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On Frattini subloops and normalizers of commutative Moufang loops

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Abstract. Let L be a commutative Moufang loop (CML) with the multiplication group $\mathfrak{M}$, and let $\mathfrak{F}(L)$, $\mathfrak{F}(\mathfrak{M})$ be the Frattini subloop of L and Frattini subgroup of $\mathfrak{M}$. It is proved that $\mathfrak{F}(L) = L$ if and only if $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, and the structure of this CML is described. The notion of normalizer for subloops in CML is defined constructively. Using this it is proved that if $\mathfrak{F}(L) \neq L$, then L satisfies the normalizer condition and that any divisible subgroup of $\mathfrak{M}$ is an abelian group and serves as a direct factor for $\mathfrak{M}$.

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It is known that in many classes of algebras the Frattini subalgebras essentially determine the structure of these algebras. In this paper this dependance is considered in the class of commutative Moufang loops (CML) and their multiplication groups. Let L be a CML with the multiplication group $\mathfrak{M}$, let $\mathfrak{F}(L)$ and $\mathfrak{F}(\mathfrak{M})$ denote the Frattini subloop of L and the Frattini subgroup of $\mathfrak{M}$. It is proved that $\mathfrak{F}(L) = L$ if and only if $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, and the structure of this CML and groups is described. In particular, if L has the exponent 3, then $\mathfrak{F}(L) = L$ if and only if $L = L'$, where L' denotes the associator subloop of L (Theorem 1). The existence of CML with $L' = L$ is proved in [1].

The normalizer $N_L(H)$ is defined constructively for subloop H of commutative Moufang loop L which, in general, has the same role as a normalizer for subgroups. The normalizer $N_L(H)$ is the unique maximal subloop of L such that H is normal in $N_L(H)$. By analogy with the group theory the notion of CML with normalizer condition is defined: every proper subloop of CML differs from his normalizer. Using essentially Theorem 1, it is proved that if a CML L satisfies the inequality $\mathfrak{F}(L) \neq L$ then L satisfies the normalizer condition. It is proved also that for multiplication groups of CML an analogous situation does not take place. There exists a CML L with multiplication group $\mathfrak{M}$ such that $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$, but $\mathfrak{M}$ does not satisfy the normalizer condition.

Again, using essentially Theorem 1 it is proved that every divisible subgroup of multiplication group $\mathfrak{M}$ of any CML is an abelian group and serves as a direct factor for $\mathfrak{M}$ (Theorem 2). A similar result for divisible subloops of CML is proved in [2]. We note that in general case Theorem 2 is not true. In [3, Theorem 2.7

and Example 2.2] there is an example of a divisible non-periodic and non-abelian ZA-group.

At last we note that the material of present article has been published earlier, in [4].

1 Preliminaries

Let us bring some notions and results on the loop theory from [5, 6].

The *multiplication group* $\mathfrak{M}(L)$ of a loop L is the group generated by all *translations* $L(x)$, $R(x)$, where $L(x)y = xy$, $R(x)y = yx$. The subgroup $\mathfrak{I}(L)$ of group $\mathfrak{M}(L)$, generated by all *inner mappings* $L(x,y) = L^{-1}(xy)L(x)L(y)$, $R(x,y) = R^{-1}(xy)R(y)R(x)$, $T(x) = L^{-1}(x)R(x)$ is called the *inner mapping group* of loop L. A subloop H of a loop L is called *normal* in L if $\mathfrak{I}(L)H = H$. The set of all elements $x \in L$ which commute and associate with all elements of L so that for all $a, b \in L$ $ax = xa$, $ab \cdot x = a \cdot bx$, $ax \cdot b = a \cdot xb$, $xa \cdot b = x \cdot ab$ is a normal subloop $Z(L)$ of L, its *centre*.

Lemma 1 (see [5, p. 63]). *Let H be a normal subloop of loop L with the multiplication group $\mathfrak{M}$. Then $\mathfrak{M}(L/H) \cong \mathfrak{M}/H^*$ where $H^* = \{\alpha \in \mathfrak{M} | (\alpha x)H = xH \quad \forall x \in L\}$. Conversely, every normal subgroup $\mathfrak{N}$ of $\mathfrak{M}$ determines a normal subloop $H = \mathfrak{N}1 = \{\alpha 1 | \alpha \in \mathfrak{N}\}$ of L and $\mathfrak{N} \subseteq H^*$.*

Proposition 1. *Let $(L, \cdot, 1)$ be a loop with centre $Z(L)$, let $\mathfrak{M}$ be its multiplication group with centre $Z(\mathfrak{M})$ and let $\tilde{Z}(L) = \{\varphi 1 | \varphi \in Z(\mathfrak{M})\}$, $\tilde{Z}(\mathfrak{M}) = \{L(\varphi 1) | \varphi \in Z(\mathfrak{M})\}$, $\overline{Z}(\mathfrak{M}) = \{L(a) | a \in Z(L)\}$. Then $\overline{Z}(L) = Z(L) \cong \overline{Z}(\mathfrak{M}) = \tilde{Z}(\mathfrak{M}) = Z(\mathfrak{M})$.*

Proof. Let $a \in Z(Q)$ and $x, y \in L$. Then $R(a) = L(a)$, $a \cdot a^{-1}x = x$, $L(a)L(a^{-1})x = x$ $L(a^{-1}) = L^{-1}(a)$ and $a \cdot xy = ax \cdot y$, $L(a)L(y)x = L(y)L(a)x$, $L(a)L(y) = L(y)L(a)$. Similarly, for $a^{-1} \in Z(L)$ we obtain that $L(a^{-1})R(y) = R(y)L(a^{-1})$. Then $(L(a^{-1})R(y))^{-1} = (R(y)L(a^{-1}))^{-1}$, $R^{-1}(y)L^{-1}(a^{-1}) = L^{-1}(a^{-1})R^{-1}(y)$, $R^{-1}(y)L(a) = L(a)R^{-1}(y)$. Analogously, from $yx \cdot a = y \cdot xa$ and $R(a) = L(a)$ we get $L(a)L(y) = L(y)L(a)$, $L(a)L^{-1}(y) = L(y)^{-1}L(a)$. Then from the definition of the group $\mathfrak{M}(L)$ it follows that $L(a) \in Z(\mathfrak{M})$. Similarly, for $a^{-1} \in Z(L)$ we get that $L(a^{-1}) = L^{-1}(a) \in Z(\mathfrak{M})$. We also have $L(a)L(b) = L(ab)$ for $a, b \in Z(L)$. Then the set $\overline{Z}(\mathfrak{M})$ is a subgroup of $\mathfrak{M}$, $\overline{Z}(\mathfrak{M}) \subseteq Z(\mathfrak{M})$ and the isomorphism $Z(L) \cong \overline{Z}(\mathfrak{M})$, defined by $u \to L(u)$, $u^{-1} \to L^{-1}(u)$, $u \in L$, follows from the equality $L(a)L(b) = L(ab)$.

