BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(70), 2012, Pages 38–44
ISSN 1024–7696

Properties of covers in the lattice of group topologies for nilpotent groups

V. I. Arnautov

Abstract. A nilpotent group $\widehat{G}$ and two group topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ on $\widehat{G}$ are constructed such that $\widehat{\tau}*$ is a coatom in the lattice of all group topologies of the group $\widehat{G}$ and such that between $\inf\{\widehat{\tau}'', \widehat{\tau}_d\}$ and $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$ there exists an infinite chain of group topologies.

Mathematics subject classification: 22A05.
Keywords and phrases: Nilpotent group, group topology, lattice of group topologies, unrefinable chains, coatom, infimum of group topologies.

1 Introduction

As is known, in any modular lattice, the lengths of any finite unrefinable chains with the same ends are equal. Moreover, the lengths of finite unrefinable chains do not become greater if we take the infimum or the supremum in these lattices.

The lattice of all group topologies for a nilpotent group need not be modular [1]. However, as is shown in [2], in the lattice of all group topologies on a nilpotent group, the lengths of any finite unrefinable chains which have the same ends are equal. Moreover, in the same article it is shown that the lengths of any finite unrefinable chains do not become greater if we take the supremum.

Given the above, it was natural to expect that the lengths of any finite unrefinable chains do not become greater if in the lattice of all group topologies for a nilpotent group we take the infimum. However, as shown in this article, it is not the case.

To present the further results we need the following known result (see [3], page 203):

Theorem 1. *Let $\mathcal{B}$ be a collection of subsets of a group $G(\cdot)$ such that the following conditions are satisfied:*

1) $e \in V$ for any $V \in \mathcal{B}$, where e is the unity element in the group $G(\cdot)$;

2) for any $V_1, V_2 \in \mathcal{B}$ there exists $V_3 \in \mathcal{B}$ such that $V_3 \subseteq V_1 \bigcap V_2$;

3) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2 \cdot V_2 \subseteq V_1$;

4) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2^{-1} \subseteq V_1$;

5) for any $V_1 \in \mathcal{B}$ and any element $g \in G$ there exists $V_2 \in \mathcal{B}$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.

Then there exists a unique group topology τ on the group $G(\cdot)$ for which the collection $\mathcal{B}$ is a basis of neighborhoods of the unity element e [1] *(see [3], page 26).*

From Theorem 1 follows easily:

Corollary 2. Let group topologies τ_1 and τ_2 be defined on a group $G(\cdot)$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of neighborhoods of the unity element in topological groups (G,τ_1) and (G,τ_2), respectively, then the collection $\mathcal{B} = \{U \bigcap V \mid U \in \mathcal{B}_1, V \in \mathcal{B}_2\}$ is a basis of neighborhoods of the unity element in the topological group (G,τ), where $\tau = \sup\{\tau_1,\tau_2\}$.

2 Basic results

To state basic results we need the following notations:

Notations 3.

3.1. $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}(+,\cdot)$ is the field of real numbers;

3.2. G is the set of all matrices $\begin{pmatrix}1 & a_{1,2} & a_{1,3}\\ 0 & 1 & a_{2,3}\\ 0 & 0 & 1\end{pmatrix}$ of the dimension 3×3 over the field $\mathbb{R}$ of real numbers such that $a_{i,i} = 1$ for $1 \le i \le 3$ and $a_{i,j} = 0$ for $1 \le j < i \le 3$,

$G' = \left\{\begin{pmatrix}1 & a_{1,2} & a_{1,3}\\ 0 & 1 & a_{2,3}\\ 0 & 0 & 1\end{pmatrix} \in G \mid a_{1,3} = a_{2,3} = 0\right\}$;

$G'' = \left\{\begin{pmatrix}1 & a_{1,2} & a_{1,3}\\ 0 & 1 & a_{2,3}\\ 0 & 0 & 1\end{pmatrix} \in G \mid a_{1,2} = a_{1,3} = 0\right\}$;

$G(A) = \left\{\begin{pmatrix}1 & a_{1,2} & a_{1,3}\\ 0 & 1 & a_{2,3}\\ 0 & 0 & 1\end{pmatrix} \in G \mid a_{1,2} = 0 \text{ and } a_{1,3} \in A\right\}$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$;

3.3. $G_i(\cdot) = G(\cdot)$, $G'_i(\cdot) = G'(\cdot)$ and $G''_i(\cdot) = G''(\cdot)$ for every natural number i;

