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On a Method for Estimation of Risk Premiums Loaded by a Fraction of the Variance of the Risk

Virginia Atanasiu

Abstract. In this paper we have obtained linear approximations which are unbiased estimates for the expected value part, respectively for the variance part and finally for the fluctuation part of the loading from the variance premium, using the greatest accuracy theory. The article provides a means to approximate the separate parts of the variance loaded premium by linear non-homogeneous credibility estimators. Apart from the purpose of this paper, which is to simply add "credibility" like estimators for the separate parts of the variance premium, we have presented some basic theorems from statistics and some basic results on finding estimators with minimal mean squared error from probability theory. The fact that it is based on complicated mathematics, involving conditional expectations, needs not bother the user more than it does when he applies statistical tools like, discriminating analysis and scoring models.

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Introduction

It is an original paper which describes techniques for estimating premiums for risks, containing a fraction of the variance of the risk as a loading on the net risk premium. An approach "in this sense" is to consider the variance premium. The problem under discussion is to get linear approximations, which are unbiased estimates for the expected value part, variance part, fluctuation part, i.e. for the separate parts of the variance premium, using the classical model of Bühlmann and the credibility for the Esscher premiums. The present article contains a method to estimate risk premiums loaded by a fraction of the variance of the risk, as opposed to the net premiums studied thus far in the credibility theory.

The first section shows that the Esscher premium approaches the variance principle and that this premium is derived as an optimal estimator minimizing a suitable loss function. In the first section it is shown that the Esscher premium can be used as an approximation to the variance loaded premium, by truncating the development of a power series. Also, the approach of the problem of Esscher premium, followed in the first section is to consider the best linear credibility estimator which minimizes the exponentially weighted squared error loss function. The second section analyses and presents the linear non-homogeneous credibility estimators for the separate parts of the variance premium.

It turns out that the linear credibility approximations for each of the parts in the variance premium to coincide with the unbiased estimates for the expected value part, the variance part and the fluctuation part from the variance premium.

The approach of the problem of loaded premiums, followed in the second section is to simply add credibility - like estimators for the separate parts of the variance premium.

1 Techniques for estimating premiums for risks, containing a fraction of the variance of the risk as a loading on the net risk premium

1.1 The classical model of Bühlmann

Consider a portfolio of contracts $j = 1, \ldots, k$ satisfying the constraints (B_1) and (B_2). The index contract j is a random vector consisting of the structural variables θ_j and the observable variables: $X_{j1}, \ldots X_{jt}$, where $j = 1, \ldots, k$.

(B_1) $E[X_{jr}|\theta_j] = \mu(\theta_j)$ - the net premium for a contract with risk parameter θ_j-,Cov$[\underline{X}_j|\theta_j] = \sigma^2(\theta_j)I^{(t,t)}$, $j = 1, \ldots, k$, and:

(B_2) the contracts $j = 1, \ldots, k$ are independent, the variables $\theta_1, \ldots, \theta_k$ are identically distributed, and the observations X_{jr} have finite variance, then the optimal non-homogeneous linear estimators $\hat{\mu}(\theta_j)$ for $\mu(\theta_j)$, $j = 1, \ldots, k$, in the least squares sense read: $\hat{\mu}(\theta_j) = (1 - z)m + zM_j$, where $M_j = \dfrac{1}{t}\sum\limits_{s=1}^{t} X_{js}$ denotes the individual estimator for $\mu(\theta_j)$. The resulting credibility factor z which appears in the credibility adjusted estimator $\hat{\mu}(\theta_j)$ is found as: $z = at/(s^2 + at)$, with the structural parameters m, a and s^2 as defined by the following formulae:

$$m = E[X_{jr}] = E[\mu(\theta_j)],\ a = \mathrm{Var}[\mu(\theta_j)],\ s^2 = E[\sigma^2(\theta_j)],\ j = 1, \ldots, k.$$

Here the identity or unit matrix I denotes a matrix with unities on the diagonal and zeros elsewhere.

