Boundedness for Vector-Valued Multilinear Singular Integral Operator on $L^p$ Spaces with Variable Exponent

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Abstract. In this paper, we prove the boundedness for some vector-valued multilinear singular integral operators on $L^p$ spaces with variable exponent by using a sharp estimate of the multilinear operators.


Keywords and phrases: Vector-valued multilinear operator, singular integral operator, BMO, variable $L^p$ space.

1 Introduction and Theorems

As the development of the Calderón-Zygmund singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [4, 9, 16–19]). Let $T$ be the Calderón-Zygmund singular integral operator. In [1–3], Cohen and Gosselin studied the $L^p$ ($p > 1$) boundedness of the multilinear singular integral operator $T^A$ defined by

$$T^A(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y) f(y) dy,$$

where

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha.$$

In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In the last years, a theory of $L^p$ spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity (see [5–8,15] and their references). Karlovich and Lerner have studied the boundedness of the commutators of singular integral operators on $L^p$ spaces with variable exponent (see [13]). In this paper, we will study the boundedness properties for some vector-valued multilinear singular integral operators on $L^p$ spaces with variable exponent, whose definition is the following.

Fix $\varepsilon > 0$. Let $S$ and $S'$ be Schwartz space and its dual and $T : S \to S'$ be a linear operator. If there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

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for every bounded and compactly supported function $f$, where $K$ satisfies:

$$|K(x, y)| \leq C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|\varepsilon|x - z|^{-n - \varepsilon}$$

when $2|y - z| \leq |x - z|$. Let $m_j$ be positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and $A_j$ be functions on $R^n$ ($j = 1, \cdots, l$). For $1 < s < \infty$, the vector-valued multilinear operator related to $T$ is defined by

$$|T_A(f)(x)|_s = \left( \sum_{i=1}^{\infty} |T_A(f_i)(x)|^s \right)^{1/s},$$

where

$$T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^{l} R_{m_j+1}(A_j; x, y) K(x, y) f_i(y) dy$$

and

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha.$$

We also denote

$$|T(f)(x)|_s = \left( \sum_{i=1}^{\infty} |T(f_i)(x)|^s \right)^{1/s} \quad \text{and} \quad |f|_s = \left( \sum_{i=1}^{\infty} |f_i(x)|^s \right)^{1/s}.$$

Suppose that $|T|$ is weakly $(L^1, L^1)$-bounded.

Note that when $m = 0$, $|T_A|$ is just the vector-valued multilinear commutator of $T$ and $A$ (see [19]). While when $m > 0$, $|T_A|$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1–4, 9]). In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [18], Pérez and Trujillo-Gonzalez proved a sharp estimate for some multilinear commutator. The main purpose of this paper is to prove the boundedness for the vector-valued multilinear singular integral operators $|T_A|$ on $L^p$ spaces with variable exponent. To do this, we first prove a sharp inequality for the vector-valued multilinear singular integral operators.

Now, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $R^n$ with sides parallel to the axes. For any locally integrable function $f$ and $\delta > 0$, the sharp function of $f$ is defined by

$$f_{\#}^{\delta}(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^{\delta} dy \right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [11,20])

$$f_{\#}^{\delta}(x) \approx \sup_{Q \ni x, c \in C} \left( \frac{1}{|Q|} \int_Q |f(y) - c|^{\delta} dy \right)^{1/\delta}.$$
We write $f^\# = f_1^\#$ if $\delta = 1$. We say that $f$ belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. Let $M$ be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$ 

For $k \in N$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ when $k \geq 2$.

Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$. We denote by $\Phi$-average for a function $f$ of $\lambda$ for $\lambda$ of all Lebesgue measurable functions $\lambda = 1$ by $\Phi(f)$ and define $\Phi(f) = \Phi(f(x)) - \Phi(f(x))$. Following \[16-19\], we know the generalized Hölder’s inequality:

$$\frac{1}{|Q|} \int_Q |f(y)| g(y) dy \leq \|f\|_{\Phi,Q} \|g\|_{\tilde{\Phi},Q}$$

and the following inequalities, for $r, r_j \geq 1, j = 1, \ldots, l$, with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$\|f\|_{L^{\phi\log\lambda L^{1/r},Q}} \leq M_{L^{\phi\log\lambda L^{1/r},Q}}(f) \leq CM_{L^{\phi\log\lambda L^{1/r},Q}}(f) \leq CM^{l+1}(f),$$

$$\|b - b_Q\|_{expL^{1/r},Q} \leq C \|b\|_{BMO},$$

$$|b_{2k+1}Q - b_{2Q}| \leq Ck \|b\|_{BMO}.$$ 

The non-increasing rearrangement of a measurable function $f$ on $R^n$ is defined by

$$f^*(t) = \inf \{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leq t \} \ (0 < t < \infty).$$