3.4. $G_i(A) = G(A)$ for every natural number i and any subgroup $A(+)$ of the group $\mathbb{R}(+)$;

3.5. $\widehat{G} = \sum\limits_{i=1}^{\infty} G_i$, $\widehat{G'} = \sum\limits_{i=1}^{\infty} G'_i$ and $\widehat{G''} = \sum\limits_{i=1}^{\infty} G''_i$;

3.6. $\widetilde{V_n} = \{\widetilde{g} \in \widetilde{G} \mid pr_i(\widetilde{g}) = e_i \text{ if } i \le n\}$ for any $n \in \mathbb{N}$;

3.7. $\widehat{G}_k(A) = \{\widehat{g} \in \widehat{G} | pr_k(\widehat{g}) \in G''_k(A) \text{ and } pr_j(\widehat{g}) = \{e\} \text{ if } j \ne k\}$, where $k \in \mathbb{N}$ and $A(+)$ is a subgroup of the group $\mathbb{R}(+)$;

3.8. $\widehat{G}(A,S) = \{\widehat{g} \in \widehat{G} \mid pr_i(\widehat{g}) \in G_i(A) \text{ if } i \in S \text{ and } pr_j(\widehat{g}) \in G''_j \text{ if } j \notin S\}$, where $A(+)$ is a subgroup of the group $\mathbb{R}(+)$ and $S \subseteq \mathbb{N}$;

[1] As usual, the set V is called a neighborhood of an element a in the topological space (X,τ) if $a \in U \subseteq V$ for some $U \in \tau$.

3.9. τ_i is discrete in the group G_i and $\widetilde{\tau}$ is Tikhonov topology of the direct product $\widetilde{G} = \prod\limits_{i=1}^{\infty}(G_i, \tau_i)$.

Remark 4. It is easy to see that G with the usual operation of matrix multiplication is a group.

Since $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & a \cdot c - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$, then

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -y \cdot c + a \cdot z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and as the center of the group G contains any matrix of the form $\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $d \in \mathbb{R}$, then $G(\cdot)$ is a nilpotent group and its nilpotency index is 2.

In addition, since $\begin{pmatrix} 1 & a & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & b_{1,3} \\ 0 & 1 & b_{2,3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & a_{1,3} + b_{1,3} \\ 0 & 1 & a_{2,3} + b_{2,3} \\ 0 & 0 & 1 \end{pmatrix}$, then $G'(\cdot)$, $G''(\cdot)$ and $G(A)(\cdot)$ are subgroups of the group $G(\cdot)$ for any subgroup $A(+)$ of the additive group of the field $\mathbb{R}(+, \cdot)$.

Proposition 5. *For the group $\widehat{G}$ the following statements are true:*

Statement 5.1. *The collection $\mathcal{B}' = \{\widetilde{V}_i \bigcap \widehat{G}' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, it is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}'$ on the group $\widehat{G}$;*

Statement 5.2. *The collection $\mathcal{B}'' = \{\widetilde{V}_i \bigcap \widehat{G}'' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}''$ on the group $\widehat{G}$;*

Statement 5.3. *If A is a subgroup of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+, \cdot)$, and $\mathcal{F}$ is the Frechet filter*[2] *on the set $\mathbb{N}$, then the collection $\mathcal{B}(A, \mathcal{F}) = \{\widehat{G}(A, F) \bigcap \widetilde{V_n} | F \in \mathcal{F}, n \in \mathbb{N}\}$ satisfies all the conditions of Theorem 1, and hence, it is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}(A, \mathcal{F})$ on the group $\widehat{G}$.*

Proof. Since $\widehat{G}(A, F \bigcap S) \subseteq \widehat{G}(A, F) \bigcap \widehat{G}(A, S)$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$ and any subsets $S \subseteq \mathbb{N}$ and $F \subseteq \mathbb{N}$ for which $\widetilde{V}_i \subseteq \widetilde{V}_k$ if $k \leq i$, then any of the mentioned collections satisfies condition 2 of Theorem 1.

In addition, taking into consideration the definitions of sets $\widetilde{V}_n$, $\widehat{G}'$, $\widehat{G}''$, and $\widehat{G}(A, F)$ we obtain that any set from the collection $\mathcal{B}' \bigcup \mathcal{B}'' \bigcup \mathcal{B}(A, \mathcal{F})$ is a subgroup of the group $\widehat{G}(\cdot)$, and hence, any collection $\mathcal{B}'$, $\mathcal{B}''$, and $\mathcal{B}(A, \mathcal{F})$ satisfies conditions 1, 3 and 4 of Theorem 1.