1.2 The credibility for the Esscher premiums

Minimizing weighted mean squared error

When X and Y are two random variables, and Y must be estimated using a function $g(X)$ of X, the choice yielding the minimal weighted mean squared error $E[(Y - g(X))^2 e^{hY}]$ is the quantity:

$$E[Ye^{hY}|X]/E[e^{hY}|X].$$

Indeed:

$$E[(Y - g(X))^2 e^{hY}] = E\{E[(Y - g(X))^2 e^{hY}|X]\} =$$
$$= \int E[(Y - g(x))^2 e^{hY}|X = x] \cdot dF_X(x).$$

For a fixed x, the integrand can be written as: $E[(Z-p)^2e^{hZ}]$, with $p=g(x)$ and Z distributed as Y, given $X=x$ ($Z\overset{(P)}{\equiv}[Y|(X=x)]$). This quadratic form in p is minimized taking $p=E[Ze^{hZ}]/E[e^{hZ}]$ or what is the same $g(x)=E[Ye^{hY}|X=$ $=x]/E[e^{hY}|X=x]$.

Indeed:

$$\varphi(p)\overset{not}{=}E[(Z-p)^2e^{hZ}]=E(Z^2e^{hZ})+p^2E(e^{hZ})-2pE(Ze^{hZ}),$$

so $\varphi(p)$ is the following quadratic form in $p:E[(Z-p)^2e^{hZ}]$. We have to solve the following minimization problem: $\underset{p}{\mathrm{Min}}\varphi(p)$. Since this problem is the minimum of a positive definite quadratic form, it suffices to find a solution with the first derivative equal to zero. Taking the first derivative with respect to p, we get the equation: $2pE(e^{hZ})-2E(Ze^{hZ})=0$. So: $p=E(Ze^{hZ})/E(e^{hZ})$, because: $\varphi''(p)=2E(e^{hZ})>0$. If the integrand is chosen minimal for each x, the integral over all x is minimized, too.

Definition. The quantity $E[Ye^{hY}|X]/E[e^{hY}|X]$, denoted by $H[Y|X]$ and which minimizes the weighted mean squared error $E[(Y-g(X))^2e^{hY}]$ in the above theoretical result, entitled "Minimizing weighted mean squared error" is called the Esscher premium for Y, given X.

Applying the formula $H[Y|X]=E[Ye^{hY}|X]/E[e^{hY}|X]$ to $Y=X_{t+1},j$ and $X=\underline{X}_j=(X_{j1},\ldots,X_{jt})'$, we see that the best risk premium - in the sense of minimal weighted mean squared error - to charge for period $(t+1)$ is the Esscher premium for $X_{t+1,j}$, given $\underline{X}_j=(X_{j1},X_{j2},\ldots,X_{jt})'$:

$$H[X_{t+1,j}|\underline{X}_j]\overset{not}{=}g(X_j)=E[X_{t+1,j}e^{hX_{t+1,j}}|\underline{X}_j]/E[e^{hX_{t+1,j}}|\underline{X}_j]. \tag{1.1}$$

Apart from the optimal credibility result (1.1) for this situation we can obtain the Esscher premium as an optimal estimator minimizing a suitable loss function.

The linear credibility formula for exponentially weighted squared error loss function requires not just the knowledge of a few natural structure parameters, but it is necessary that for the structure function some values of the moment generating function are known.

This is why the less refined approach followed in Section 2, is more useful in practice.

2 The credibility for the variance premiums

For a small h, the optimal credibility estimated for the variance loaded premium can be approximated as:

$$\begin{aligned}
&g(X_j)\cong(E[X_{t+1,j}|\underline{X}_j]+hE[X^2_{t+1,j}|\underline{X}_j]+O(h^2))/(1+hE[X_{t+1,j}|\underline{X}_j]+O(h^2))\cong\\
&\approx(E[X_{t+1,j}|\underline{X}_j]+hE[X^2_{t+1,j}|\underline{X}_j]+O(h^2))(1-hE[X_{t+1,j}|\underline{X}_j]+O(h^2))=\\
&=E[X_{t+1,j}|\underline{X}_j]+h\mathrm{Var}[X_{t+1,j}|\underline{X}_j]+O(h^2)\cong E[X_{t+1,j}|\underline{X}_j]+\\
&+h\mathrm{Var}[X_{t+1,j}|\underline{X}_j]=E[\mu(\theta_j)|\underline{X}_j]+h\{E[\sigma^2(\theta_j)|\underline{X}_j]+\mathrm{Var}[\mu(\theta_j)|\underline{X}_j\}
\end{aligned} \tag{2.1}$$

approximating numerator and denominator of $g(X_j)$ up to the first order in h.

