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The Ricci-flat spaces related to the Navier-Stokes equations

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Abstract. Examples of the Riccii-flat metrics associated with the equations of Navier-Stokes are constructed. Their properties are investigated.

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1 Introduction

Properties of solutions of the Navier-Stokes equations to the incompressible fluid can be studied by geometric methods [1].

For this purpose we rewrite the NS- equations in equivalent form of conservation laws

$$\begin{aligned} &U_t + (U^2 - \mu U_x + P)_x + (UV - \mu U_y)_y + (UW - \mu U_z)_z = 0, \\ &V_t + (UV - \mu V_x)_x + (V^2 - \mu V_y + P)_y + (VW - \mu V_z)_x = 0, \\ &W_t + (UW - \mu W_x)_x + (VW - \mu W_y)_y + (W^2 - \mu W_z + P)_z = 0, \\ &U_x + V_y + W_z = 0, \end{aligned} \tag{1}$$

where U, V, W and P are components of the velocity and the pressure of the fluid. The relations (1) can be considered as conditions of the equality to zero the Ricci tensor of 14-dimensional space D^{14} in local coordinates $X = (x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n) = (\vec{x}, t, \eta, \rho, m, \Psi_l)$, $l = 1, ..., 7,$ endowed with the Riemann metric

$${}^{14}ds^2 = -2\Gamma^i_{jk}(\vec{x}, t)\Psi_i dx^j dx^k + 2d\Psi_l dx^l. \tag{2}$$

The metric (2) is the metric of the Riemann extension [2] of 7-dimensional space D^7 of affine connection in local coordinates $(x, y, z, t, \eta, \rho, m)$ and the components of connection $\Gamma^i_{jk}(\vec{x}, t)$.

In explicit form it looks as follows

$$\begin{aligned} {}^{14}ds^2 = 2\,dx\,du + 2\,dy\,dv + 2\,dz\,dw + (-V(\vec{x},t)v - W(\vec{x},t)w - U(\vec{x},t)u)\,dt^2 + \\ + \Big(-u\,(U(\vec{x},t))^2 + uP(\vec{x},t) + u\mu\,U_x(\vec{x},t) - vU(\vec{x},t)V(\vec{x},t) - U(\vec{x},t)p\Big)\,d\eta^2 + \\ + (v\mu\,U_y(\vec{x},t) - wU(\vec{x},t)W(\vec{x},t) + w\mu\,U_z(\vec{x},t))\,d\eta^2 + 2\,d^2\eta\,\xi + \end{aligned}$$

$$+(-uU(\vec{x},t)V(\vec{x},t)+vP(\vec{x},t)-V(\vec{x},t)p+u\mu\,V_x(\vec{x},t)-wV(\vec{x},t)W(\vec{x},t))\,d\rho^2+$$
$$+\left(v\mu\,V_y(\vec{x},t)+w\mu\,V_z(\vec{x},t)-v\,(V(\vec{x},t))^2\right)d\rho^2+2\,d^2\rho\,\chi+$$
$$+\left(wP(\vec{x},t)-W(\vec{x},t)p-w\,(W(\vec{x},t))^2+w\mu\,W_z(\vec{x},t)+v\mu\,W_y(\vec{x},t)\right)dm^2+$$
$$+(-vV(\vec{x},t)W(\vec{x},t)+u\mu\,W_x(\vec{x},t)-uU(\vec{x},t)W(\vec{x},t))\,dm^2+2\,dm\,dn+2\,dt\,dp. \quad (3)$$

The main property of the space withe the metric (3) is that it is Ricci-flat if the functions U, V, W and P satisfy the NS-equations (1).

Despite the fact that all scalar invariants of the space D^{14} are equal to zero its geometric properties can be studied with the help of equations of geodesics and corresponding invariant differential operators.

2 Geodesics

The full system of geodesics of the metrics (2) consists of two parts

$$\ddot{x}^k+\Gamma^k_{ij}\dot{x}^i\dot{x}^j=0, \quad \frac{\delta^2\Psi_k}{ds^2}+R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l=0,$$

where $\dfrac{\delta\Psi_k}{ds}=\dot{\Psi}_k-\Gamma^l_{jk}\Psi_l\dot{x}^j$.