If $\varphi \in Z(\mathfrak{M})$, then $\varphi L(x) = L(x)\varphi$, $\varphi L(x)y = L(x)\varphi y, \varphi(xy) = x \cdot \varphi y$ and $\varphi R(x) = R(x)\varphi$, $\varphi R(x)y = R(x)\varphi y$, $\varphi(yx) = \varphi y \cdot x$ for any $x, y \in L$. Hence $\varphi(xy) = x \cdot \varphi y$ and $\varphi(yx) = \varphi y \cdot x$. Let $y = 1$. Then $\varphi x = x \cdot \varphi 1$, $\varphi x = \varphi 1 \cdot x$, i.e. $x \cdot \varphi 1 = \varphi 1 \cdot x$. Now, using the equality $\varphi(xy) = x \cdot \varphi y$ we obtain that $xy \cdot \varphi 1 = \varphi(xy \cdot 1) = \varphi(xy) = x \cdot \varphi y = x \cdot \varphi(y \cdot 1) = x(y \cdot \varphi 1)$ and using the equality $\varphi(yx) = \varphi y \cdot x$ we obtain that $\varphi 1 \cdot xy = \varphi(1 \cdot xy) = \varphi(xy) = \varphi x \cdot y = \varphi(1 \cdot x)y = (\varphi 1 \cdot x)y$. Hence, if $\varphi \in Z(\mathfrak{M})$ then $\varphi 1 \in Z(L)$, i.e. $\tilde{Z}(L) \subseteq Z(L)$. Conversely, let $a \in Z(L)$. Then $L(a) \in Z(\mathfrak{M})$ and $a = L(a)1 \in \tilde{Z}(L)$. Hence $Z(L) \subseteq \tilde{Z}(L)$ Consequently, $Z(L) = \tilde{Z}(L)$ and therefore $\overline{Z}(\mathfrak{M}) = \tilde{Z}(\mathfrak{M})$.

Let $\mathfrak{I}(L)$ be the inner mapping group of $\mathfrak{M}$. In the proof of Lemma IV.1.2 from [5] it is shown that each element $\alpha \in \mathfrak{M}$ has the form $\alpha = L(\alpha 1)\theta$ where $\theta \in \mathfrak{I}(L)$; moreover $\alpha \in \mathfrak{I}(L)$ if and only if $L(\alpha 1) = e$ where e is the unit of $\mathfrak{M}$. Let $\mathfrak{J}(L) = \mathfrak{I}(L) \cap Z(\mathfrak{M})$. If $\alpha \in Z(\mathfrak{M})$ then, by the cases considered above, $L(\alpha 1) \in \overline{Z}(\mathfrak{M})$. Then $\theta \in \mathfrak{J}(L)$. The subgroups $\overline{Z}(\mathfrak{M}) \subseteq Z(\mathfrak{M})$ and $\mathfrak{J}(L) \subseteq Z(\mathfrak{M})$ are normal in $\mathfrak{M}$. As $\overline{Z}(\mathfrak{M}) \cap \mathfrak{J}(L) = \varepsilon$ then $Z(\mathfrak{M}) = \overline{Z}(\mathfrak{M}) \times \mathfrak{J}(L)$. By Lemma 1 $\mathfrak{J}(L)1 = 1$ is a normal subloop of L and $\mathfrak{J}(L) \subseteq 1^*$ where $1^* = \{\alpha \in \mathfrak{M} | (\alpha x)1 = x1 \quad \forall x \in L\}$. But $1^* = e$, hence $\mathfrak{J}(L) = e$ and $Z(\mathfrak{M}) = \overline{Z}(\mathfrak{M})$, as required. □

A system Σ of subloops of loop L will be called a *subnormal system* if:

1) it contains 1 and L;

2) it is linearly ordered by inclusion, i. e. for all A, B from Σ either $A \subseteq B$, or $B \subseteq A$;

3) it is closed with respect to the unions and intersections, in particular, together with each $A \neq L$ it contains the intersection $A^\sharp$ of all $H \in \Sigma$ with the condition $H \supset A$ and together with each $B \neq 1$ it contains the union $B^\flat$ of all $H \in \Sigma$ with the condition $H \subset B$;

4) it satisfies the condition: A is normal in $A^\natural$ for all $A \in \Sigma$, $A \neq L$.

A system Σ is called *ascending* (respect. *descending*) if $A^\sharp \neq A$ (respect. $B^\flat \neq B$) for all $A \in \Sigma$, $A \neq L$ (respect. $B \in \Sigma$, $B \neq 1$) and is called *normal* if the subloops $A \in \Sigma$ are normal in L.

A loop L may be called an *SD-loop* if it has a descending subnormal system Σ such that the quotient loops $A^\sharp/A$ are abelian groups for all $A \in \Sigma$, $A \neq L$. If a loop L has an ascending normal system such that $A^\sharp/A \subseteq Z(L/A)$ for all $A \in \Sigma$, $A \neq L$ then L is called a *ZA-loop*.

If the upper central series of the ZA-loop has a finite length, then the loop is called *centrally nilpotent*. The least such length is called the *class* of the central nilpotency. If the loop L is centrally nilpotent of class k then the upper central series of L has the form

$$1 = Z_0(L) \subset Z_1(L) \subset \cdots \subset Z_k(L) = L, \tag{1}$$

where $Z_1(L) = Z(L)$, $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$.

A *commutative Moufang loop* (*CML*) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$. The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The identities

$$L(x, y)z = z(z, y, x), \tag{2}$$

$$(x, y, z) = (y, z, x) = (y^{-1}, x, z) = (y, x, z)^{-1}, \tag{3}$$

$$(xy, u, v) = (x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v), y, x) \tag{4}$$

hold in the CML.

If A, B, C are subsets of CML L, (A, B, C) denotes the set of all associators (a, b, c), $a \in A$, $b \in B$, $c \in C$. If $A = B = C = L$, then the normal subloop $L' = (L, L, L)$ is called the *associator subloop* of CML L.

