theta_1,\theta_2)$. According to the definition $(x^*, y^*) \in NE[\Gamma(\theta_1,\theta_2)]$ if and only if $\left\{\begin{array}{l} \widetilde{H}_1(x^*,y^*) \geqslant \widetilde{H}_1(x,y^*)\ \forall x \in X, \\ \widetilde{H}_2(x^*,y^*) \geqslant \widetilde{H}_2(x^*,y)\ \forall y \in Y. \end{array}\right.$ So we have

$$\left\{\begin{array}{l} \dfrac{3}{2}\theta_1(y^*)\theta_2(x^*) - (\theta_1(y^*))^2 \geqslant \dfrac{3}{2}\theta_1(y^*)\theta_2(x) - (\theta_1(y^*))^2\ \ \forall x \in X, \\ \dfrac{3}{2}\theta_1(y^*)\theta_2(x^*) - (\theta_1(y^*))^2 \geqslant \dfrac{3}{2}\theta_1(y)\theta_2(x^*) - (\theta_2(x^*))^2\ \ \forall y \in Y, \end{array}\right.$$

and finally

$$\left\{\begin{array}{l} \dfrac{27}{32}x^*y^* - \dfrac{9}{16}y^{*2} = \max\limits_{x\in[0,1]}\left\{\dfrac{27}{32}xy^* - \dfrac{9}{16}(y^*)^2\right\}, \\ \dfrac{27}{32}x^*y^* - \dfrac{9}{16}y^{*2} = \max\limits_{y\in[0,1]}\left\{\dfrac{27}{32}x^*y - \dfrac{9}{16}(x^*)^2\right\}. \end{array}\right.$$

Thus the Nash equilibrium profile is $(x^*, y^*) = (1,1)$, that is $NE[\Gamma(\theta_1,\theta_2)] = \{(1,1)\}$ while $NE[\Gamma] = \{(0,0)\}$.

Example 2. We consider the following bimatricial game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 =$

$\begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$. We construct the normal form game generated by the informational extended strategies of type (3)-(4) and we determine the Nash equilibrium profiles.

Solution. The strategies of the type (3)-(4) in the game $\Gamma\left(2 \overset{\inf}{\leftrightarrows} 1\right)$ are $i^*(j) = \arg\max_i a_{ij} = \begin{cases} 1 \text{ if } j = 3, \\ 2 \text{ if } j = 1, 2 \end{cases}$ and $j^*(i) = \arg\max_j b_{ij} = \begin{cases} 2 \text{ if } i = 1, \\ 1 \text{ if } i = 2. \end{cases}$ We construct the game $\Gamma(i^*, j^*)$ according to Definition 1. In the table below the correspondence between the strategies profile in the informational non-extended game (initial game) and the strategies profile generated by the informational extended strategies i^* and j^* is presented

(i,j)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$(i^*(j), j^*(i))$	$(2,2)$	$(2,2)$	$(1,2)$	$(2,1)$	$(2,1)$	$(1,1)$

Then the payoff matrices will be $\widetilde{H}_1 = \left\| a_{i^*(j)j^*(i)} \right\|_{i \in I}^{j \in J} = \begin{pmatrix} 7 & 7 & 5 \\ 6 & 6 & 3 \end{pmatrix}$,
$\widetilde{H}_2 = \left\| b_{i^*(j)j^*(i)} \right\|_{i \in I}^{j \in J} = \begin{pmatrix} 3 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix}$ and so we have the following normal form game

$$\Gamma(i^*,, j^*) = \left\langle I = \{1,2\}, J = \{1,2,3\},\ \widetilde{H}_1 = \begin{pmatrix} 7 & 7 & 5 \\ 6 & 6 & 3 \end{pmatrix},\ \widetilde{H}_2 = \begin{pmatrix} 3 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix} \right\rangle.$$

The game is done in the following way. In case the players choose the informational extended strategies i^* and j^*, then the game (for players 1 and 2) like "if-then" starts, i.e. "if the player 1 chooses the line 1, then the player 2, knowing this, chooses the column 2 and simultaneously, if the player 2 chooses the column 1, then the player 1, knowing this, chooses the line 2 etc. We note that the equilibrium profile in the game $\Gamma(i^*, j^*)$ is $(i^e, j^e) = (1,3)$ and $\widetilde{H}_1(1,3) = 5$, $\widetilde{H}_2(1,3) = 5$. To this profile corresponds the following profile in the informational extended strategy $(i^*(j^e), j^*(i^e)) = (i^*(3), j^*(1)) = (1,2)$ for which we have that $H_1(1,2) = 5$, $H_2(1,2) = 5$. According to the definition of the Nash equilibrium profile (i^e, j^e) we have that

$$\begin{cases} a_{i^*(j^e)j^*(i^e)} \geqslant a_{i^*(j^e)j^*(i)} \text{ for all } i = 1,2, \\ b_{i^*(j^e)j^*(i^e)} \geqslant b_{i^*(j)j^*(i^e)} \text{ for all } j = 1,2,3, \end{cases}$$

from which we deduce the following relations:

$$\left\{ \widetilde{a}_{13} = 5 = a_{i^*(3)j^*(1)} > \widetilde{a}_{23} = 3 = a_{i^*(3)j^*(2)} = a_{11} = 3, \right.$$

$$\begin{cases} \widetilde{b}_{13} = 5 = b_{i^*(3)j^*(1)} > \widetilde{b}_{11} = 3 = b_{i^*(1)j^*(1)} = b_{22} = 3, \\ \widetilde{b}_{13} = 5 = b_{i^*(3)j^*(1)} > \widetilde{b}_{12} = 3 = b_{i^*(2)j^*(1)} = b_{22} = 3. \end{cases}$$

So we have shown that $NE\left[\Gamma(i^*, j^*)\right] = \{(1,3)\}$.