In the considered case the first group of equations is

$$\ddot{x}+1/2\,U(\vec{x},t)\dot{t}^2+1/2\,\dot{\eta}U(\vec{x},t)^2-1/2\,\dot{\eta}^2\mu\,U_x(\vec{x},t)-1/2\,\dot{\eta}^2P(\vec{x},t)+$$
$$+1/2\,\dot{\rho}^2U(\vec{x},t)V(\vec{x},t)-1/2\,\dot{\rho}^2\mu\,V_x(\vec{x},t)+1/2\,\dot{m}^2U(\vec{x},t)W(\vec{x},t)-1/2\,\dot{m}(s)\mu\,W_x(\vec{x},t)=0,$$
$$\ddot{y}+1/2\,V(\vec{x},t)\dot{t}^2+1/2\,\dot{\eta}(s)^2U(\vec{x},t)V(\vec{x},t)-1/2\,\dot{\eta}^2\mu\,U_y(\vec{x},t)+1/2\,\dot{\rho}^2V(\vec{x},t)^2-$$
$$-1/2\,\dot{\rho}^2\mu\,V_y(\vec{x},t)-1/2\,\dot{\rho}(s)P(\vec{x},t)+1/2\,\dot{m}^2V(\vec{x},t)W(\vec{x},t)-1/2\,\dot{m}^2\mu\,W_y(\vec{x},t)=0,$$
$$\ddot{z}+1/2\,W(\vec{x},t)\dot{t}^2+1/2\,\dot{\eta}(s)^2U(\vec{x},t)W(\vec{x},t)-1/2\,\dot{\eta}^2\mu\,U_z(\vec{x},t)+1/2\,\dot{\rho}^2V(\vec{x},t)W(\vec{x},t)-$$
$$-1/2\,\dot{\rho}^2\mu\,V_z(\vec{x},t)+1/2\,\dot{m}^2W(\vec{x},t)^2-1/2\,\dot{m}^2\mu\,W_z(\vec{x},t)-1/2\,\dot{m}^2P(\vec{x},t)=0,$$
$$\ddot{t}+1/2\,U(x,y,z,t)\dot{\eta}^2+1/2\,V(x,y,z,t)\dot{\rho}^2+1/2\,W(x,y,z,t)\dot{m}^2=0,$$
$$\ddot{\eta}(s)=0, \quad \ddot{\rho}(s)=0, \quad \ddot{m}(s)=0.$$

In the particular case of $2D$-potential flow $U=\phi_y$, $V=-\phi_x$, $W=0$, $P=Q(x,y,t)$ this system takes the form

$$2\,\ddot{x}+\phi_y\dot{t}^2+\alpha 1^2\phi_y^2-{\alpha_1}^2\mu\,\phi_{xy}-{\alpha_1}^2Q-{\alpha_2}^2\phi_y\phi_x+{\alpha_2}^2\mu\,\phi_{xx}=0,$$
$$2\,\ddot{y}-\phi_x\dot{t}^2-{\alpha_1}^2\phi_y\phi_x-{\alpha_1}^2\mu\,\phi_{yy}+{\alpha_2}^2\phi_x^2+{\alpha_2}^2\mu\,\phi_{xy}-{\alpha_2}^2Q=0,$$
$$2\,\ddot{z}-Q{\alpha_3}^2=0, \quad 2\,\ddot{t}+\phi_y{\alpha_1}^2-\phi_x{\alpha_2}^2=0,$$
$$\eta(s)=\alpha_1 s, \;\; \rho(s)=\alpha_2 s, \;\; m(s)=\alpha_3 s.$$

Remark. The coefficients of the geodesics are the components Γ^k_{ij} of affine connection of the 7-dimensional manifold in the local coordinates $(\vec{x}, t, \eta, \rho, m)$.

It is Ricci-flat

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{im}\Gamma^m_{kj} = 0$$

on solutions of the NS-equations (1) and its properties can be studied independently of the enclosing 14-dimensional Riemann space with the metric (3).

The linear part of geodesic has the form of linear system of equations with variable coefficients

$$\ddot{\Psi}_i = A^k_i \dot{\Psi}_k + B^k_i \Psi_k,$$

where $\Psi_k = [u, v, w, p, \xi, \chi, n]$ are vector-functions, $A^k_i = A^k_i(\vec{x}, t)$ and $B^k_i = B^k_i(\vec{x}, t)$ are matrix-functions depending on coordinates $X^a = (\vec{x}, t)$.

3 Parameters of Beltrami

The solutions of the equations

$$\Delta_2 \psi = g^{ij}\left(\frac{\partial^2 \psi}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial \psi}{\partial x^k}\right) = 0, \;\; \Delta_1 \psi = g^{ij}\frac{\partial \psi}{\partial x^i}\frac{\partial \psi}{\partial x^j} = 1 \tag{4}$$

play important role in geometry of Riemann spaces and can be used to study properties of solutions of the NS-equations.

As example in the case of two dimensional potential flow of fluid the metric of associated Riemann space has the form

$$\begin{gathered} ds^2 = 2\,dx\,du + 2\,dy\,dv + 2\,dz\,dw + (-\phi_y u + \phi_x v)\,dt^2 + 2\,dt\,dp + \\ + \left(-u\phi_y^2 + u\mu\,\phi_{xy} + uQ(x,y,t) + v\phi_y\phi_x + v\mu\,\phi_{yy} - \phi_y p\right) d\eta^2 + \\ +2\,d\eta\,d\xi + \left(u\phi_y\phi_x - u\mu\,\phi_{xx} - v\phi_x^2 - v\mu\,\phi_{xy} + vQ(x,y,t) + \phi_x p\right) d\rho^2 + \\ +2\,d\rho\,d\chi + wQ(x,y,t)dm^2 + 2\,dm\,dn. \end{gathered} \tag{5}$$

