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A Note on the Affine Subspaces of Three-Dimensional Lie Algebras

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Abstract. A classification of full-rank affine subspaces of (real) three-dimensional Lie algebras is presented. In the context of invariant control affine systems, this is exactly a classification of all full-rank systems evolving on three-dimensional Lie groups.

Mathematics subject classification: 17B20, 17B30, 93A10, 93B27.
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1 Introduction

In this note we exhibit a classification, under $\mathfrak{L}$-equivalence, of full-rank affine subspaces of (real) three-dimensional Lie algebras. Two affine subspaces are $\mathfrak{L}$-equivalent, provided there exists a Lie algebra automorphism mapping one to the other. This classification is presented in three parts (see Theorems 1, 2, and 3). Proofs are omitted. However, a full treatment of each part will appear elsewhere [4–6]. Tables detailing these results are included as an appendix.

It turns out that two left-invariant control affine systems are detached feedback equivalent if (and only if) their traces are $\mathfrak{L}$-equivalent. Therefore, a classification under $\mathfrak{L}$-equivalence induces one under detached feedback equivalence.

2 Three-dimensional Lie algebras

The classification of three-dimensional Lie algebras is well known. The classification over $\mathbb{C}$ was done by S. Lie (1893), whereas the standard enumeration of the real cases is that of L. Bianchi (1918). In more recent times, a different (method of) classification was introduced by C. Behr (1968) and others (see [12–14] and the references therein). This is customarily referred to as the *Bianchi-Behr classification*, or even the "Bianchi-Schücking-Behr classification". Accordingly, any real three-dimensional Lie algebra is isomorphic to one of eleven types (in fact, there are nine algebras and two parametrised infinite families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$[E_2, E_3] = n_1E_1 - aE_2$$
$$[E_3, E_1] = aE_1 + n_2E_2$$

$$[E_1, E_2] = n_3 E_3.$$

The (Bianchi-Behr) structure parameters a, n_1, n_2, n_3 for each type are given in Table 1.

Type	Notation	a	n_1	n_2	n_3	Representatives
I	$3\mathfrak{g}_1$	0	0	0	0	$\mathbb{R}^3$
II	$\mathfrak{g}_{3.1}$	0	1	0	0	$\mathfrak{h}_3$
$III = VI_{-1}$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	0	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
IV	$\mathfrak{g}_{3.2}$	1	1	0	0	
V	$\mathfrak{g}_{3.3}$	1	0	0	0	
VI_0	$\mathfrak{g}_{3.4}^0$	0	1	-1	0	$\mathfrak{se}(1,1)$
$VI_h,\ {}^{h<0}_{h\neq -1}$	$\mathfrak{g}_{3.4}^h$	$\sqrt{-h}$	1	-1	0	
VII_0	$\mathfrak{g}_{3.5}^0$	0	1	1	0	$\mathfrak{se}(2)$
$VII_h,\ h>0$	$\mathfrak{g}_{3.5}^h$	$\sqrt{h}$	1	1	0	
$VIII$	$\mathfrak{g}_{3.6}$	0	1	1	-1	$\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(2,1)$
IX	$\mathfrak{g}_{3.7}$	0	1	1	1	$\mathfrak{su}(2)$, $\mathfrak{so}(3)$

Table 1. Bianchi-Behr classification

We note that for the two infinite families, VI_h and VII_h, each value of the parameter h yields a distinct (i. e., non-isomorphic) Lie algebra. Furthermore, for the purposes of this paper, type $III = VI_{-1}$ will be considered as part of VI_h.

3 Affine subspaces and classification

An affine subspace Γ of a Lie algebra $\mathfrak{g}$ is written as

$$\Gamma = A + \Gamma^0 = A + \langle B_1, B_2, \ldots, B_\ell \rangle$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Let Γ_1 and Γ_2 be two affine subspaces of $\mathfrak{g}$. We say that Γ_1 and Γ_2 are $\mathfrak{L}$-*equivalent* if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. $\mathfrak{L}$-equivalence is a genuine equivalence relation. An affine subspace Γ is said to have *full rank* if it generates the whole Lie algebra (i.e., the smallest Lie algebra containing Γ is $\mathfrak{g}$). Note that the full-rank property is invariant under $\mathfrak{L}$-equivalence.