For $\lambda \in (0, 1)$ and a measurable function $f$ on $R^n$, the local sharp maximal function of $f$ is defined by

$$M_{\lambda}^\#(f)(x) = \sup_{Q \ni x} \inf_{c \in C} ((f - c)\chi_Q)^*(\lambda|Q|).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions $f$ on $R^n$ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f, p) = \int_{R^n} |f(x)|^{p(x)} dx.$$
The sets becomes Banach spaces with respect to the following norm

$$
\|f\|_{L^p(\cdot)} = \inf\{\lambda > 0 : m(f/\lambda, p) \leq 1\}.
$$

Denote by $M(R^n)$ the set of all measurable functions $p : R^n \rightarrow [1, \infty)$ such that the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(\cdot)(R^n)$ and the following holds

$$
1 < p_- = \text{ess inf} \inf_{x \in R^n} p(x), \quad \text{ess sup} \sup_{x \in R^n} p(x) = p_+ < \infty.
$$

In recent years, the boundedness of classical operators on spaces $L^p(\cdot)(R^n)$ have attracted a great attention (see [5–8,15]). In this paper, we shall prove the following theorems.

**Theorem 1.** Let $1 < s < \infty$ and $D^\alpha A_j \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in L^\infty_0(R^n)$, $0 < \delta < 1$ and $\tilde{x} \in R^n$,

$$
(|T_A(f)|_s)_\tilde{x} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha_j A_j\|_{BMO} \right) M^{l+1}(\|f\|_s)(\tilde{x}).
$$

**Theorem 2.** Let $1 < s < \infty$, $p(\cdot) \in M(R^n)$ and $D^\alpha A_j \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T_A|_s$ is bounded on $L^p(\cdot)(R^n)$, that is

$$
\| |T_A(f)|_s \|_{L^p(\cdot)} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^\alpha_j A_j\|_{BMO} \right) \|f\|_s \|_{L^p(\cdot)}.
$$

**Remark 1.** Let $T$ be the Calderon-Zygmund operator (see [4, 11, 20]). Then Theorem 1 and Theorem 2 hold for $T$.

## 2 Some Lemmas

We begin with some preliminary lemmas.

**Lemma 1 (see [3]).** Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for all $\alpha$ with $|\alpha| = m$ and some $q > n$. Then

$$
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x, y)|} \int_{Q(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},
$$

where $\tilde{Q}$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$. 
Lemma 2 (see [11, p. 485]). Let $0 < p < q < \infty$. We define that, for any function $f \geq 0$ and $1/r = 1/p - 1/q$,
\[
||f||_{W^{L^q}} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \quad N_{p,q}(f) = \sup_E ||f \chi_E||_{L^p}/||\chi_E||_{L^r},
\]
where the sup is taken for all measurable sets $E$ with $0 < |E| < \infty$. Then
\[
||f||_{W^{L^q}} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{W^{L^q}}.
\]

Lemma 3 (see [18]). Let $r_j \geq 1$ for $j = 1, \cdots, l$ and we denote $1/r = 1/r_1 + \cdots + 1/r_l$. Then
\[
\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x)g(x)|dx \leq ||f||_{L^{r_1,1},Q} \cdots ||f||_{L^{r_l,1},Q} ||g||_{L(\log L)^{1/r},Q}.
\]

Lemma 4 (see [13]). Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Then $L_0^\infty(R^n)$ is dense in $L^p(\cdot)(R^n)$.