[2] i.e. $\mathbb{N} \setminus \{1, \dots, k\} \in \mathcal{F}$ for every $k \in \mathbb{N}$.

To complete the proof of the theorem it remains to verify that for any of the mentioned collections condition 5 of Theorem 1 is also satisfied.

Let $\widehat{g} \in \widehat{G}$, then there exists a natural number n such that $pr_i(\widehat{g}) = e_i$ for $i > m$.

If $\widetilde{V_k} \bigcap \widehat{G}' \in \mathcal{B}'$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widetilde{V}_m \bigcap \widehat{G}'$, and hence,

$$\widehat{g} \cdot (\widetilde{V}_m \bigcap \widehat{G}') \cdot \widehat{g}^{-1} = \widetilde{V}_m \bigcap \widehat{G}' \subseteq \widetilde{V_k} \bigcap \widehat{G}',$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}'$.

Analogously, if $\widetilde{V_k} \bigcap \widehat{G}'' \in \mathcal{B}''$ and $M = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widetilde{V}_m \bigcap \widehat{G}''$, and hence,

$$\widehat{g} \cdot (\widetilde{V}_m \bigcap \widehat{G}'') \cdot \widehat{g}^{-1} = \widetilde{V}_m \bigcap \widehat{G}'' \subseteq \widetilde{V_k} \bigcap \widehat{G}'',$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}''$.

If $\widehat{V}(A, F) \bigcap \widetilde{V}_k \in \mathcal{B}(A\mathcal{F})$ and $m = max\{n, k\}$, then $\widehat{V}(A, F) \bigcap \widetilde{V}_m \subseteq \widehat{V}(A, F) \bigcap \widetilde{V}_k$ and $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widehat{V}(A, F) \bigcap \widetilde{V}_m$, and hence,

$$\widehat{g} \cdot (\widehat{V}(A, F) \bigcap \widetilde{V_m}) \cdot \widehat{g}^{-1} = \widehat{V}(A, F) \bigcap \widetilde{V_m} \subseteq \widehat{V}(A, F) \bigcap \widetilde{V_k}$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}(A, \mathcal{F})$.

By this, the proposition is completely proved. $\square$

Proposition 6. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be group topologies on the group $\widehat{G}$, defined in Proposition 5, and $n \in \mathbb{N}$. If τ is a non-discrete group topology on the group $\widehat{G}$ such that $\tau \geq \widehat{\tau}'$, then for any neighborhood W of the unity element $\widehat{e}$ in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$ there exists a natural number $k \geq n$ such that (see 3.7) $\widehat{G}_k(\mathbb{R}) \subseteq W$.*

Proof. Let W be a neighborhood of the unity element in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$, and let W_1 be a neighborhood of the unity element in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$ such that $W_1 \cdot (W_1 \cdot W_1 \cdot (W_1)^{-1} \cdot (W_1)^{-1}) \subseteq W$.

Then W_1 is a neighborhood of the unity element in each of the topological groups $(\widehat{G}, \tau)$ and $(\widehat{G}, \widehat{\tau}'')$, and hence, there exists a natural number $n_0 \in \mathbb{N}$ such that $n_0 \geq n$ and $\widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$.

Since $\tau \geq \widehat{\tau}'$, then $\widehat{G}' \bigcap \widetilde{V}_{n_0}$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \tau)$. Then, $\widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \tau)$.

Since τ is a non-discrete topology, then $\widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1 \neq \{0\}$. If $0 \neq \widehat{g}_0 \in \widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1 \neq \{0\}$, then there exists a natural number $k \geq n_0 \geq n$ such that $pr_k(\widehat{g}_0) \neq 0$.

Since $\widehat{g}_0 \in \widehat{G}'$, then $pr_k(\widehat{g}_0) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $a \neq 0$.