The purpose of this section is to get linear approximations for each of the terms in the right-hand side. We will derive unbiased estimates for the:

$$\begin{cases} \text{expected value part} & E[\mu(\theta_j)|\underline{X}_j] & (2.2)\\ \text{variance part} & E[\sigma^2(\theta_j)|\underline{X}_j] & (2.3)\\ \text{fluctuation part} & \mathrm{Var}[\mu(\theta_j)|\underline{X}_j] & (2.4)\end{cases}$$

Remark. Another problem appears if we want to find an estimate for the random variable:

$$p(\theta) := \mu(\theta) + \alpha\sigma^2(\theta).$$

Minimizing the squared error would lead to the following credibility estimator:

$$E[p(\theta)|\underline{X}] = E[\mu(\theta)|\underline{X}] + \alpha E[\sigma^2(\theta)|\underline{X}], \tag{2.5}$$

without the fluctuation part, because there is the following basic result on finding estimators with minimal mean squared error.

Minimizing mean squared error for conditional distributions

When X and Y are random variables, the function $g(\cdot)$ of X estimating Y with minimal mean squared error is:

$$g^*(X) = E[Y|X].$$

Applying this theorem to $Y = p(\theta)$ and $X = \underline{X} = (X_1, \dots, X_t)'$ we obtain that the verification of the equality (2.5) is readily performed.

One might argue that this premium is more reasonable, since the policyholder, having himself a fixed though unknown risk parameter, should not pay for the uncertainty concerning his own risk parameter, only for the variation of his claims.

2.1 The main results of this paper

Here and as follows we present **the main results** leaving the detailed calculation to the reader.

A) An approximation for the expected value part:

The expected value part has been dealt with in Subsection 1.1. We recall the result:

$$\hat{\mu}(\theta_j) = (1-z)m + zM_j \tag{2.6}$$

where

$$z = at/(s^2+at), \quad M_j = \frac{1}{t}\sum_{r=1}^{t} X_{jr}, \quad a = \mathrm{Var}[\mu(\theta_j)], \quad s^2 = E[\sigma^2(\theta_j)], \ (j = \overline{1,k})$$

(for more details, see [5] or [6]). One could approximate the expected value part by its best linear non-homogeneous credibility estimator (2.6).

B) An approximation of the fluctuation part:

Next we consider the fluctuation part:

$$\mathrm{Var}[\mu(\theta_j)|\underline{X}_j] \stackrel{def}{=} E\{(\mu(\theta_j) - E[\mu(\theta_j)|\underline{X}_j])^2|\underline{X}_j\}. \tag{2.7}$$

It is difficult to estimate this expression because of the appearance of $E[\mu(\theta_j)|\underline{X}_j]$.

However, one could approximate this expectation by its best linear non-homogeneous credibility estimator (2.6), and try to estimate:

$$E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\} \tag{2.8}$$

(see (2.6)), where $z = at/(s^2 + at)$.

To obtain **an approximation for the fluctuation part**, this quantity is averaged once more over the entire collective (the averaging is representative for the conditioned variance, because:

$$\mathrm{Var}[\mu(\theta_j)|\underline{X}_j] \stackrel{(2.7)}{=} E\{(\mu(\theta_j) - E[\mu(\theta_j)|\underline{X}_j])^2|\underline{X}_j\}$$
$$\stackrel{(2.6)}{=} E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}) :$$

