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Boundedness for Vector-Valued Multilinear Singular Integral Operator on L^p Spaces with Variable Exponent

Jing Du, Chuangxia Huang, Lanzhe Liu

Abstract. In this paper, we prove the boundedness for some vector-valued multilinear singular integral operators on L^p spaces with variable exponent by using a sharp estimate of the multilinear operators.

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1 Introduction and Theorems

As the development of the Calderón-Zygmund singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [4, 9, 16–19]). Let T be the Calderón-Zygmund singular integral operator. In [1–3], Cohen and Gosselin studied the L^p $(p > 1)$ boundedness of the multilinear singular integral operator T^A defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y) dy,$$

where

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In the last years, a theory of L^p spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity(see [5–8,15] and their references). Karlovich and Lerner have studied the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent (see [13]). In this paper, we will study the boundedness properties for some vector-valued multilinear singular integral operators on L^p spaces with variable exponent, whose definition is the following.

Fix $\varepsilon > 0$. Let S and S' be Schwartz space and its dual and $T : S \to S'$ be a linear operator. If there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x,y)| \le C|x-y|^{-n}$$

and

$$|K(y,x)-K(z,x)|+|K(x,y)-K(x,z)| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

when $2|y-z| \le |x-z|$. Let m_j be positive integers ($j=1,\cdots,l$), $m_1+\cdots+m_l=m$ and A_j be functions on R^n ($j=1,\cdots,l$). For $1<s<\infty$, the vector-valued multilinear operator related to T is defined by

$$|T_A(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^s\right)^{1/s},$$

where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} K(x,y) f_i(y)dy$$

and

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^{\alpha}A_j(y)(x-y)^{\alpha}.$$

We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^s\right)^{1/s} \text{ and } |f|_s = \left(\sum_{i=1}^{\infty} |f_i(x)|^s\right)^{1/s}.$$

Suppose that $|T|_s$ is weakly (L^1, L^1)-bounded.

Note that when $m=0$, $|T_A|_s$ is just the vector-valued multilinear commutator of T and A (see [19]). While when $m>0$, $|T_A|_s$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1–4, 9]). In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [18], Pérez and Trujillo-Gonzalez proved a sharp estimate for some multilinear commutator. The main purpose of this paper is to prove the boundedness for the vector-valued multilinear singular integral operators $|T_A|_s$ on L^p spaces with variable exponent. To do this, we first prove a sharp inequality for the vector-valued multilinear singular integral operators.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta>0$, the sharp function of f is defined by

$$f_\delta^{\#}(x) = \sup_{Q\ni x} \left(\frac{1}{|Q|}\int_Q |f(y)-f_Q|^{\delta}dy\right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [11,20])

$$f_\delta^{\#}(x) \approx \sup_{Q\ni x} \inf_{c\in C} \left(\frac{1}{|Q|}\int_Q |f(y)-c|^{\delta}dy\right)^{1/\delta}.$$

We write $f^\# = f^\#_\delta$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. We denote the Φ-average for a function f by

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q \ni x} ||f||_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $|| \cdot ||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [16-19], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|} \int_Q |f(y)g(y)|dy \leq ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q}$$

and the following inequalities, for $r, r_j \geq 1, j = 1, \cdots, l$, with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leq M_{L(logL)^{1/r}}(f) \leq CM_{L(logL)^l}(f) \leq CM^{l+1}(f),$$

$$||b - b_Q||_{expL^r,Q} \leq C||b||_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck||b||_{BMO}.$$

The non-increasing rearrangement of a measurable function f on R^n is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leq t\} \quad (0 < t < \infty).$$

For $\lambda \in (0, 1)$ and a measurable function f on R^n, the local sharp maximal function of f is defined by

$$M^\#_\lambda(f)(x) = \sup_{Q \ni x} \inf_{c \in C} ((f - c)\chi_Q)^*(\lambda|Q|).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f, p) = \int_{R^n} |f(x)|^{p(x)}dx.$$

The sets becomes Banach spaces with respect to the following norm

$$||f||_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f/\lambda, p) \leq 1\}.$$

Denote by $M(R^n)$ the set of all measurable functions $p : R^n \to [1, \infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following holds

$$1 < p_- = ess \inf_{x \in R^n} p(x), \qquad ess \sup_{x \in R^n} p(x) = p_+ < \infty. \tag{1}$$

In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [5–8,15]). In this paper, we shall prove the following theorems.