Clearly, if Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent, then they are necessarily of the same dimension. Furthermore, $0 \in \Gamma_1$ if and only if $0 \in \Gamma_2$. We shall find it convenient to refer to an ℓ-dimensional affine subspace Γ as an $(\ell, 0)$-affine subspace when $0 \in \Gamma$ (i.e., Γ is a vector subspace) and as an $(\ell, 1)$-affine subspace, otherwise.

Remark 1. No $(1,0)$-affine subspace has full rank. A $(1,1)$-affine subspace has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A $(2,0)$-affine subspace has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any $(2,1)$-affine subspace or $(3,0)$-affine subspace has full rank.

There is only one affine subspace whose dimension coincides with that of the Lie algebra $\mathfrak{g}$, namely the space itself. From the standpoint of classification, this case is trivial and hence will not be covered explicitly.

Let us fix a three-dimensional Lie algebra $\mathfrak{g}$ (together with an ordered basis). In order to classify the affine subspaces of $\mathfrak{g}$, one requires the (group of) automorphisms of $\mathfrak{g}$. These are well known (see, e. g., [7, 8, 14]); a summary is given in Table 2. For each type of Lie algebra, one constructs class representatives (by considering the action of automorphisms on a typical affine subspace). Finally, one verifies that none of these representatives are equivalent.

Type	Commutators	Automorphisms
II	$[E_2, E_3] = E_1$ $[E_3, E_1] = 0$ $[E_1, E_2] = 0$	$\begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$; $yw \neq vz$
IV	$[E_2, E_3] = E_1 - E_2$ $[E_3, E_1] = E_1$ $[E_1, E_2] = 0$	$\begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix}$; $u \neq 0$
V	$[E_2, E_3] = -E_2$ $[E_3, E_1] = E_1$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix}$; $xv \neq yu$
VI_0	$[E_2, E_3] = E_1$ $[E_3, E_1] = -E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x & y & u \\ -y & -x & v \\ 0 & 0 & -1 \end{bmatrix}$; $x^2 \neq y^2$
VI_h	$[E_2, E_3] = E_1 - aE_2$ $[E_3, E_1] = aE_1 - E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$; $x^2 \neq y^2$
VII_0	$[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x & y & u \\ y & -x & v \\ 0 & 0 & -1 \end{bmatrix}$; $x^2 \neq -y^2$
VII_h	$[E_2, E_3] = E_1 - aE_2$ $[E_3, E_1] = aE_1 + E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$; $x^2 \neq -y^2$
$VIII$	$[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = -E_3$	$M^\top J M = J$ $J = \mathrm{diag}(1, 1, -1)$ $\det M = 1$
IX	$[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = E_3$	$M^\top M = I$ $I = \mathrm{diag}(1, 1, 1)$ $\det M = 1$