Lemma 2 (see [5]). *Let L be a CML with centre $Z(L)$ and let $a \in L$. Then $a^3 \in Z(L)$.*

Lemma 3. *If $Z_2(L) \neq Z_1(L)$ for a CML L then $L' \neq L$.*

Proof. If $z \in Z_2(L) \backslash Z_1(L)$ then $((x, y, z), u, v) = 1$ for all $x, y, u, v \in L$ and there exist elements $x_0, y_0 \in L$ such that $(x_0, y_0, z) \neq 1$. From (4) it follows that $(uv, y_0, z) = (u, y_0, z)(v, y_0, z)$, which shows that the mapping $\varphi : u \to (u, y_0, z)$ is a homomorphism of L into $Z_1(L)$. The centre $Z_1(L)$ is an associative subloop and as $(x_0, y_0, z) \neq 1$ then $L' \subseteq \ker\varphi$ and $L/\ker\varphi$ is non-unitary. Hence $L' \neq L$, as required. □

Lemma 4. *Let L be a CML with the multiplication group $\mathfrak{M}$, let L' be the associator subloop of L and let $\mathfrak{M}'$ be the commutator subgroup of $\mathfrak{M}$. Then $L' \subseteq \mathfrak{F}(L)$ and $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$.*

Proof. The inclusion $L' \subseteq \mathfrak{F}(L)$ is proved in [7]. The group $\mathfrak{M}$ is locally nilpotent [2], then the proof of inclusion $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$ can be found, for example, in [8]. □

2 Frattini subloops

If $S, T, \ldots$ are subsets of elements of a loop L, let $< S, T, \ldots >$ denote the subloop of L generated by $S, T, \ldots$. An element x of a loop L is a *non-generator* of L if, for every subset S of L, $< x, S >= L$ implies $< S >= L$. The non-generators of L form the *Frattini subloop*, $\mathfrak{F}(L)$, of L. If L has at least one maximal proper subloop, then $\mathfrak{F}(L)$ is the intersection of all maximal proper subloops of L. In the contrary case, $\mathfrak{F}(L) = L$ [5].

Lemma 5. *Let θ be a homomorphism of the loop L into a loop and let $\mathfrak{F}(L) = L$. Then $\mathfrak{F}(\theta L) = \theta L$.*

Proof. In [5] it is proved that if φ is a homomorphism of the loop L into a loop, then $\varphi(\mathfrak{F}(L)) \subseteq \mathfrak{F}(\varphi(L))$. In our case we have $\theta L = \theta(\mathfrak{F}(L)) \subseteq \mathfrak{F}(\theta L) \subseteq \theta L$. Hence $\mathfrak{F}(\theta L)) = \theta L$, as required. □

Lemma 6. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$; 2) $\mathfrak{F}(L) = L$.*

Proof. 1) $\Rightarrow$ 2). Let $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ and we assume that $\mathfrak{F}(L) \neq L$. Then L has at least one maximal proper subloop H and $\mathfrak{F}(L)$ is the intersection of all such subloops. By Lemma 4 the associator subloop L' lies in $\mathfrak{F}(L)$. Hence H is a normal subloop of L and the quotient loop L/H is a cyclic group of prime order p. Then

$\mathfrak{M}(L/H)$ is a cyclic group of order p too, and by Lemma 1 $\mathfrak{M}/H^*$ is a cyclic group of order p. Consequently, H^* is a maximal proper subgroup of $\mathfrak{M}$. Then $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Contradiction. Hence 1) implies 2).

Conversely, let $\mathfrak{F}(L) = L$ and we assume that $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Let $\mathfrak{N}$ be a maximal proper subgroup of $\mathfrak{M}$. $\mathfrak{M}$ is a locally nilpotent group [2], then by Lemma 4 the commutator subgroup $\mathfrak{M}'$ lies in $\mathfrak{F}(\mathfrak{M})$. Then $\mathfrak{N}$ is a normal subgroup of $\mathfrak{M}$ and $\mathfrak{M}/\mathfrak{N}$ is a cyclic group of prime order p. By Lemma 1 $\mathfrak{N}1 = H$ is a normal subloop of L and $\mathfrak{N} \subseteq H^*$. Then from $\mathfrak{M}(L/H) \cong \mathfrak{M}/H^*$ it follows that L/H is a cyclic group of order p. Hence H is a maximal subloop of L. Then $\mathfrak{F}(L) \neq L$. Contradiction. Hence 2) implies 1). □

Let $M(H)$ denote the subgroup of multiplication group of CML L, generated by $\{L(x) | \forall x \in H\}$, where H is a subset of L.

Lemma 7 (see [9]). *Let L be a CML with the multiplication group $\mathfrak{M}(L)$ and the inner mapping group $\mathfrak{I}(L)$. Then $\mathfrak{M}(L)' =<\mathfrak{I}(L), M(L')>= (L')^\star = \overline{\mathfrak{I}(L)}$, where $(L')^\star = \{\alpha \in \mathfrak{M}(L) | \alpha x \cdot L' = xL' \quad \forall x \in L\}$, $\overline{\mathfrak{I}(L)}$ is the normal subgroup of $\mathfrak{M}(L)$, generated by $\mathfrak{I}(L)$.*

Proposition 2. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(L) = L$ and L satisfies the identity $x^3 = 1$; 2) $L = L'$; 3) $\mathfrak{M} = \mathfrak{M}'$; 4) $\mathfrak{F}(L) = L$ and $Z(L) = \{1\}$; 5) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ and $Z(\mathfrak{M}) = \{e\}$.*

Proof. 1) $\Leftrightarrow$ 2). As $\mathfrak{F}(L) = L$ then by Lemma 5 $\mathfrak{F}(L/L') = L/L'$. In [10] it is proved that for an abelian group G $\mathfrak{F}(G) = G$ if and only if G is a divisible group. The abelian group L/L' satisfies the identity $x^3 = 1$. L/L' is a divisible group, then L/L' is a unitary group. Hence $L' = L$, i.e. 1) implies 2). Conversely, let $L' = L$. By [5] the associator subloop L' satisfies the identity $x^3 = 1$ and from the relations $L = L' \subseteq \mathfrak{F}(L) \subseteq L$ it follows that $\mathfrak{F}(L) = L$. Hence 2) implies 1).