Components of the Ricci-tensor to the metric (5) are equal to zero $R_{\eta\eta} = 0$, $R_{\rho\rho} = 0$ on solutions of $2D$ NS-equations

$$\phi_y\phi_{xy} - \mu\,\phi_{xxy} - Q_x - \phi_{yy}\phi_x - \mu\,\phi_{yyy} + \phi_{yt} = 0,$$

$$-\phi_y\phi_{xx} + \mu\,\phi_{xxx} + \mu\,\phi_{xyy} - Q_y - \phi_{xt} + \phi_{xy}\phi_x = 0,$$

where the function $\phi(x, y, t)$ satisfies the condition of compatibility

$$(\phi_{xx} + \phi_{yy})_t + \phi_y(\phi_{xx} + \phi_{yy})_x - \phi_x(\phi_{xx} + \phi_{yy})_y - \mu\Delta(\phi_{xx} + \phi_{yy}) = 0. \tag{6}$$

As is known [3] with the help of solutions equation (4) can be studied geodesics of the metric to the arbitrary Riemann space.

To the metric (5) the second equation (4) takes the form

$$2\,\psi_x\psi_u+2\,\psi_y\psi_v+2\,\psi_z\psi_w+2\,\psi_t\psi_p+2\,\psi_\eta\psi_\xi+2\,\psi_\rho\psi_\chi+2\,\psi_m\psi_n+\psi_p^2\phi_y u-$$
$$-\psi_p^2\phi_x v+\psi_\xi^2 u\phi_y^2-\psi_\xi^2 u\mu\,\phi_{xy}-\psi_\xi^2 uQ-\psi_\xi^2 v\phi_y\phi_x-\psi_\xi^2 v\mu\,\phi_{yy}+\psi_\xi^2\phi_y p-$$
$$-\psi_\chi^2 u\phi_y\phi_x+\psi_\chi^2 u\mu\,\phi_{xx}+\psi_\chi^2 v\phi_x^2+\psi_\chi^2 v\mu\,\phi_{xy}-\psi_\chi^2 vQ-\psi_\chi^2\phi_x p-wQ\psi_n^2-1=0. \quad (7)$$

After the separation of variables in the equation (7) we find

$$\psi(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)=c_3z+c_5\eta+c_6\rho+c_7m+c_{12}\xi+$$
$$+c_{13}\chi+c_{14}n+F(x,y,t,u,v,w,p)$$

where the function $F(x,y,t,u,v,w,p)$ satisfies the equation

$$2\,F_xF_u+2\,F_yF_v+2\,c_3F_w+2\,F_tF_p+2\,c_5\,c_{12}+2\,c_6\,c_{13}+2\,c_7\,c_{14}+$$
$$+F_p^2\phi_y u-F_p^2\phi_x v+{c_{12}}^2u\phi_y^2-{c_{12}}^2u\mu\,\phi_{xy}-{c_{12}}^2uQ-{c_{12}}^2v\phi_y\phi_x-$$
$$-{c_{12}}^2v\mu\,\phi_{yy}+{c_{12}}^2\phi_y p-{c_{13}}^2u\phi_y\phi_x+{c_{13}}^2u\mu\,\phi_{xx}+{c_{13}}^2v\phi_x^2+$$
$$+{c_{13}}^2v\mu\,\phi_{xy}-{c_{13}}^2vQ-{c_{13}}^2\phi_x p-wQ\,{c_{14}}^2-1=0. \quad (8)$$

In particular case

$$F(x,y,t,u,v,w,p)=A(x,y,t)p+uB(x,y,t),\,c_{14}=0,\,c_{13}=0,\,c_{12}=1/2\,{c_5}^{-1}$$

from the equation (8) we get three equations on two functions $A(x,y,t)$, $B(x,y,t)$ having solutions if the functions $\phi(x,y,t)$, $Q(x,y,t)$ satisfy the relation

$$H(\phi,\phi_x,\phi_y,\phi_t,...Q)=0. \quad (9)$$

As example in the case of the Euler system of equations, ($\mu=0$) the relation (9) takes the form

$$\phi_{yt}\phi_{xy}^2\phi_y-4\,\phi_{yt}\phi_y\phi_{xy}\phi_{xyt}+2\,\phi_y\phi_{xxy}\phi_{yt}^2-4\,\phi_{yt}\phi_y^2\phi_{xxy}-$$
$$-3\,\phi_{xy}^2\phi_y^2+4\,\phi_y^2\phi_{xy}\phi_{xyt}+2\,\phi_y^3\phi_{xxy}+2\,\phi_y\phi_{xy}^2\phi_{ytt}-Q\phi_{xy}^3=0.$$

References

[1] Dryuma V. *On spaces related to the Navier-Stokes equation.* Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2010, No. 3(64), 107–110.

[2] Dryuma V. *The Riemann extensions in theory of differential equations and their applications.* Matematicheskaya fizika, analiz, geometriya, 2003 **10**, 307–325.

[3] Boccaletti D., Catoni F., Cannata R., Zampetti P. Integrating the geodesic equations in the Schwarzschild and Kerr space-times using Beltrami geometrical method. *arXiv:gr-qc/0502051 v2 3 Apr*, **11** (2008) 1–12.

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