$$\begin{aligned}
&E[E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}] = E[E\{[\mu(\theta_j) - m - z(M_j - m)]^2|\underline{X}_j\}] = \\
&= E\{[\mu(\theta_j) - m - z(M_j - m)]^2\} = E[\mu^2(\theta_j) + m^2 + z^2(M_j - m)^2 - 2m\mu(\theta_j) + \\
&+2mz(M_j - m) - 2\mu(\theta_j)z(M_j - m)] = E[\mu^2(\theta_j)] + m^2 + z^2E[(M_j - m)^2] - \\
&-2mE[\mu(\theta_j)] + 2mzE[(M_j - m)] - 2zE[\mu(\theta_j)(M_j - m)] = E[\mu^2(\theta_j)] + m^2 + \\
&+z^2E\{[M_j - E(M_j)]^2\} - 2m\cdot m - 2zE\{[\mu(\theta_j) - m][M_j - m]\} = E[\mu^2(\theta_j)] - \\
&-m^2 + z^2\mathrm{Var}(M_j) - 2zE\{(\mu(\theta_j) - E[\mu(\theta_j)])[M_j - E(M_j)]\} = E[\mu^2(\theta_j)] - \\
&-E^2[\mu(\theta_j)] + z^2\mathrm{Var}(M_j) - 2z\mathrm{Cov}[\mu(\theta_j), M_j] = \mathrm{Var}[\mu(\theta_j)] + z^2\mathrm{Var}(M_j) - \\
&-2z\mathrm{Cov}[\mu(\theta_j), M_j] = \mathrm{Var}[\mu(\theta_j)] - 2z\mathrm{Cov}[\mu(\theta_j), M_j] + z^2\mathrm{Var}(M_j) = \\
&= a - 2za + z^2(a + s^2/t) = a(1 - 2z + z^2) + z^2s^2/t = a(1-z)^2 + z^2s^2/t,
\end{aligned} \tag{2.9}$$

because:

$$E(M_j) = m \tag{2.10}$$

$$E[\mu(\theta_j)] = E[E(X_{jr}|\theta_j)] = E(X_{jr}) \tag{2.11}$$

$$\mathrm{Var}[\mu(\theta_j)] = a \tag{2.12}$$

(see the definition of the structure parameter a).

$$\mathrm{Cov}[\mu(\theta_j), M_j] = \frac{1}{t}\sum_{r=1}^{t}\mathrm{Cov}[\mu(\theta_j), X_{jr}] = \frac{1}{t}\sum_{r=1}^{t} a = \frac{1}{t}ta = a, \tag{2.13}$$

$$\operatorname{Var}(M_j) = \operatorname{Cov}(M_j, M_j) = \frac{1}{t^2}\sum_{r,r'}\operatorname{Cov}(X_{jr}, X_{jr'}) = \frac{1}{t^2}\sum_{r,r'}(a + \delta_{rr'}s^2) =$$

$$= \frac{1}{t^2}\sum_{r}\left[(a+\delta_{rr}s^2) + \sum_{r',r'\neq r}(a+\delta_{rr'}s^2)\right] = \frac{1}{t^2}\sum_{r=1}^{t}[(a+s^2)+(t-1)a] = \tag{2.14}$$

$$= \frac{1}{t^2}\sum_{r=1}^{t}(s^2+at) = \frac{t(s^2+at)}{t^2} = \frac{s^2+at}{t} = a + \frac{s^2}{t}.$$

But inserting the value of the credibility factor z in the right hand side of (2.9) shows that it equals $(1-z)a$, so:

$$\begin{aligned}&\operatorname{Var}[\mu(\theta_j)|\underline{X}_j] \cong E[E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}] =\\ &= a(1-z)^2 + \frac{z^2s^2}{t} = a\left(1 - \frac{at}{at+s^2}\right)^2 + \frac{s^2}{t}\cdot\frac{a^2t^2}{(at+s^2)^2} =\\ &= \frac{as^4 + a^2s^2t}{(at+s^2)^2} = \frac{as^2(s^2+at)}{(at+s^2)^2} = \frac{as^2}{at+s^2} = a(1-z) = (1-z)a,\end{aligned} \tag{2.15}$$

C) An approximation for the variance part:

For **the variance part**, there is in analogy with the expected value part, $E[\sigma^2(\theta_j)|\underline{X}_j]$ that is approximated as a non-homogeneous linear combination:

$$E[\sigma^2(\theta_j)|\underline{X}_j] \cong c_0 + c_1 S_j^2 \tag{2.16}$$

where

$$S_j^2 = \sum_{s=1}^{t}(X_{js} - \overline{X}_j)^2/(t-1). \tag{2.17}$$

The following distance will be minimized:

$$E\{[\sigma^2(\theta_j) - c_0 - c_1 S_j^2]^2\}. \tag{2.18}$$

So, for each $j = \overline{1,k}$ we have to solve the following minimization problem:

$$\operatorname*{Min}_{c_0,c_1} E\{[\sigma^2(\theta_j) - c_0 - c_1 S_j^2]^2\}. \tag{2.19}$$

As (2.19) is the minimum of a positive definitive quadratic form, it is enough to find a solution with all partial derivates equal to zero. Taking the partial derivative with respect to c_0 results in:

$$c_0 = E[\sigma^2(\theta_j)](1-c_1), \tag{2.20}$$

because if:

$$\begin{aligned}&f(c_0,c_1) \overset{not.}{=} E\{[\sigma^2(\theta_j) - c_0 - c_1S_j^2]^2\} = E\{[\sigma^2(\theta_j)]^2\} + c_0^2 + c_1^2E[(S_j^2)^2] -\\ &-2c_0E[\sigma^2(\theta_j)] + 2c_0c_1E(S_j^2) - 2c_1E[\sigma^2(\theta_j)S_j^2],\end{aligned}$$

then $\dfrac{\partial f}{\partial c_0} = 0$ implies: $2c_0 - 2E[\sigma^2(\theta_j)] + 2c_1 E(S_j^2) = 0$, that is the verification of the equality (2.20) that is readily performed (see (2.24)). Inserting the result (2.20) in (2.19) we obtain:

$$\operatorname*{Min}_{c_1} E\{[\sigma^2(\theta_j) - E(\sigma^2(\theta_j))(1-c_1) - c_1 S_j^2]^2\}. \tag{2.21}$$

Taking the derivative with respect to c_1, gives:

$$\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Cov}[S_j^2, S_j^2], \tag{2.22}$$

because if:

$$\begin{aligned}
&f(c_1) \overset{not.}{=} E\{[\sigma^2(\theta_j) - E(\sigma^2(\theta_j))(1-c_1) - c_1 S_j^2]^2\} = E\{[\sigma^2(\theta_j)]^2 + E^2[\sigma^2(\theta_j)]\cdot \\
&\cdot(1-c_1)^2 + c_1^2(S_j^2)^2 - 2\sigma^2(\theta_j)E[\sigma^2(\theta_j)](1-c_1) - 2\sigma^2(\theta_j)c_1 S_j^2 + \\
&+2E[\sigma^2(\theta_j)](1-c_1)c_1 S_j^2\} = E\{[\sigma^2(\theta_j)]^2\} + E^2[\sigma^2(\theta_j)](1-c_1)^2 + c_1^2 E[(S_j^2)^2] - \\
&-2E^2[\sigma^2(\theta_j)](1-c_1) - 2c_1 E[\sigma^2(\theta_j)S_j^2] + 2E[\sigma^2(\theta_j)](1-c_1)c_1 E(S_j^2),
\end{aligned}$$

then $\dfrac{\partial f}{\partial c_1} = 0$ implies:

$$\begin{gathered}
-2E^2[\sigma^2(\theta_j)](1-c_1) + 2c_1 E[(S_j^2)^2] + 2E^2[\sigma^2(\theta_j)] - 2E[\sigma^2(\theta_j)S_j^2] + \\
+2E[\sigma^2(\theta_j)]E(S_j^2)\cdot(1-2c_1) = 0,
\end{gathered}$$

that is:

$$\begin{gathered}
-E^2[\sigma^2(\theta_j)] + c_1 E^2[\sigma^2(\theta_j)] + c_1 E[(S_j^2)^2] + E^2[\sigma^2(\theta_j)] - E[\sigma^2(\theta_j)S_j^2] + \\
+E[\sigma^2(\theta_j)]E(S_j^2) - 2E[\sigma^2(\theta_j)]E(S_j^2)c_1 = 0.
\end{gathered} \tag{2.23}$$