Remark 2. If the normal form $\Gamma(i^*, j^*)$ has already been constructed, then the equilibrium profile is determined using the matrices $\widetilde{H}_1$ and $\widetilde{H}_2$, otherwise, using the elements $a_{i^*(j)j^*(i)}$, $b_{i^*(j)j^*(i)}$ of the matrices of the game Γ.

We begin by proving Nash's Theorem about the existence of a strategy equilibrium profile in the normal form game $\Gamma(\theta_i, \theta_j)$ first giving some remarks about the Kakutani's fixed point theorem. **Kakutani's theorem states** [4]: *Let S be a non-empty, compact and convex subset of the Euclidean space R^n. Let $\varphi : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $\varphi(x)$ is non-empty and convex for all $x \in S$. Then φ has a fixed point.*

The Kakutani fixed point theorem is a fixed-point theorem for point-to-set mapping. It provides sufficient conditions for a point-to-set mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. Kakutani theorem extends this to point-to-set mapping.

Mathematician **John Nash** used the Kakutani fixed point theorem to prove a major result in game theory. Stated informally, the theorem implies the existence of a Nash equilibrium in every finite game with mixed strategies for any number of players. In this case, S is the set of tuples of strategies chosen by each player in a game. The function $\varphi(x)$ gives a new tuple where each player's strategy is his best response to other players' strategies at x. Since there may be a number of responses which are equally good, φ is set-valued rather than single-valued. Then the Nash equilibrium of the game is defined as a fixed point of φ, i.e. a tuple of strategies where each player's strategy is a best response to the strategies of the other players. Kakutani's theorem ensures that this fixed point exists.

Let us prove the following theorem.

Theorem. *Let* $\Gamma(\theta_i, \theta_j) = \left\langle I, \{X_p\}_{p\in I}, \left\{\widetilde{H}_p\right\}_{p\in I} \right\rangle$ *be the normal form of the informational non-extended game generated by the informational extended strategies using the* $i \overset{\inf}{\leftrightarrows} j$ *type flow of information, where for all* $x_i \in X_i$, $x_j \in X_j$, $x_{[-ij]} \in X_{[-ij]}$ *we have* $\widetilde{H}_p\left(x_i, x_j, x_{[-ij]}\right) \equiv H_p\left(\theta_i(x_j), \theta_j(x_i), x_{[-ij]}\right)$. *Let this game satisfy the following conditions:*

1) *the X_p is a non-empty compact and convex subset of the finite-dimensional Euclidean space for all $p \in I$;*

2) *the functions θ_i (correspondingly θ_j) are continuous on X_j (correspondingly on X_i) and the functions H_p are continuous on X for all $p \in I$.*

3) *the functions θ_i (correspondingly θ_j) are quasi-concave on X_j (correspondingly on X_i), the functions H_p are quasi-concave on X_p, $p \in I \backslash \{i, j\}$ and monotonically increasing on $X_i \times X_j$.*

Then $NE\left[\Gamma\left(\theta_i,\theta_j\right)\right]\neq\emptyset$.

Proof. If we define the following correspondence (point-to-set mapping) $Br : X \rightarrow X$ such that $Br\left(x\right) = \left(Br_1\left(x_{[-1]}\right),...,Br_i\left(x_{[-i]}\right),...,Br_n\left(x_{[-n]}\right)\right)$ then if $x^* \in Br\left(x^*\right)$, then $x_i^* \in Br_i\left(x^*_{[-i]}\right)$ for all $i \in I$ and hence $x^* \in NE$. To prove this theorem we can show that: a) the X is a non-empty compact and convex subset of the Euclidean finite-dimensional space and b) the set-valued mapping $Br : X \rightarrow X$ has a closed graph, that is, if $\{x^k, y^k\} \rightarrow \{x, y\}$ with $y^k \in Br\left(x^n\right)$, then $y \in Br\left(x\right)$, and the set $Br\left(x\right)$ is nonempty, convex and compact for all $x \in X$. According to the Tikhonov's theorem: *a product of a family of compact topological spaces* $X = \prod\limits_{p\in I} X_p$ *is compact,* the item a) is fulfilled. For all $x_{[-i]}$ the set $Br_i\left(x_{[-i]}\right)$ is non-empty because according to conditions 1) and 2) $\widetilde{H}_i$ is continuous and X_i is compact (Weierstrass's theorem). According to condition 3) $Br_i\left(x_{[-i]}\right)$ is also convex because $\widetilde{H}_i$ is quasi-concave on X_i. Hence the set $Br\left(x\right)$is nonempty convex and compact for all $x \in X$. The mapping Br has a closed graph because each function $\widetilde{H}_p$ is continuous on X for all $p \in I$. Hence by Kakutani's theorem, the set-valued mapping Br has a fixed point. As we have noted, any fixed point is a Nash equilibrium. □

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Received July 04, 2012