1) $\Rightarrow$ 3). By Lemma 6 $\mathfrak{F}(L) = L$ implies $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$. Like in the previous case from here it follows that $\mathfrak{M}/\mathfrak{M}'$ is a divisible abelian group. By definition the group $\mathfrak{M}$ is generated by translations $L(x), x \in L$. Then from the identity $x^3 = 1$ for L and diassociativity of L it follows that the divisible abelian group $\mathfrak{M}/\mathfrak{M}'$ satisfies the identity $x^3 = 1$. Then $\mathfrak{M}/\mathfrak{M}'$ is a unitary group. Hence $\mathfrak{M}' = \mathfrak{M}$, i.e. 1) $\Rightarrow$ 3).

Conversely, let $\mathfrak{M}' = \mathfrak{M}$. By Lemma 4 $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$. Then from the relations $\mathfrak{M} = \mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M}) \subseteq \mathfrak{M}$ it follows that $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$. By Lemma 1 $\mathfrak{M}(L/L') \cong \mathfrak{M}/(L')^*$. $\mathfrak{M}(L/L')$ is an abelian group. Then $\mathfrak{M}' \subseteq (L')^*$ and from the relation $\mathfrak{M}' = \mathfrak{M}$ it follows that $\mathfrak{M}(L/L')$ is unitary group. Hence $L' = L$. Consequently, 3) implies 2).

2) $\Rightarrow$ 4). We consider the homomorphism $\alpha : L \to L/Z(L)$. The elements of quotient loop have the form $aZ(L)$, $a \in L$. From $L = L'$ it follows that the element a is a product of associators (u, v, w), $u, v, w \in L$. From the equalities $(u, v, w)Z(L) = (uZ(L), v, w) = (u, v, w)$, $ab \cdot Z(L) = a \cdot bZ(L) = aZ(L) \cdot b$ it follows that if $aZ(L) = bZ(L)$ then $a = b$. But this means that α is an isomorphism. Then $Z(L) = \{1\}$. Consequently, 2) implies 4).

Conversely, let $Z(L) = \{1\}$. Then from Lemma 2 it follows that CML L satisfies the identity $x^3 = 1$. Hence 4) implies 1). Further, the equivalence of statements 4), 5) follows from Lemma 6 and Proposition 1, as required. □

Theorem 1. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(L) = L$; 2) L is a direct product $L = L' \times L^3$, where L' is the associator subloop of L and $L^3 = \{x^3 | x \in L\}$ is a divisible abelian group; 3) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$; 4) $\mathfrak{M}$ is a direct product $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$, where $\mathfrak{M}'$ is the commutator subgroup of $\mathfrak{M}$ and $\mathfrak{D}$ is a divisible abelian group. In these cases $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$, $Z(L) = L^3$, $L^3 \cong \mathfrak{D}$, $Z(\mathfrak{M}) = \mathfrak{D}$, $\mathfrak{F}(\mathfrak{M}') = \mathfrak{M}' = M(L') = (L')^\star = \overline{\mathfrak{I}(L)}$, where $(L')^\star = \{\alpha \in \mathfrak{M}(L) | \alpha x \cdot L' = xL' \quad \forall x \in L\}$, $\overline{\mathfrak{I}(L)}$ is the normal subgroup of $\mathfrak{M}(L)$, generated by the inner mapping group $\mathfrak{I}(L)$, $Z(\mathfrak{M}') = \{e\}$.*

Proof. 1) $\Rightarrow$ 2). Using the diassociativity of CML it is easy to prove that L^3 is a subloop of L. By Lemma 2 $L^3 \subseteq Z(L)$. Then L^3 is a normal associative subloop of L. The quotient loop L/L^3 satisfies the identity $x^3 = 1$. By Lemma 5 from $\mathfrak{F}(L) = L$ it follows that $\mathfrak{F}(L/L^3) = L/L^3$. Then by Proposition 1 $L/L^3 = (L/L^3)'$. But $(L/L^3)' = L'L^3/L^3$. Then from $L/L^3 = L'L^3/L^3$ it follows that $L = L'L^3$. Hence $L/L^3 = L'L^3/L^3 \cong L'/(L' \cap L^3)$. We have $\mathfrak{F}(L'/(L' \cap L^3)) = L'/(L'/(L' \cap L^3))$ and $L' \cap L^3 \subseteq Z(L)$. Then by analogy with the proof of implication 1) $\Rightarrow$ 4) of Proposition 2 it is easy to prove that $L' \cap L^3 = \{1\}$. But $L = L'L^3$. Then $L = L' \times L^3$. Further, by Lemma 5 we get that $\mathfrak{F}(L') \cong \mathfrak{F}(L/L^3) = \mathfrak{F}(L)/L^3 = L/L^3 \cong L'$, $\mathfrak{F}(L') = L'$ and by Proposition 2 $Z(L') = \{1\}$. Analogously, $\mathfrak{F}(L^3) = L^3$. The subloop L^3 is associative. Then from $\mathfrak{F}(L^3) = L^3$ it follows that L^3 is a divisible abelian group [9]. Consequently, 1) implies 2) and $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$. Further, from $L = L\prime \times L^3$, $Z(L') = \{1\}$ it follows that $Z(L) = L^3$.

Conversely, let $L = L' \times L^3$ and let $L^3 \subseteq Z(L)$ be a divisible group. Then $\mathfrak{F}(L^3) = L^3$ and $L' = (L' \times L^3)' = (L')'$. By Proposition 2 $\mathfrak{F}(L') = L'$. Hence $L = \mathfrak{F}(L') \times \mathfrak{F}(L^3)$. $\mathfrak{F}(L')$ and $\mathfrak{F}(L^3)$ do not have maximal proper subloops. From here it is easy to see that L does not have a maximal proper subloop, either. Then $\mathfrak{F}(L) = L$. Hence 2) implies 1) and, consequently, the statements 1), 2) are equivalent.

The equivalence of statements 1), 3) follows from Lemma 6.

2) $\Leftrightarrow$ 4). Let $L = L' \times L^3$. From here it follows that any element $a \in L$ has the form $a = ud$, where $u \in L'$, $d \in L^3$. As by Lemma 2 $L^3 \subseteq Z(L)$, then $L(a) = L(u)L(d)$, therefore, $\mathfrak{M} = M(L')M(L^3)$. Any element $\alpha \in M(L^3)$ has the form $\alpha = L(v)$, where $v \in L^3$. Let $\alpha \in M(L') \cap M(L^3)$. Then $\alpha 1 \in L' \cap L^3 = \{1\}$, $\alpha 1 = 1$, $L(v)1 = 1$, $v = 1$, $L(v) = e$, $M(L') \cap M(L^3) = \{e\}$. Further, by Proposition 1 $Z(L) = L^3$ implies $Z(\mathfrak{M}) = M(L^3) \cong Z(L) \cong \mathfrak{D}$. $M(L^3)$ is a normal subgroup of $\mathfrak{M}$. Then from $\mathfrak{M} = M(L')M(L^3)$ it follows that $M(L')$ is also normal in $\mathfrak{M}$. Hence $\mathfrak{M} = M(L') \times \mathfrak{D}$. The quotient loop $\mathfrak{M}/M(L')$ is abelian. Then $\mathfrak{M}' \subseteq M(L')$. By Lemma 7 $M(L') \subseteq \mathfrak{M}'$. Hence $M(L') = \mathfrak{M}'$. Consequently, $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$, i.e. 2) implies 4). Conversely, if $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$ then $\mathfrak{M} = M(L') \times M(L^3)$, $\mathfrak{M}1 = M(L')1 \times M(L^3)1$. Hence, 4) implies 2).