Theorem 1. *Let $1 < s < \infty$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in L_0^\infty(R^n)$, $0 < \delta < 1$ and $\tilde{x} \in R^n$,*

$$(|T_A(f)|_s)^{\#}_{\delta}(\tilde{x}) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M^{l+1}(|f|_s)(\tilde{x}).$$

Theorem 2. *Let $1 < s < \infty$, $p(\cdot) \in \mathrm{M}(R^n)$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T_A|_s$ is bounded on $L^{p(\cdot)}(R^n)$, that is*

$$|||T_A(f)|_s||_{L^{p(\cdot)}} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |||f|_s||_{L^{p(\cdot)}}.$$

Remark 1. Let T be the Calderón-Zygmund operator (see [4, 11, 20]). Then Theorem 1 and Theorem 2 hold for T.

2 Some Lemmas

We begin with some preliminary lemmas.

Lemma 1 (see [3]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 (see [11, p. 485]). *Let $0 < p < q < \infty$. We define that, for any function $f \geq 0$ and $1/r = 1/p - 1/q$,*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q}, \ N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 3 (see [18]). *Let $r_j \geq 1$ for $j = 1, \cdots, l$ and we denote $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_l(x)g(x)|dx \leq ||f||_{expL^{r_1},Q}\cdots ||f||_{expL^{r_l},Q}||g||_{L(logL)^{1/r},Q}.$$

Lemma 4 (see [13]). *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). Then $L_0^\infty(R^n)$ is dense in $L^{p(\cdot)}(R^n)$.*

Lemma 5 (see [14]). *Let $f \in L^1_{loc}(R^n)$ and g be a measurable function satisfying*

$$|\{x \in R^n : |g(x)| > \alpha\}| < \infty \quad \text{for all} \quad \alpha > 0.$$

Then

$$\int_{R^n} |f(x)g(x)|dx \leq C_n \int_{R^n} M^{\#}_{\lambda_n}(f)(x)M(g)(x)dx.$$

Lemma 6 (see [14]). *Let $\delta > 0$, $0 < \lambda < 1$ and $f \in L^\delta_{loc}(R^n)$. Then*

$$M^{\#}_\lambda(f)(x) \leq (1/\lambda)^{1/\delta} f^{\#}_\delta(x).$$

Lemma 7 (see [13]). *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). If $f \in L^{p(\cdot)}(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$ with $p'(x) = p(x)/(p(x)-1)$, then fg is integrable on R^n and*

$$\int_{R^n} |f(x)g(x)|dx \leq C||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}.$$

Lemma 8 (see [13]). *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying (1). Set*

$$||f||'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} |f(x)g(x)|dx : f \in L^{p(\cdot)}(R^n), g \in L^{p'(\cdot)}(R^n)\right\}.$$

Then $||f||_{L^{p(\cdot)}} \leq ||f||'_{L^{p(\cdot)}} \leq C||f||_{L^{p(\cdot)}}$.

3 Proof of Theorem

It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q ||T_A(f)(x)|_s - C_0|^\delta dx\right)^{1/\delta} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^{l+1}(|f|_s)(\tilde{x}).$$

Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}_j(x)=A_j(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_m(A_j;x,y)=R_m(\tilde{A}_j;x,y)$ and $D^\alpha\tilde{A}_j=D^\alpha A_j-(D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We split $f=g+h=\{g_i\}+\{h_i\}$ for $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
&T_A(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy\\
&=\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)h_i(y)dy\\
&+\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy,
\end{aligned}$$

then, by Minkowski' inequality,

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q \left||T_A(f)(x)|_s-|T_{\tilde{A}}(h)(x_0)|_s\right|^\delta dx\right]^{1/\delta}\\
\le\ &\left[\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^\infty |T_A(f_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^s\right)^{\delta/s}dx\right]^{1/\delta}\\
\le\ &\left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^\infty\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^s\right)^{\delta/s}dx\right]^{1/\delta}\\
+\ &[\frac{C}{|Q|}\int_Q(\sum_{i=1}^\infty|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}
\end{aligned}$$

$$\begin{aligned}
& \times D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & [\frac{C}{|Q|}\int_Q(\sum_{i=1}^{\infty}|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} \\
& \times D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & [\frac{C}{|Q|}\int_Q(\sum_{i=1}^{\infty}|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} \\
& \times K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & \left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)\right|^s\right)^{\delta/s}dx\right]^{1/\delta} \\
:= \quad & I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j;x,y)\le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO},$$

thus, by Lemma 2 and the weak type (1,1) of $|T|_s$, we obtain

$$\begin{aligned}
I_1 \quad & \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_s^{\delta}dx\right)^{1/\delta} \\
& = \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}\frac{|||T(g)|_s\chi_Q||_{L^\delta}}{|Q|^{1/\delta-1}} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}|||T(g)|_s||_{WL^1} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}|||g|_s||_{L^1} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_2, by Lemma 2 and generalized Hölder's inequality, we get

$$\begin{aligned}
I_2 \quad & \le \quad C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_s^{\delta}dx\right)^{1/\delta} \\
& \le \quad C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1g)(x)|_s\chi_Q||_{WL^1}
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)||g(x)|_s dx\\
&\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL,\tilde{Q}}|||f|_s||_{L(logL),\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_3, similarly to the proof of I_2, we get