Lemma 5 (see [14]). Let $f \in L_{loc}^1(R^n)$ and $g$ be a measurable function satisfying
\[
|\{x \in R^n : |g(x)| > \alpha\}| < \infty \quad \text{for all} \quad \alpha > 0.
\]
Then
\[
\int_{R^n} |f(x)g(x)|dx \leq C_n \int_{R^n} M_{\lambda,n}^\#(f(x))M(g)(x)dx.
\]

Lemma 6 (see [14]). Let $\delta > 0$, $0 < \lambda < 1$ and $f \in L_{loc}^\delta(R^n)$. Then
\[
M_{\lambda}^\#(f)(x) \leq (1/\lambda)^{1/\delta} M_{\delta}^\#(x).
\]

Lemma 7 (see [13]). Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). If $f \in L^p(\cdot)(R^n)$ and $g \in L^p(\cdot)(R^n)$ with $p'(x) = p(x)/(p(x)-1)$, then $fg$ is integrable on $R^n$ and
\[
\int_{R^n} |f(x)g(x)|dx \leq C||f||_{L^p(\cdot)} ||g||_{L^{p'}(\cdot)}.
\]

Lemma 8 (see [13]). Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Set
\[
||f||'_{L^p(\cdot)} = \sup \left\{ \int_{R^n} |f(x)g(x)|dx : f \in L^p(\cdot)(R^n), g \in L^p(\cdot)(R^n) \right\}.
\]
Then $||f||_{L^p(\cdot)} \leq ||f||'_{L^p(\cdot)} \leq C||f||_{L^p(\cdot)}$. 

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3 Proof of Theorem

It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q ||T_A(f)(x)|_s - C_0||^\delta dx\right)^{1/\delta} \leq C \prod_{|\alpha_j|=m_j} \||D^{\alpha_j}A_j||_{BMO}\right)^{M+1}(|f|_s)(\bar{x}).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\bar{x} \in Q$. Let $\bar{Q} = 5\sqrt{n}Q$ and $\bar{A}_j(x) = A_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A_j)\bar{Q} x^\alpha$, then $R_m(A_j; x, y) = R_m(\bar{A}_j; x, y)$ and $D^\alpha \bar{A}_j = D^\alpha A_j - (D^\alpha A_j)\bar{Q}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\bar{Q}}$ and $h_i = f_i \chi_{R^n \setminus \bar{Q}}$. Write

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{mj+1}(\bar{A}_j; x, y)}{|x - y|^m} K(x, y) f_i(y) dy$$

$$= \int_{R^n} \frac{\prod_{j=1}^2 R_{mj+1}(\bar{A}_j; x, y)}{|x - y|^m} K(x, y) h_i(y) dy$$

$$+ \int_{R^n} \frac{\prod_{j=1}^2 R_{mj}(\bar{A}_j; x, y)}{|x - y|^m} K(x, y) g_i(y) dy$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\bar{A}_2; x, y)(x - y)^{\alpha_1}}{|x - y|^m} D^{\alpha_1} \bar{A}_1(y) K(x, y) g_i(y) dy$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\bar{A}_1; x, y)(x - y)^{\alpha_2}}{|x - y|^m} D^{\alpha_2} \bar{A}_2(y) K(x, y) g_i(y) dy$$

$$+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x - y)^{\alpha_1 + \alpha_2} D^{\alpha_1} \bar{A}_1(y) D^{\alpha_2} \bar{A}_2(y)}{|x - y|^m} K(x, y) g_i(y) dy,$$

then, by Minkowski’ inequality,
\[ \times D^{\alpha_1}A_1(y)K(x, y)g_i(y)dy|x|^\delta/s dx|^{1/\delta} \]
\[ + \left| \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} \times D^{\alpha_2}A_2(y)K(x, y)g_i(y)dy|x|^\delta/s dx \right)^{1/\delta} \]
\[ + \left| \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}A_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} \times K(x, y)g_i(y)dy|x|^\delta/s dx \right)^{1/\delta} \]
\[ + \left| \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty |T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^s dx \right)^{\delta/s} \right]^{1/\delta} \]
\[ := I_1 + I_2 + I_3 + I_4 + I_5. \]

Now, let us estimate \( I_1, I_2, I_3, I_4 \) and \( I_5 \), respectively. First, for \( x \in Q \) and \( y \in \tilde{Q} \), by Lemma 1, we get
\[ R_{m_j}(\tilde{A}_j; x, y) \leq C|x-y|^m_j \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}, \]
thus, by Lemma 2 and the weak type \((1,1)\) of \(|T|_s\), we obtain
\[ I_1 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) \left( \frac{1}{|Q|} \int_Q |T(g)(x)|^{\delta/s} dx \right)^{1/\delta} \]
\[ = C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |Q|^{-1} ||T(g)_s|_\chi_Q|_{L^1} |Q|^{1/\delta - 1} \]
\[ \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |Q|^{-1} ||T(g)_s|_{W_1} \]
\[ \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M(|f|_s)(\tilde{x}). \]