Finally, the equality $\mathfrak{F}(\mathfrak{M}') = \mathfrak{M}'$ follows, by Lemma 5, from the relations $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, $\mathfrak{M}/M(L^3) \cong \mathfrak{M}'$, the equalities $\mathfrak{M}' = (L')^\star = \overline{\mathfrak{I}(L)}$ follow from

Lemma 7 and the equality $Z(\mathfrak{M}') = \{e\}$ follows from equalities $\mathfrak{M} = \mathfrak{M}' \times M(L^3)$, $Z(\mathfrak{M}) = M(L^3)$. This completes the proof of Theorem 1. □

3 Normalizer condition

Let M be a subset, H be a subgroup of group G. The subgroup $N_H(M) = \{h | h \in H, h^{-1}Mh = M\}$ is called *the normalizer* of the set M in the subgroup H [8]. Now, constructively, we define the notion of normalizer for the subloops of CML. Let H, K, where $H \subseteq K$ be subloops of CML L. We define inductively the sequences of sets $\{P_\alpha\}$ and $\{D_\alpha\}$ as follows:

i) $P_1 = \{x \in K | (H, H, x) \subseteq H\}$ and $D_1 = \{x \in K | (H, x, P_1) \subseteq H\}$;

ii) for any ordinal α, $P_{\alpha+1} = \{x \in K | (H, D_\alpha, x) \subseteq H\}$ and $D_{\alpha+1} = \{x \in K | (H, x, P_{\alpha+1}) \subseteq H\}$;

iii) if α is a limit ordinal, $P_\alpha = \bigcap_{\beta<\alpha} P_\beta$ and $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$.

Further, we will also denote the conditions of item ii) by $(H, D_\alpha, \overline{P}_{\alpha+1})$ and $(H, \overline{D}_{\alpha+1}, P_{\alpha+1})$ respectively. H is a subloop of CML L, then from $(H, H, H) \subseteq H$, $(H, H, \overline{P})$ it follows that $H \subseteq P_1$, from $H \subseteq P_1$, $(H, \overline{D}_1, P_1)$, $(H, H, \overline{P}_1)$ it follows that $H \subseteq D_1$, from $(H, H, \overline{P}_1)$, $(H, D_1, \overline{P}_2)$, $H \subseteq D_1$ it follows that $P_1 \supseteq P_2$, from $(H, \overline{D}_1, P_2)$, $(H, \overline{D}_2, P_2)$, $P_1 \supseteq P_2$ it follows that $D_1 \subseteq D_2$. Further, let α be a non-limit ordinal and we suppose by inductive hypothesis that $D_\alpha \supseteq D_{\alpha+1}$ and $P_\alpha \subseteq P_{\alpha+1}$. Then from $D_\alpha \supseteq D_{\alpha+1}$, $(H, D_\alpha, \overline{P}_{\alpha+1})$, $(H, D_{\alpha+1}, \overline{P}_{\alpha+2})$ it follows that $P_{\alpha+1} \subseteq P_{\alpha+2}$ and from $P_{\alpha+1} \subseteq P_{\alpha+2}$, $(H, \overline{D}_{\alpha+2}, P_{\alpha\alpha+1})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha\alpha+2})$ it follows that $D_{\alpha+1} \supseteq D_{\alpha+2}$. Hence, if consider also item iii), we get a sequence of subsets

$$P_1 \supseteq P_2 \supseteq \ldots \supseteq P_\alpha \supseteq \ldots$$

$$D_1 \subseteq D_2 \subseteq \ldots \subseteq D_\alpha \subseteq \ldots. \tag{5}$$

The construction process of subsets P_α, D_α from (5) shall end with an ordinal number, whose cardinality does not exceed the cardinality of CML K itself. We suppose that $P_{\alpha+1} = P_{\alpha+2} = \ldots$. From $(H, \overline{D}_{\alpha+1}, P_{\alpha+1})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha+2})$ it follows that $D_{\alpha+1} = D_{\alpha+2}$. Then from $(H, D_{\alpha+1}, \overline{P}_{\alpha+2})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha+2})$ it follows that $(H, \overline{D}_{\alpha+2}, \overline{P}_{\alpha+2})$. We remind that the inscriptions $\overline{D}_{\alpha+2}$, $\overline{P}_{\alpha+2}$ denote the biggest subsets $D_{\alpha+2}$ and $P_{\alpha+2}$ such that the relation $(H, D_{\alpha+2}, P_{\alpha+2}) \subseteq H$ holds true. H is a subloop of CML L, then from (3) it follows that $D_{\alpha+2} = P_{\alpha+2}$ and using (3), (4) it is easy to prove that $D_{\alpha+2}$ is a subloop of CML L. Hence $D_{\alpha+2}$ is the biggest (and the only) subloop of CML K where by (2) H is a normal subloop. By analogy with group theory the subloop D_α will be called *the normalizer* of subloop H in subloop K of CML L and will be denoted by $N_K(H)$. If the subgroup where the normalizer is taken from is not indicated, it means that it is taken from the entire CML L. Consequently, from the construction of normalizer follows

Proposition 3. *Let H, L, K, where $H \subseteq L \subseteq K$, be subloops of CML L and let H be a normal subloop of L. Then $L \subseteq N_K(H)$.*

The group theory contains studies of the group that satisfies the normalizer condition (see, for example,[8]). These are such groups, where every proper subgroup differs from its normalizer. A similar notion can be introduced for CML. We will say that a *CML satisfies the normalizer condition* or, in short, is an N-loop if every proper subloop differs from its normalizer.