For any numbers $r, x \in \mathbb{R}$ consider matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (a^{-1}) \cdot x \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then (see Remark 4)

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix}^{-1} \right).$$

For any numbers $r, x \in \mathbb{R}$ we consider elements $\widehat{g}_r, \widehat{g}_x \in \widehat{G}$ such that

$$pr_k(\widehat{g}_r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \text{ and } pr_i(\widehat{g}) = e_i \text{ for } i \neq k,$$

$$pr_k(\widehat{g}_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } pr_i(\widehat{g}) = e_i \text{ for } i \neq k.$$

Since $k \geq n_0$, then $\widehat{g}_r \in \widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$ and $\widehat{g}_x \in \widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$ for any numbers $r, x \in \mathbb{R}$. Then $\widehat{g}_r \cdot (\widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g_0}^{-1} \cdot \widehat{g_x}^{-1}) \in W_1 \cdot W_1 \cdot W_1 \cdot (W_1)^{-1} \cdot (W_1)^{-1} \subseteq W$ for any numbers $r, x \in \mathbb{R}$, and hence, $\widehat{G}_k(\mathbb{R}) = \{\widehat{g}_r \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g_0}^{-1} \cdot \widehat{g_x}^{-1} | r, x \in \mathbb{R}\} \subseteq W$.

As $pr_k(\widehat{g}_r \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g_0}^{-1} \cdot \widehat{g_x}^{-1}) =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix}^{-1} \right) =$$

$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$ for any $r, x \in \mathbb{R}$, and $pr_i(\widehat{g}_i \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g_0}^{-1} \cdot \widehat{g_x}^{-1}) = e_i$ for any $R, x \in \mathbb{R}$ and for any $i \neq k$, then $pr_k(\widehat{A}_k) = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} | r \text{ and } x \in \mathbb{R} \right\} = G_k(\mathbb{R})$ and $pr_i(\widehat{A}_k) = \{e_i\}$ for $i \neq k$.

By this, the proposition is completely proved. □

Theorem 7. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be group topologies on the group $\widehat{G}$, defined in Proposition 5, and $\mathcal{F}$ be the Frechet filter. Then the following statements are true:*

Statement 7.1. *If τ is a group topology on the group $\widehat{G}$ such that $\tau \geq \widehat{\tau}'$, then*

$$\sup\{\widehat{\tau}(A,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\}.$$

for any subgroups $A \subset B$ of the group $\mathbb{R}(+)$.

Statement 7.2. *If $\widehat{\tau}_d$ is the discrete topology on the group $\widehat{G}$, and $\widehat{\tau}*$ is a coatom in the lattice of all group topologies on the group $\widehat{G}$ such that $\widehat{\tau}* \geq \widehat{\tau}'$, then between the topologies $\inf\{\widehat{\tau}_d, \widehat{\tau}''\}$ and $\inf\{\widehat{\tau}*, \widehat{\tau}''\}$, there exists a chain of group topologies on the group $\widehat{G}$ which is infinitely decreasing and infinitely increasing.*

Proof. Proof of Statement 7.1. Since $A \subset B$, then (see the notation at the beginning of this article) $\widehat{V}(A,S) \subseteq \widehat{V}(B,S)$ for any $S \in \mathcal{F}$. Then (see Proposition 5) $\widehat{\tau}(A,\mathcal{F}) \geq \widehat{\tau}(B,\mathcal{F})$, and hence, $\sup\{\widehat{\tau}(A,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\} \geq \sup\{\widehat{\tau}(B,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\}$.

We will show that $\sup\{\widehat{\tau}(A,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\}$.

Assume the contrary, i.e. that

$$\sup\{\widehat{\tau}(A,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\} = \sup\{\widehat{\tau}(B,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\},$$

and let $S_0 \in \mathcal{F}$. Then $\widehat{V}(A,S_0)$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \widehat{\tau}(A,\mathcal{F}))$, and hence, $\widehat{V}(A,S_0)$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \sup\{\widehat{\tau}(A,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\}) = (\widehat{G}, \sup\{\widehat{\tau}(B,\mathcal{F}), \inf\{\widehat{\tau}'', \tau\}\})$. Then there exists a neighborhood W of the unity element in the topological group $(\widehat{G}, \inf\{\widehat{\tau}'', \tau\})$ such that $W \bigcap (\widehat{V}(B,S_1) \bigcap \widetilde{V}_n) \subseteq \widehat{V}(A,S_0)$ for some $S_1 \in \mathcal{F}$ and a natural number $n \in \mathbb{N}$.

Since $\mathcal{F}$ is the Frechet filter, then there exists a natural number $m \in \mathbb{N}$ such that $\{i \in \mathbb{N} \mid i > m\} \subseteq S_0 \bigcap S_1$.

By Proposition 6, there exists a natural number $k \geq \max\{n, m\}$ such that $\widehat{G}_k(\mathbb{R}) \subseteq W$, and hence, $\widehat{G}_k(B) \subseteq \widehat{G}_k(\mathbb{R}) \subseteq W$.