But

$$\sigma^2(\theta_j) = E(S_j^2|\theta_j) \quad \text{and so} \quad E[\sigma^2(\theta_j)] = E[E(S_j^2|\theta_j)] = E(S_j^2). \tag{2.24}$$

Now after plugging (2.24) in (2.23) we obtain:

$$E[\sigma^2(\theta_j)S_j^2] - E[\sigma^2(\theta_j)]E(S_j^2) = c_1\{E[(S_j^2)^2] - E^2(S_j^2)\},$$

that is

$$\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Cov}(S_j^2, S_j^2), \quad \text{or} \quad \mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Var}(S_j^2)$$

and so the verification of the equality (2.22) is readily performed. But:

$$\begin{aligned}
&\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = E\{\mathrm{Cov}[\sigma^2(\theta_j), S_j^2|\theta_j]\} + \mathrm{Cov}\{E[\sigma^2(\theta_j)|\theta_j], E(S_j^2|\theta_j)\} = \\
&= E\{E[\sigma^2(\theta_j)S_j^2|\theta_j] - E[\sigma^2(\theta_j)|\theta_j]E(S_j^2|\theta_j)\} + \mathrm{Cov}[\sigma^2(\theta_j), \sigma^2(\theta_j)] = \\
&= E[\sigma^2(\theta_j)E(S_j^2|\theta_j) - \sigma^2(\theta_j)\sigma^2(\theta_j)] + \mathrm{Var}[\sigma^2(\theta_j)] = E[\sigma^2(\theta_j)\sigma^2(\theta_j) - \\
&-\sigma^2(\theta_j)\sigma^2(\theta_j)] + \mathrm{Var}[\sigma^2(\theta_j)] = \mathrm{Var}[\sigma^2(\theta_j)],
\end{aligned} \tag{2.25}$$

$$\mathrm{Var}(S_j^2) = \mathrm{Var}[E(S_j^2|\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)] = \mathrm{Var}[\sigma^2(\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)]. \quad (2.26)$$

Inserting (2.25) and (2.26) in (2.22), the value of c_1 follows as:

$$c_1 = \mathrm{Var}[\sigma^2(\theta_j)]/\{\mathrm{Var}[\sigma^2(\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)]\}. \quad (2.27)$$

We have

$$\mathrm{Var}(S_j^2|\theta_j) = 2\sigma^4(\theta_j)/(t-1) + O(t^{-2}) \cong 2\sigma^4(\theta_j)/(t-1), \quad (2.28)$$

for large values of t, and under the assumption of normality we get:

$$\mu_4(\theta_j) = 3\sigma^4(\theta_j), \quad (2.29)$$

because from statistics we recall some basic theorems:

(I) Suppose that X is a random variable, with Normal (μ, σ^2) distribution and in addition for all $r \in \mathbf{N}$:

$$\mu_{2r} = E[(X-\mu)^{2r}],$$

then:

$$\mu_{2r} = \frac{(2r)!}{2^r r!}\sigma^{2r}.$$

(II) Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with the same expectations μ and the variance σ^2, and in addition for each r:

$$\mu_4 = E[(X_r - \mu)^4].$$

Let $\tilde{S}^2$ be defined as: $\tilde{S}^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \bar{X})^2$ the sample variance, where $\bar{X}$ is the sample mean of n i. i. d. random variables $X_1, X_2, \ldots, X_n$, that is:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then the following relation is valid:

$$\mathrm{Var}\,(\tilde{S}^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right).$$

Here $(X_{js}|\theta_j)$, $s = \overline{1,t}$ are n i. i. d. random variables, with: $E(X_{js}|\theta_j) = \mu(\theta_j)$, $\mathrm{Var}\,(X_{js}|\theta_j) = \sigma^2(\theta_j)$ and $E\{[X_{js} - E(X_{js}|\theta_j)]^4|\theta_j\} = \mu_4(\theta_j)$ for all $s = \overline{1,t}$.