The CML L will be a N-loop if and only if an ascending subnormal system $\{H_\alpha\}$ *passes through each subloop H of CML L.*

Really, we denote $H_0 = 1, H_1 = H$ (respect. $\mathfrak{N}_0 = e$, $\mathfrak{N}_1 = \mathfrak{N}$). Further, for non-limit α we take as H_α the normalizer of subloop $H_{\alpha-1}$, and for limit α H_α will be the union of all H_β for $\beta < \alpha$. This ascending subnormal system, obviously, reaches CML L itself. Conversely, if all subloops of CML L are contained in some ascending subnormal system, then all proper subloop will be normal in some bigger subloop, and, consequently, by Proposition 3, will differ from its normalizer.

Using this result it is easy to prove that *all subloops and all quotient loops of N-loop will be N-loops themselves.* Really, let A be a subloop of N-loop L, and let B be a subloop of L such that $B \subseteq A$. By the aforementioned, an ascending subnormal system $\{B_\alpha\}$ passes through B. Then $\{B_\alpha \cap A\}$ after removing the repetitions will be an ascending subnormal system of A, passing through B. Hence A will be an N-loop. The second statement is proved by analogy.

Theorem 2. *If a CML L with the Frattini subloop* $\mathfrak{F}(L)$ *satisfies the inequality* $\mathfrak{F}(L) \neq L$ *then it satisfies the normalizer condition.*

Proof. As $\mathfrak{F}(L) \neq L$ then the CML L has a maximal proper subloop. Let H be an arbitrary proper subloop of CML L. If H is a maximal subloop of L then by [5] H is normal in L. Hence $H \neq N_L(H) = L$. Let now the subloop H be a non-maximal subloop. By Zorn's Lemma let M be a maximal subloop of L with respect to the property $H \subseteq M$ and let $a \notin L \backslash M$. We suppose that $a^3 = 1$. Let $K = < H, a >$. M is a maximal proper subloop of L, then by [5] the subloop M is normal in L. Let φ be a restriction on K of homomorphism $L \to L/M$. Obviously, $\mathrm{Ker}\varphi = M \cap K$. As $a^3 = 1$ then $M \cap < a > = 1$. Hence $K \backslash < a > = H$ and then $M \cap K = H$. Consequently, H is a normal subloop of K, and as $H \neq K$ then by Proposition 1 $H \neq Z_L(H)$, as required.

Let now $a^3 \neq 1$. By Lemma 1 $a^3 \in Z(L)$, hence $< a^3 >$ is a normal subloop of L. Let $a^3 \in H$. We denote $L/ < a^3 > = \overline{L}$. From $a^3 \in M$, $a \notin M$ it follows that $\overline{M}$ is a maximal proper subloop of $\overline{L}$. Hence $\mathfrak{F}(\overline{L}) \neq \overline{L}$. Further, $\overline{a}^3 = \overline{1}$, then by the previous cases $\overline{H} \neq N(\overline{H})$. As $a^3 \in H$ and $a^3 \in N(H)$ then the inverse images of $\overline{H}$ and $N(\overline{H})$ will be H and $N(H)$ respectively. Hence from $\overline{H} \neq N(\overline{H})$ it follows that $H \neq N(H)$, as required.

If $a^3 \notin H$, then $H \neq H < a^3 >$. By (3) and Lemma 1 we get $(H, H < a^3 >, H < a^3 >) = (H, H, H) \subseteq H$. This means by Proposition 1 that $H < a^3 > \subseteq N(H)$. Hence $H \neq N(H)$. This completes the proof of Theorem 2. □

Any subloop of a ZA-loop is a ZA-loop. From Lemma 3 it follows that a non-associative commutative Moufang ZA-loop has a non-trivial associative quotient loop. Hence it differs from its associator subloop. Hence *any commutative Moufang ZA-loop is a SD-loop.*

Corollary 1. *For a CML L let $Z_2(L) \neq Z_1(L)$. In particular, let L be a ZA-loop or a SD-loop. Then the CML L satisfies the normalizer condition.*

Proof. We suppose that $\mathfrak{F}(L) = L$. Then by Theorem 1 $L = L' \times Z(L)$, $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$. From here it follows that $Z_2(L) = Z_1(L)$. Contradiction. Hence $\mathfrak{F}(L) \neq L$ and by Theorem 2 the CML L satisfies the normalizer condition, as required. □

In [5] it is proved that CML L is centrally nilpotent of class n if and only if the group $\mathfrak{M}$ is nilpotent of class $2n-1$. Then Corollary 1 for $\mathfrak{M}$ follows from the known result about subnormal subgroups of nilpotent group (see, for example,[8]).

Proposition 4. *If L is a centrally nilpotent CML of class n, then for any subloop $H \subseteq L$ (respect. subgroup $\mathfrak{N}$ of group $\mathfrak{M}$) the sequence of consecutive normalizers reaches L (respect. $\mathfrak{M}$) not later that after n (respect. $2n-1$) steps.*

Proof. Let (1) be the upper central series of CML L. We denote $H_0 = H$, $H_{i+1} = N_L(H_i)$. It is sufficient to check that $Z_i(L) \subseteq H_i$. For $i = 0$, this is obvious. We suppose that $Z_i(L) \subseteq H_i$. From the relation $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L)$ it follows that $(Z_{i+1}(L), L, L) \subseteq Z_i(L)$. In particular, $(Z_i(L), Z_{i+1}(L), Z_{i+1}(L)) \subseteq Z_i(L)$. As $Z_i(L) \subseteq H_i$, then $(H_i, Z_{i+1}(L), Z_{i+1}(L)) \subseteq H_i$. But this means that $Z_{i+1}(L)$ normalizes H_i. Hence $Z_{i+1}(L) \subseteq H_{i+1}$. This completes the proof of Proposition 4. □

Remark. Theorem 1 (see, also, Theorem 3) reveals a strong analogy between the Frattini subloops of CML and the Frattini subgroups of the multiplication groups of CML. However for the multiplication group of CML the statement, analogous to Theorem 2, is not true. In [5] there is an example of CML G of exponent 3, such that $G' \neq G$ and $Z(G) = 1$. By Proposition 2 $\mathfrak{F}(G) \neq G$. Then by Proposition 1 $Z(\mathfrak{M}) = e$ and by Lemma 6 $\mathfrak{F}(M) \neq \mathfrak{M}$, where $\mathfrak{M}(G)$ denotes the multiplication group of G. In [6] J. D. H. Smith showed that no group with trivial centre and satisfying the normalizer condition can be the multiplication group of a quasigroup. Hence the multiplication group $\mathfrak{M}$ satisfies the inequality $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$ but it does not satisfy the normalizer condition.