Table 2. Automorphisms of three-dimensional Lie algebras

Type	Notation	(ℓ, ε)	Equivalence representative	Parameter
II	$\mathfrak{g}_{3.1}$	$(1,1)$	$E_2 + \langle E_3 \rangle$	
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $E_3 + \langle E_1, E_2 \rangle$	
IV	$\mathfrak{g}_{3.2}$	$(1,1)$	$E_2 + \langle E_3 \rangle$ $\alpha E_3 + \langle E_2 \rangle$	$\alpha \neq 0$
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $E_2 + \langle E_3, E_1 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	
V	$\mathfrak{g}_{3.3}$	$(1,1)$	$\emptyset$	$\alpha \neq 0$
		$(2,0)$	$\emptyset$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	
VI_0	$\mathfrak{g}_{3.4}^0$	$(1,1)$	$E_2 + \langle E_3 \rangle$ $\alpha E_3 + \langle E_2 \rangle$	$\alpha > 0$
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $E_1 + \langle E_1 + E_2, E_3 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	
VI_h	$\mathfrak{g}_{3.4}^h$	$(1,1)$	$E_2 + \langle E_3 \rangle$ $\alpha E_3 + \langle E_2 \rangle$	$\alpha \neq 0$
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $E_1 + \langle E_1 + E_2, E_3 \rangle$ $E_1 + \langle E_1 - E_2, E_3 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	
VII_0	$\mathfrak{g}_{3.5}^0$	$(1,1)$	$E_2 + \langle E_3 \rangle$ $\alpha E_3 + \langle E_2 \rangle$	$\alpha > 0$
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	
VII_h	$\mathfrak{g}_{3.5}^h$	$(1,1)$	$E_2 + \langle E_3 \rangle$ $\alpha E_3 + \langle E_2 \rangle$	$\alpha \neq 0$
		$(2,0)$	$\langle E_2, E_3 \rangle$	
		$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$ $\alpha E_3 + \langle E_1, E_2 \rangle$	

Table 3. Affine subspaces (types II to VII, solvable)

We present our results for the solvable Lie algebras (types I to VII) in the following two theorems; a summary is given in Table 3. The classification of type I is trivial and is therefore omitted.

Theorem 1. *Any full-rank affine subspace of* $\mathfrak{g}_{3.1}$ *(type II) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2 + \langle E_3\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, *and* $E_3 + \langle E_1, E_2\rangle$. *Any full-rank affine subspace of* $\mathfrak{g}_{3.2}$ *(type IV) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2 + \langle E_3\rangle$, $\alpha E_3 + \langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, $E_2 + \langle E_3, E_1\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$. *Any full-rank affine subspace of* $\mathfrak{g}_{3.3}$ *(type V) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_1 + \langle E_2, E_3\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$. *Here* $\alpha \neq 0$ *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

The automorphisms of $\mathfrak{g}^0_{3.4}$, $\mathfrak{g}^h_{3.4}$ (including $h = -1$), $\mathfrak{g}^0_{3.5}$, and $\mathfrak{g}^h_{3.5}$ are very similar. Due to this similarity, we treat these types separately.

Theorem 2. *Any full-rank affine subspace of* $\mathfrak{g}^0_{3.5}$ *or* $\mathfrak{g}^h_{3.5}$ *(type* VII_0 *or* VII_h*, respectively) is* $\mathfrak{L}$*-equivalent (with respect to the different ordered bases) to exactly one of* $E_2 + \langle E_3\rangle$, $\alpha E_3 + \langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$, *where* $\alpha > 0$ *for* $\mathfrak{g}^0_{3.5}$ *and* $\alpha \neq 0$ *for* $\mathfrak{g}^h_{3.5}$. *Any full-rank affine subspace of* $\mathfrak{g}^0_{3.4}$ *(type* VI_0*) is* $\mathfrak{L}$*-equivalent to exactly one of the above formal list for* $\mathfrak{g}^0_{3.5}$ *or* $E_1 + \langle E_1 + E_2, E_3\rangle$. *Any full rank-affine subspace of* $\mathfrak{g}^h_{3.4}$ *(type* VI_h*) is* $\mathfrak{L}$*-equivalent to exactly one of the above formal list for* $\mathfrak{g}^h_{3.5}$*, or one of* $E_1 + \langle E_1 + E_2, E_3\rangle$ *and* $E_1 + \langle E_1 - E_2, E_3\rangle$. *Here* α *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 2. The Lie algebras of types II, III, IV, V, VI_0, and VI_h are completely solvable, whereas those of types VII_0 and VII_h are not.