Let j be fixed. Under the assumption of normality we get: $(X_{js}|\theta_j) \in N(\mu(\theta_j), \sigma^2(\theta_j))$ for all $s = \overline{1,t}$. Applying result **(I)** to $X = (X_{js}|\theta_j)$, for all $s = \overline{1,t}$ and $r = 2$ we have:

$$\mu_4(\theta_j) = \frac{(2\cdot 2)!}{2^2 \cdot 2!}(\sigma^2(\theta_j))^2 = 3\sigma^4(\theta_j).$$

So the verification of the equality (2.29) is readily performed. Applying result **(II)** to $\tilde{S}^2 = (S_j^2|\theta_j)$ we obtain:

$$\mathrm{Var}\,(S_j^2|\theta_j) = \frac{1}{t}\left[\mu_4(\theta_j) - \frac{t-3}{t-1}\sigma^4(\theta_j)\right] = \frac{1}{t}\left[3\sigma^4(\theta_j) - \frac{t-3}{t-1}\sigma^4(\theta_j)\right] =$$

$$= \frac{1}{t} \cdot \frac{3(t-1)-t+3}{t-1} \cdot \sigma^4(\theta_j) = \frac{2t}{t(t-1)}\sigma^4(\theta_j) = \frac{2\sigma^4(\theta_j)}{t-1}.$$

So the verification of the equality (2.28) is already performed.

Let $a^* = \mathrm{Var}[\sigma^2(\theta_j)]$, $s^{2*} = E[\sigma^4(\theta_j)]$, then we obtain for large values of t the following approximation:

$$c_1 \overset{(2.27)}{=} a^*/\{a^*+E[\mathrm{Var}(S_j^2|\theta_j)]\} \overset{(2.28)}{=} a^*/\{a^*+2E[\sigma^4(\theta_j)]/(t-1)\} = a^*/\{a^*+2s^{2^*}/(t-1)\}.$$

Consequently one obtains the following linear estimator for the variance part of the loading, i.e. the conditional expectation of $\sigma^2(\theta_j)$:

$$E[\sigma^2(\theta_j)|\underline{X}_j] \cong (1-c_1)E[\sigma^2(\theta_j)] + c_1 S_j^2$$

(see (2.16) and (2.20)).

3 Conclusions

In this paper we have obtained linear approximations which are unbiased estimates for the expected value part (i.e. for the conditional expectation of $\mu(\theta_j)$), respectively for the variance part (i.e. for the conditional expectation of $\sigma^2(\theta_j)$) and finally for the fluctuation part (the conditional variance of $\mu(\theta_j)$) of the loading from the variance premium, using the greatest accuracy theory.

The present article contains a method to estimate risk premiums loaded by a fraction of the variance of the risk, as opposed to the net premiums studied thus far in the credibility theory.

The first section shows that the Esscher premium approaches the variance principle and that this premium is derived as an optimal estimator minimizing a suitable loss function. So, in the first section it is shown that it can be used as an approximation to the variance loaded premium, by truncating a series expansion. Also, the approach of the problem of Esscher premium, followed in the first section is to consider the best linear credibility estimator which minimizes the exponentially weighted squared error loss function.

The second section analyses and presents the linear non-homogeneous credibility estimators for the separate parts of the variance premium. It happens that the linear credibility approximations for each of the parts in the variance premium to coincide with the unbiased estimates for the expected value part, the variance part and the fluctuation part from the variance premium. The approach of the problem of loaded premiums, followed in the second section is to simply add "credibility" like estimators for the separate parts of the variance premium.

So, the problem under discussion is to get linear approximations, which are unbiased estimates for the expected value part, variance part, fluctuation part, i.e. for the separate parts of the variance premium, using the classical model of Bühlmann and the credibility for the Esscher premiums.

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Virginia Atanasiu
Department of Mathematics
Academy of Economic Studies
Bucharest, Romania
E-mail: *virginia_atanasiu@yahoo.com*

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