4 Divisible subgroups of multiplication group

We remind ([8] (respect.[2]) that the group (respect. CML) G is called *divisible* or *complete* (by terminology of [3] *radically complete*) if the equality $x^n = a$ has at least one solution in G, for any number $n > 0$ and any element $a \in G$.

Theorem 3. *Any divisible subgroup $\mathfrak{N}$ of a multiplication group $\mathfrak{M}$ of a CML L is an abelian group and serves as a direct factor for $\mathfrak{M}$, i.e. $\mathfrak{M} = \mathfrak{N} \times \mathfrak{C}$ for a certain subgroup $\mathfrak{C}$ of $\mathfrak{M}$.*

Proof. If $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ then the statement follows from Theorem 1. Hence let $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Then $\mathfrak{M}$ has a maximal proper subgroups. The group $\mathfrak{M}$ is locally nilpotent [2], then the maximal proper subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$ [8]. Let $\mathfrak{H}$ be a maximal proper subgroup of $\mathfrak{M}$ such that $\varrho \notin \mathfrak{H}$ for some $\varrho \in \mathfrak{N}$. We will consider two cases: ϱ has a finite order and ϱ has an infinite order.

Let the element ϱ have a finite order n. Then the element $\alpha = \varrho^{n/p}$, where p is a prime divisor of n, has the order p. The subgroup $\mathfrak{N}$ is divisible. Then there exists a sequence $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ of elements in $\mathfrak{N}$ such that $\alpha_1^p = e$, $\alpha_{k+1}^p = \alpha_k$, where e is the unit of $\mathfrak{M}$. From here it follows that $\alpha_k^{p^k} = e$, $k = 1, 2, \ldots$.

We denote by $\mathfrak{C}$ the subgroup of $\mathfrak{N}$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$. It is easy to prove that any element $\alpha \in \mathfrak{C}$ is a power of some generator α_k, i.e. $\alpha = \alpha_k^n$, and the cyclic groups $< \alpha_k >$ form a sequence

$$e \subset < \alpha_1 > \subset < \alpha_2 > \subset \ldots \subset < \alpha_k > \subset \ldots .$$

We prove that $\mathfrak{C} \cap \mathfrak{H} = e$. $< \alpha_1 >$ is a cyclic group of order p and $\alpha_1 \notin \mathfrak{H}$. Then $< \alpha_1 > \cap \mathfrak{H} = e$. We suppose that $< \alpha_k > \cap \mathfrak{H} = e$. We have $< \alpha_{k+1} > = \{\alpha_{k+1}, \alpha_{k+1}^p, \ldots, \alpha_{k+1}^{p^{k+1}-1}\} \cup < \alpha_k >$. We suppose that $\alpha_{k+1}^n \in \mathfrak{H}$ ($n = 1, 2, \ldots, p^{k+1} - 1$). Then $(\alpha_{k+1}^n)^p \in \mathfrak{H}$. But $(\alpha_{k+1}^n)^p = (\alpha_{k+1}^p)^n = \alpha_k^n$. Hence $\alpha_k^n \in \mathfrak{H}$. But this contradicts the supposition $< \alpha_k > \cap \mathfrak{H} = e$. Hence $< \alpha_{k+1} > \cap \mathfrak{H} = e$ and, consequently, $\mathfrak{C} \cap \mathfrak{H} = e$. Let $\mathfrak{M}'$ denote the commutator subgroup of group $\mathfrak{M}$. By Lemma 4 $\mathfrak{M}' \subseteq \mathfrak{H}$. Then $\mathfrak{M}' \cap \mathfrak{C} = e$, $\mathfrak{C}' = e$, hence $\mathfrak{C}$ is an abelian group. More concretely, $\mathfrak{C}$ is isomorphic to a quasicyclic p-group. Further, the subgroup $\mathfrak{H}$ as maximal in $\mathfrak{M}$ is normal in $\mathfrak{M}$. Then from $\mathfrak{M} = \mathfrak{H}\mathfrak{C}$, $\mathfrak{H} \cap \mathfrak{C} = e$ it follows that $\mathfrak{C}$ is normal in $\mathfrak{M}$. Hence $\mathfrak{M} = \mathfrak{H} \times \mathfrak{C}$.

Let now $\varrho \in \mathfrak{N}$ be an element of infinite order. If $Z(\mathfrak{M})$ denotes the centre of $\mathfrak{M}$ then $\mathfrak{M}/Z(\mathfrak{M})$ is a locally finite 3-group [5]. Hence $\varrho^n \in Z(\mathfrak{M})$ for some n. Let $\mathfrak{H}$ be a maximal subloop of $\mathfrak{M}$ such that $\varrho^n \notin \mathfrak{H}$. $\mathfrak{N}$ is a divisible group. Then there exists a sequence $\varrho^n = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ of elements in $\mathfrak{N}$ such that $\alpha_{k+1}^{k+1} = \alpha_k$, $k = 1, 2, \ldots$. We denote by $\mathfrak{Q}$ the subgroup of $\mathfrak{N}$, generated by $\varrho^n = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$. As $\alpha_1 \in Z(\mathfrak{M})$ then it is easy to see that $\mathfrak{Q} \subseteq Z(\mathfrak{M})$. Hence the subgroup $\mathfrak{Q}$ is normal in $\mathfrak{M}$. The subgroup $\mathfrak{Q}$ is without torsion. In [4] it is proved that the commutator subgroup of the multiplication group of any CML is a locally finite 3-group. Then $\mathfrak{Q} \cap \mathfrak{M}' = e$. From here it follows that $\mathfrak{Q}' = e$, i.e. $\mathfrak{Q}$ is an abelian group. More concretely, $\mathfrak{Q}$ is isomorphic to the additive group of rationales.

Thus, in both cases in group $\mathfrak{M}$ there exists an abelian normal subgroup $\mathfrak{D} \subseteq \mathfrak{N}$, which is isomorphic to quasicyclic p-group or additive group of rationales such that $\mathfrak{M} = \mathfrak{H} \times \mathfrak{D}$. We will use this procedure of separating the divisible subgroup from $\mathfrak{N}$ as direct factor for defining the subgroups $\mathfrak{M}_\beta$, $\mathfrak{A}_\beta$ of group $\mathfrak{M}_{\beta-1}$.