$$I_3\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

Similarly, for I_4, taking $r,r_1,r_2\ge 1$ such that $1/r=1/r_1+1/r_2$, we obtain, by Lemma 3,

$$\begin{aligned}
I_4 &\le C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|_s^{\delta}dx\right)^{1/\delta}\\
&\le C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)|_s\chi_Q||_{WL^1}\\
&\le C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)||g(x)|_s dx\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left|\left|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right|\right|_{expL^{r_j},\tilde{Q}}\cdot|||f|_s||_{L(logL)^{1/r},\tilde{Q}}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0) &= \int_{R^n}\left(\frac{K(x,y)}{|x-y|^m}-\frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\Big)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\Big)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\Big[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)\\
&-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\Big]D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y) \\
& -\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)]D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
& +\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] \\
& \times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
= \ & I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

By Lemma 1 and the following inequality(see [20])

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_{m_j}(\tilde{A}_j;x,y)| &\le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}(||D^{\alpha_j}A||_{BMO}+|(D^{\alpha_j}A)_{\tilde{Q}(x,y)}-(D^{\alpha_j}A)_{\tilde{Q}}|) \\
&\le Ck|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A||_{BMO}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by the conditions on K,

$$
\begin{aligned}
|I_5^{(1)}| &\le C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) \\
&\quad\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f_i(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=1}^\infty k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f_i(y)|dy,
\end{aligned}
$$

thus, by Minkowski' inequality,

$$
\begin{aligned}
\left(\sum_{i=1}^\infty\left|I_5^{(1)}\right|^s\right)^{1/s} &\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=1}^\infty k^2(2^{-k}+2^{-\varepsilon k}) \\
&\quad\times\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s dy
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)=\sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)|\leq C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\left|I_5^{(2)}\right|^s\right)^{1/s} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\\ &\quad\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\backslash 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|_s dy\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(3)}\right|^s\right)^{1/s}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\begin{aligned}&\left(\sum_{i=1}^{\infty}\left|I_5^{(4)}\right|^s\right)^{1/s}\\ \leq\ & C\sum_{|\alpha_1|=m_1}\int_{R^n\backslash\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\ &\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy\\ &+C\sum_{|\alpha_1|=m_1}\int_{R^n\backslash\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\ &\times\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k}+2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k}+2^{-\varepsilon k}) \\
&\quad \times ||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL,2^k\tilde{Q}}|||f|_s||_{L(logL),2^k\tilde{Q}} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) M^2(|f|_s)(\tilde{x}).
\end{aligned}
$$

Similarly,

$$
\left(\sum_{i=1}^{\infty}\left|I_5^{(5)}\right|^s\right)^{1/s} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) M^2(|f|_s)(\tilde{x}).
$$

For $I_5^{(6)}$, we obtain

$$
\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left|I_5^{(6)}\right|^s\right)^{1/s} \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \int_{R^n\setminus\tilde{Q}} \left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right| \\
&\quad \times |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \prod_{j=1}^{2}\left|\left|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right|\right|_{expL^{r_j},2^k\tilde{Q}} \cdot |||f|_s||_{L(logL)^{1/r},2^k\tilde{Q}} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) M^3(|f|_s)(\tilde{x}).
\end{aligned}
$$

Thus

$$
|I_5| \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) M^3(|f|_s)(\tilde{x}).
$$

This completes the proof of Theorem 1.

By Lemmas 4–7, we get, for $f=\{f_i\}\in L_0^{\infty}(R^n)$ and $g\in L^{p'(\cdot)}(R^n)$,

$$
\begin{aligned}
\int_{R^n} |T_A(f)(x)|_s g(x)|dx &\leq C\int_{R^n} M_{\lambda_n}^{\#}(T_A(f)|_s)(x)M(g)(x)|dx \\
&\leq C\int_{R^n} (T_A(f)|_s)_{\delta}^{\#}(x)M(g)(x)dx
\end{aligned}
$$

$$\begin{aligned}
&\leq C\int_{R^n} M^{l+1}(|f|_s)(x)M(g)(x)dx\\
&\leq C||M^{l+1}(|f|_s)||_{L^{p(\cdot)}}||M(g)||_{L^{p'(\cdot)}}\\
&\leq C|||f|_s||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}},
\end{aligned}$$

thus, by Lemma 8,

$$|||T_A(f)(x)|_s||_{L^{p(\cdot)}} \leq |||f|_s||_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2.

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Jing Du, Chuangxia Huang, Lanzhe Liu
College of Mathematics
Changsha University of Science and Technology
Changsha, 410077
China

E-mail: *17719360@qq.com; cxiahuang@126.com; lanzheliu@163.com*

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