For \( I_2 \), by Lemma 2 and generalized Hölder’s inequality, we get
\[ I_2 \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1g)(x)|^{\delta/s} dx \right)^{1/\delta} \]
\[ \leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1} ||T(D^{\alpha_1}\tilde{A}_1g)(x)|_{s\chi_Q}|_{W_1} \]
\[
\begin{align*}
  &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{A}_1(x)||g(x)|_s dx \\
  &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} A_1 - (D^{\alpha_1} A_1)_{\tilde{Q}}\|_{expL, \tilde{Q}} \|f|_s\|_{L(log L), \tilde{Q}} \\
  &\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^2(|f|_s)(\tilde{x}).
\end{align*}
\]

For \( I_3 \), similarly to the proof of \( I_2 \), we get
\[
I_3 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^2(|f|_s)(\tilde{x}).
\]

Similarly, for \( I_4 \), taking \( r, r_1, r_2 \geq 1 \) such that \( 1/r = 1/r_1 + 1/r_2 \), we obtain, by
\[
I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)||s dx \right)^{1/\delta} \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} ||T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)||_W L^{-1} \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x)||g(x)|_s dx \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^2 \|D^{\alpha_j} A_j - (D^{\alpha_j} A_j)_{\tilde{Q}}\|_{expL, \tilde{Q}} \cdot ||f|_s\|_{L(log L)^{1/r}, \tilde{Q}} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^3(|f|_s)(\tilde{x}).
\]

For \( I_5 \), we write
\[
T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0) = \int_{R^n} \left( \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right) \prod_{j=1}^2 R_{m_i}(\tilde{A}_j; x, y) h_i(y) dy \\
+ \int_{R^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) R_{m_2}(\tilde{A}_2; x, y) \frac{K(x_0, y) h_i(y)}{|x_0 - y|^m} dy \\
+ \int_{R^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) R_{m_1}(\tilde{A}_1; x_0, y) \frac{K(x_0, y) h_i(y)}{|x_0 - y|^m} dy \\
- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left( R_{m_2}(\tilde{A}_2; x, y) \frac{K(x, y)}{|x - y|^m} \right)^{\alpha_1} dx \\
- \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x - y|^m} K(x, y) D^{\alpha_1} \tilde{A}_1(y) h_i(y) dy.
\]
thus, by Minkowski' inequality,

\[ \left\| \sum_{\alpha_2 = m_2}^{1} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}; x, y)(x - y)^{\alpha_2}}{|x - y|^{m}} K(x, y) \right\| \leq C \sum_{\alpha_1 = m_1}^{1} \int_{R^n} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[ \frac{(x - y)^{\alpha_1 + \alpha_2}}{|x - y|^{m}} K(x, y) - \frac{(x - y)^{\alpha_1 + \alpha_2}}{|x - y|^{m}} K(x, y) \right] \times D^\alpha \tilde{A}_1(y) D^\alpha \tilde{A}_2(y) h_i(y) dy \]

By Lemma 1 and the following inequality (see [20])

\[ |b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2, \]

we know that, for \( x \in Q \) and \( y \in 2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q} \),

\[ |R_{m_j}(\tilde{A}; x, y)| \leq C |x - y|^{m_j} \sum_{\alpha_1 = m_1}^{1} \left( ||D^\alpha A||_{BMO} + ||(D^\alpha A)_0 (x, y) - (D^\alpha A)_0 (x, y) || \right) \leq C k |x - y|^{m_j} \sum_{\alpha_1 = m_1}^{1} ||D^\alpha A||_{BMO}. \]