As $k \in \{i \in \mathbb{N} \mid i > m\} \subseteq S_1$ then $\widehat{G}_k(B) \subseteq \widehat{V}(B,S_1)$, and as $k \geq n$ then $\widehat{G}_k(B) \subseteq \widetilde{V}_n$. Then $\widehat{G}_k(B) \subseteq W \bigcap \widehat{V}(B,S_1) \bigcap \widetilde{V}_n \subseteq \widehat{V}(A,S_0)$.

Since $k \in \{i \in \mathbb{N} \mid i > m\} \subseteq S_0$, then (see 3.7) $G_k(B) = pr_k(\widehat{G}_k(B)) \subseteq pr_k(\widehat{V}(A,S_0)) = G_k(A)$, but this contradicts that $B \nsubseteq A$.

By this, Statement 7.1 is proved.

Proof of Statement 7.2. There exists a chain $\{A_i \mid i \in \mathbb{Z}\}$ of subgroups A_i of the group $\mathbb{R}(+)$ such that $A_i \subseteq A_{i+1}$ for any $i \in \mathbb{Z}$, i.e. this chain of subgroups is infinitely decreasing and infinitely increasing.

For any subgroup A_i let consider the topology $\widehat{\tau}(A_i,\mathcal{F})$. Since $\widehat{\tau}* \geq \widehat{\tau}'$, then by Statement 7.1,

$$\sup\{\widehat{\tau}(A_i,\mathcal{F}), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\} > \sup\{\widehat{\tau}(A_{i+1},\mathcal{F}), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\},$$

and hence, the chain of group topologies $\{\sup\{\widehat{\tau}(A_i,\mathcal{F}), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\} \mid i \in \mathbb{Z}\}$ is infinitely decreasing and infinitely increasing.

To complete the proof of the theorem it remains to verify that

$$\inf\{\widehat{\tau*}, \widehat{\tau''}\} \leq \sup\{\widehat{\tau}(A_i,\mathcal{F}), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$, where $i \in \mathbb{Z}$.

In fact, from the definition of the sets $G(A)$ and G'' (see 3.2) it follows that $G(0) = G''$, and hence, $G_k(0) = G''_k$ for any $k \in \mathbb{N}$. Then $\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n = \widehat{G}'' \bigcap \widetilde{V}_n$ for any subset $S \subseteq \mathbb{N}$ and any $n \in \mathbb{N}$, and hence, the collection $\{\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n \mid n \in \mathbb{N}\}$ is a basis of neighborhoods of the unity element in the topological group $(\widehat{G}, \widehat{\tau}'')$.

Since $\widehat{\tau}_d$ is the discrete topology on the group $\widehat{G}$, then $\inf\{\widehat{\tau}'', \widehat{\tau}_d\} = \widehat{\tau}''$, and hence, the set $\{\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n \mid n \in \mathbb{N}\}$ is a basis of neighborhoods of the unity element in the topological group $(\widehat{G}, \inf\{\widehat{\tau}'', \widehat{\tau}_d\})$. Then $\widehat{\tau}(\{0\}, \mathcal{F}) \leq \inf\{\widehat{\tau}'', \widehat{\tau}_d\}$.

So, we have proved that $\sup\{\widehat{\tau}(A_i, \mathcal{F}), \inf\{\tau*, \widehat{\tau}''\}\} \leq \inf\{\tau_d, \widehat{\tau}''\}$. Since $\{0\} \subseteq A_i$ for any $i \in \mathbb{Z}$, then

$$\inf\{\widehat{\tau}*, \widehat{\tau}''\} \leq \sup\{\widehat{\tau}(A_i, \mathcal{F}), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$.

By this, the theorem is proved. □

References

[1] Arnautov V. I., Topala A. Gh. *An example of ring with non-modular lattice of ring topologies*, Bul. Acad. Ştiinţe Repub. Moldova, Mat., 1998, No. 2(27), 130–131.

[2] Arnautov V. I. Properties of finite unrefinable chais of groups topologies, Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2009, No. 2(60), 1–17.

[3] Bourbaki N. Obshchaya topologiya. Osnovnye structury (General Topology, Fundamental Structures), Moscow, 1958, 324 pages (in Russian).

V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5, Academiei str., Chişinău MD-2028
Moldova
Email: *arnautov@math.md*

Received February 02, 2012
Revised September 21, 2012