Type	Notation	(ℓ, ε)	Equivalence representative	Parameter
$VIII$	$\mathfrak{g}_{3.6}$	$(1,1)$	$E_3 + \langle E_2 + E_3\rangle$	$\alpha > 0$
			$\alpha E_2 + \langle E_3\rangle$	
			$\alpha E_1 + \langle E_2\rangle$	
			$\alpha E_3 + \langle E_2\rangle$	
		$(2,0)$	$\langle E_1, E_2\rangle$	
			$\langle E_2, E_3\rangle$	
		$(2,1)$	$E_3 + \langle E_1, E_2 + E_3\rangle$	
			$\alpha E_1 + \langle E_2, E_3\rangle$	
			$\alpha E_3 + \langle E_1, E_2\rangle$	
IX	$\mathfrak{g}_{3.7}$	$(1,1)$	$\alpha E_1 + \langle E_2\rangle$	$\alpha > 0$
		$(2,0)$	$\langle E_1, E_2\rangle$	
		$(2,1)$	$\alpha E_1 + \langle E_2, E_3\rangle$	

Table 4. Affine subspaces (types $VIII$ and IX, semisimple)

Now, consider the case of the semisimple algebras (types $VIII$ and IX). In each of the two cases, we employ a bilinear product ω (the Lorentz product and dot product, respectively) that is preserved by automorphisms. Most of the affine subspaces can then be characterised as being tangent to a level set (submanifold) $\{A \in \mathfrak{g} : \omega(A, A) = \alpha\}$. We present our classification in the following theorem; a summary is given in Table 4.

Theorem 3. *Any full-rank affine subspace of* $\mathfrak{g}_{3.6}$ *(type* $VIII$*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_3 + \langle E_2 + E_3 \rangle$, $\alpha E_2 + \langle E_3 \rangle$, $\alpha E_1 + \langle E_2 \rangle$, $\alpha E_3 + \langle E_2 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_2, E_3 \rangle$, $E_3 + \langle E_1, E_2 + E_3 \rangle$, $\alpha E_1 + \langle E_2, E_3 \rangle$, *and* $\alpha E_3 + \langle E_1, E_2 \rangle$*. Any full-rank affine subspace of* $\mathfrak{g}_{3.7}$ *(type* IX*) is* $\mathfrak{L}$*-equivalent to exactly one of* $\alpha E_1 + \langle E_2 \rangle$, $\langle E_1, E_2 \rangle$*, and* $\alpha E_1 + \langle E_2, E_3 \rangle$*. Here* $\alpha > 0$ *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4 Control affine systems and classification

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\, u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$. Also, the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i. e., $B_1, \ldots, B_\ell$ are linearly independent). The "product" $g\,\Xi\,(\mathbf{1}, u)$ is to be understood as $T_\mathbf{1} L_g \cdot \Xi\,(\mathbf{1}, u)$, where $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ is the left translation by g. Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i. e., $\Xi\,(g, u) = g\,\Xi\,(\mathbf{1}, u)$. We shall denote such a system by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [2]).

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. For more details about (invariant) control systems see, e. g., [1, 10, 11, 15, 16].

The image set $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$, called the *trace* of Σ, is an affine subspace of $\mathfrak{g}$. Specifically, $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ is said to have *full rank* if its trace (as an affine subspace) has full rank. Henceforth, we assume that all systems under consideration have full rank. (The full-rank condition is a necessary condition for a system Σ to be controllable.)

An important equivalence relation for invariant control systems is that of detached feedback equivalence. Two systems are detached feedback equivalent if there exists a "detached" feedback transformation which transforms the first system to the second (see [3, 9]). Two detached feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrised differently by admissible controls. More precisely, let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control affine systems. Σ and Σ' are called *locally detached feedback equivalent* (shortly DF_{loc}-equivalent) if there exist open neighbourhoods N and N' of identity (in G and G', respectively) and a diffeomorphism $\Phi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. It turns out that Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$ (see [3]).

For the purpose of classification, we may assume that Σ and Σ' have the same Lie algebra $\mathfrak{g}$. Then Σ and Σ' are DF_{loc}-equivalent if and only if their traces Γ and Γ' are $\mathfrak{L}$-equivalent. This reduces the problem of classifying under DF_{loc}-equivalence to that of classifying under $\mathfrak{L}$-equivalence. Suppose $\{\Gamma_i \,:\, i \in I\}$ is an exhaustive collection of (non-equivalent) class representatives (i.e., any affine subspace is $\mathfrak{L}$-equivalent to exactly one Γ_i). For each $i \in I$, we can easily find a system $\Sigma_i = (\mathsf{G}, \Xi_i)$ with trace Γ_i. Then any system Σ is DF_{loc}-equivalent to exactly one Σ_i.