Note that \( |x - y| \sim |x_0 - y| \) for \( x \in Q \) and \( y \in R^n \setminus \tilde{Q} \), we obtain, by the conditions on \( K \),

\[ |I_5^{(1)}| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1}} + \frac{|x - x_0|^{\epsilon}}{|x_0 - y|^{n+1+\epsilon}} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}; x, y) |h_i(y)| dy \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{\alpha_1 = m_1}^{1} ||D^\alpha A_j||_{BMO} \right) \times \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|^{\epsilon}}{|x_0 - y|^{n+1+\epsilon}} \right) |f_i(y)| dy \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{\alpha_1 = m_1}^{1} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\epsilon k}) \frac{1}{2^k \tilde{Q}} \int_{2^k \tilde{Q}} |f_i(y)| dy, \]

thus, by Minkowski' inequality,

\[ \left( \sum_{i=1}^{\infty} |I_5^{(1)}|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{\alpha_1 = m_1}^{1} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\epsilon k}) \times \frac{1}{2^k \tilde{Q}} \int_{2^k \tilde{Q}} |f_i(y)| dy \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M(|f|_s)(\tilde{x}). \]

For \( I^{(2)}_5 \), by the formula (see [3]):

\[
R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y) = \sum_{|\beta| < m_j} \frac{1}{\beta!} R_{m_j-|\beta|}(D^\beta \tilde{A}; x, x_0)(x - y)^\beta
\]

and Lemma 1, we have

\[
|R_{m_j}(\tilde{A}; x, y) - R_{m_j}(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m_j} \sum_{|\alpha| = m_j} |x - x_0|^{m_j-|\beta|} |x - y|^{|\beta|} ||D^\alpha A||_{BMO},
\]

thus

\[
\left( \sum_{i=1}^{\infty} |I^{(2)}_5|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right)
\]

\[
\times \sum_{k=0}^{\infty} \int_{2k+1\tilde{Q}\setminus 2k\tilde{Q}} k \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)|_s dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M(|f|_s)(\tilde{x}).
\]

Similarly,

\[
\left( \sum_{i=1}^{\infty} |I^{(3)}_5|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M(|f|_s)(\tilde{x}).
\]

For \( I^{(4)}_5 \), similar to the proof of \( I^{(1)}_5, I^{(2)}_5 \) and \( I_2 \), we get

\[
\left( \sum_{i=1}^{\infty} |I^{(4)}_5|^s \right)^{1/s} \leq C \sum_{|\alpha| = m_1} \int_{R^n \setminus \tilde{Q}} \left| (x - y)^\alpha K(x, y) - \frac{(x_0 - y)^\alpha K(x_0, y)}{|x_0 - y|^m} \right| dy
\]

\[
\times |R_{m_2}(\tilde{A}_2; x, y)||D^\alpha \tilde{A}_1(y)||f(y)|_s dy
\]

\[
+ C \sum_{|\alpha| = m_1} \int_{R^n \setminus \tilde{Q}} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)|
\]

\[
\times \frac{|x_0 - y|^\alpha K(x_0, y)}{|x_0 - y|^m} |D^\alpha \tilde{A}_1(y)||f(y)|_s dy
\]
Similarly, 

\[
\left( \sum_{i=1}^{\infty} \left| I_5^{(5)} \right|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right)^2 \left( |f|_s \right)(\tilde{x}).
\]

For \( I_5^{(6)} \), we obtain 

\[
\left( \sum_{i=1}^{\infty} \left| I_5^{(6)} \right|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right)^2 \left( |f|_s \right)(\tilde{x}).
\]

Thus 

\[
|I_5| \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right)^3 \left( |f|_s \right)(\tilde{x}).
\]

This completes the proof of Theorem 1.

By Lemmas 4–7, we get, for \( f = \{f_i\} \in L_0^{\infty}(R^n) \) and \( g \in L^{p(1)}(R^n) \),

\[
\int_{R^n} \left| T_A(f) \right|_s g(x) dx \leq C \int_{R^n} M_{\lambda_n}^\#(T_A(f)|_s)(x) M(g)(x) dx \\
\leq C \int_{R^n} (T_A(f)|_s)^{\#}_\delta (x) M(g)(x) dx
\]
\[ \begin{align*}
\leq \ & C \int_{R^n} M^{l+1}(|f_s|(x)) M(g)(x) \, dx \\
\leq \ & C |||M^{l+1}(|f_s|)||_{L^p(\cdot)} |||M(g)||_{L^{p'}(\cdot)} \\
\leq \ & C |||f_s||_{L^p(\cdot)} |||g||_{L^{p'}(\cdot)},
\end{align*} \]

thus, by Lemma 8,
\[ |||T_A(f)(x)|_s||_{L^p(\cdot)} \leq |||f_s||_{L^p(\cdot)}. \]

This completes the proof of Theorem 2.

References


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