Let $\mathfrak{M}_0 = \mathfrak{M}$, $\mathfrak{M}_1 = \mathfrak{H}$, $\mathfrak{D}_1 = \mathfrak{D}$. For a non-limit ordinal β inductively we define $\mathfrak{M}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta$. We denote $\mathfrak{A}_\beta = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \ldots \times \mathfrak{D}_\beta$. As $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_\beta \subseteq \mathfrak{N}$ then $\mathfrak{A}_\beta \subseteq \mathfrak{N}$. Further we consider the sequences of subgroups

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \ldots \subset \mathfrak{A}_\beta \subset \ldots,$$

$$\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots \supset \mathfrak{M}_\beta \supset \ldots, \beta < \alpha,$$

where $\mathfrak{M}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta$ if β is a non-limit ordinal and $\mathfrak{A}_\beta = \cup_{\gamma<\beta}\mathfrak{A}_\gamma$, $\mathfrak{M}_\beta = \cap_{\gamma<\beta}\mathfrak{M}_\gamma$ if β is a limit ordinal.

It is clear that $\mathfrak{M}_\beta$, $\mathfrak{A}_\beta$ are normal subgroups of $\mathfrak{M}$. We prove that $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$ for any β. If β is a non-limit ordinal, then by induction $\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{D}_1 = \mathfrak{M}_1 \times \mathfrak{A}_1 = \mathfrak{M}_{\beta-1} \times \mathfrak{A}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta \times \mathfrak{A}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$. Hence $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$.

Let now β be a limit ordinal and let $e \neq \lambda \in \mathfrak{M}_\beta \cap \mathfrak{A}_\beta$. Then there exists a non-ordinal $\delta < \beta$ such that $\lambda \in \mathfrak{A}_\delta$. From $\lambda \in \mathfrak{M}_\beta = \cap_{\gamma<\beta}\mathfrak{M}_\gamma$ it follows that $\lambda \in \mathfrak{M}_\gamma$ for all $\gamma < \beta$. But $\delta < \beta$. Then $\lambda \in \mathfrak{M}_\delta \cap \mathfrak{A}_\delta$. Contradiction. Hence $\mathfrak{M}_\beta \cap \mathfrak{A}_\beta = e$ and we may consider the direct product $\mathfrak{M}_\beta \times \mathfrak{A}_\beta$.

Let $\lambda \in \mathfrak{M}\backslash(\mathfrak{M}_\beta \times \mathfrak{A}_\beta)$. Then $\lambda \notin \mathfrak{M}_\beta$, $\lambda \notin \mathfrak{A}_\beta$, i.e. $\lambda \notin \cap_{\gamma<\beta}\mathfrak{M}_\gamma$, $\lambda \notin \cup_{\gamma<\beta}A_\gamma$. Hence $\lambda \notin \mathfrak{M}_\gamma$ for all $\gamma < \beta$ and from $\lambda \in \mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}_\gamma \times \mathfrak{A}_\gamma$ it follows that $\lambda \in \cap_{\gamma<\beta}\mathfrak{A}_\gamma = \mathfrak{A}_\beta$. We wet a contradiction. Hence $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$ for all β.

The process of inductive construction of $\mathfrak{A}_\alpha$ will end on the first number γ for which $\mathfrak{A}_\gamma = \mathfrak{N}$. Consequently, $\mathfrak{M} = \mathfrak{M}_\gamma \times \mathfrak{N}$. This completes the proof of Theorem 3. □

By Theorem 3 any multiplication group $\mathfrak{M}$ of CML contains a maximal divisible associative subloop $\mathfrak{D}$ and $\mathfrak{M} = \mathfrak{D} \times \mathfrak{R}$, where obviously $\mathfrak{R}$ is a *reduced CML*, meaning that it has no non-unitary divisible subgroups. Consequently, we obtain

Corollary 2. *Any multiplication group $\mathfrak{M}$ of CML L is a direct product of a divisible abelian subgroup $\mathfrak{D}$ and a reduced subgroup $\mathfrak{R}$. The subgroup $\mathfrak{D}$ is uniquely defined, the subgroup $\mathfrak{R}$ is defined up to isomorphism.*

Proof. Let us prove the last statement. As $\mathfrak{D}$ is the maximal divisible subgroup of the multiplication group $\mathfrak{M}$, then it is invariant with respect to the endomorphisms of the group $\mathfrak{M}$. Let $\mathfrak{M} = \mathfrak{D}' \times \mathfrak{R}'$, where $\mathfrak{D}'$ is a divisible subgroup, and $\mathfrak{R}'$ is a reduced subgroup of the group $\mathfrak{M}$. We denote by φ, ψ the endomorphisms $\varphi : \mathfrak{M} \to \mathfrak{D}', \psi : \mathfrak{M} \to \mathfrak{R}'$. As $\mathfrak{D}$ is invariant with respect to the endomorphisms of the group $\mathfrak{M}$, then $\varphi\mathfrak{D}$ and $\psi\mathfrak{D}$ are subgroups of the group $\mathfrak{M}$. It follows from the inclusions $\varphi\mathfrak{D} \subseteq \mathfrak{D}'$ and $\psi\mathfrak{D} \subseteq \mathfrak{R}'$ that $\varphi\mathfrak{D} \cap \psi\mathfrak{D} = 1$. By Theorem 3 $\mathfrak{D}$ is a abelian group, therefore $\varphi\mathfrak{D}, \psi\mathfrak{D}$ are normal in $\mathfrak{D}$. Then $d = \varphi d \cdot \psi d$ $(d \in \mathfrak{D})$ gives $\mathfrak{D} = \varphi\mathfrak{D} \cdot \psi\mathfrak{D}$, so $\mathfrak{D} = \varphi\mathfrak{D} \times \psi\mathfrak{D}$. Obviously, $\varphi\mathfrak{D} \subseteq \mathfrak{D} \cap \mathfrak{D}', \psi\mathfrak{D} \subseteq \mathfrak{D} \cap \mathfrak{R}'$, then $\varphi\mathfrak{D} = \mathfrak{D} \cap \mathfrak{D}', \psi\mathfrak{D} = \mathfrak{D} \cap \mathfrak{R}'$. Hence $\mathfrak{D} = (\mathfrak{D} \cap \mathfrak{D}') \times (\mathfrak{D} \cap \mathfrak{R}')$. But $\mathfrak{D} \cap \mathfrak{R}' = 1$ as a direct factor of a divisible group of a reduced group. Therefore, $\mathfrak{D} \cap \mathfrak{D}' \subseteq \mathfrak{D}, \mathfrak{D} \subseteq \mathfrak{D}'$, i.e. $\mathfrak{D} = \mathfrak{D}'$. This completes the proof of Corollary 2. □

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