Example. The Heisenberg group

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \,:\, x, y, z \in \mathbb{R} \right\}$$

is a (nilpotent) three-dimensional Lie group. Its Lie algebra $\mathfrak{h}_3$ has (ordered) basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations are $[E_2, E_3] = E_1$, $[E_3, E_1] = 0$, and $[E_1, E_2] = 0$. Thus $\mathfrak{h}_3 \cong \mathfrak{g}_{3.1}$. Hence, any system $\Sigma = (\mathsf{H}_3, \Xi)$ is DF_{loc}-equivalent to exactly one $\Sigma_i = (\mathsf{H}_3, \Xi_i)$, where

$$\begin{aligned} &\Xi_1(g, u) = g(E_2 + uE_3) && \Xi_2(g, u) = g(u_1E_2 + u_2E_3) \\ &\Xi_3(g, u) = g(E_1 + u_1E_2 + u_2E_3) && \Xi_4(g, u) = g(E_3 + u_1E_1 + u_2E_3). \end{aligned}$$

References

[1] Agrachev A. A., Sachkov Y. L. *Control Theory from the Geometric Viewpoint.* Springer-Verlag, Berlin, 2004.

[2] Biggs R., Remsing C. C. *A category of control systems.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., 2012, **20**, 355–368.

[3] Biggs R., Remsing C. C. *On the equivalence of control systems on Lie groups.* Submitted.

[4] Biggs R., Remsing C. C. *Control affine systems on semisimple three-dimensional Lie groups.* An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N. S.), 2013, **59**, 65–80.

[5] Biggs R., Remsing C. C. *Control affine systems on solvable three-dimensional Lie groups, I.* Submitted.

[6] Biggs, R., Remsing, C. C. *Control affine systems on solvable three-dimensional Lie groups, II.* Note Mat., 2013, **33** (to appear).

[7] Ha K. Y., Lee J. B. *Left invariant metrics and curvatures on simply connected three-dimensional Lie groups.* Math. Nachr., 2009, **282**, 868–898.

[8] Harvey A. *Automorphisms of the Bianchi model Lie groups.* J. Math. Phys., 1979, **20**, 251–253.

[9] Jakubczyk B. *Equivalence and invariants of nonlinear control systems.* In: "Nonlinear Controllability and Optimal Control" (Sussmann H. J., Ed.), M. Dekker, New York, 1990, 177–218.

[10] Jurdjevic V. *Geometric Control Theory.* Cambridge University Press, Cambridge, 1997.

[11] Jurdjevic V., Sussmann H. J. *Control systems on Lie groups.* J. Differential Equations, 1972, **12**, 313–329.

[12] Krasiński A., Behr C. G., Schücking E., Estabrook F. B., Wahlquist H. D., Ellis G. F. R., Jantzen R., Kundt W. *The Bianchi classification in the Schücking-Behr approach.* Gen. Relativity Gravitation, 2003, **35**, 475–489.

[13] MacCallum M. A. H. *On the classification of the real four-dimensional Lie algebras.* In: "On Einstein's Path: Essays in Honour of E. Schücking" (Harvey A., Ed.), Springer-Verlag, New York, 1999, 299–317.

[14] Popovych R. O., Boyco V. M., Nesterenko M. O., Lutfullin M. W. *Realizations of real low-dimensional Lie algebras.* J. Phys. A, 2003, **36**, 7337–7360.

[15] Remsing C. C. *Optimal control and Hamilton-Poisson formalism.* Int. J. Pure Appl. Math., 2010, **59**, 11–17.

[16] Sachkov Y. L. *Control theory on Lie groups.* J. Math. Sci. (N. Y.), 2009, **156**, 381-439.

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