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The Generalized Lagrangian Mechanical Systems

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Dedicated to Acad. Mitrofan Choban at 70-th Anniversary

Abstract. A generalized Lagrangian mechanics is a triple $\Sigma_{GL} = (M, \mathcal{E}, F_e)$ formed by a real n-dimensional manifold M, the generalized kinetic energy $\mathcal{E}$ and the external forces F_e. The Lagrange equations (or fundamental equations) can be defined for a generalized Lagrangian mechanical system Σ_{GL}. We get a straightforward extension of the notions of Riemannian, or Finslerian, or Lagrangian mechanical systems studied in the recent book [7]. The applications of this systems in Mechanics, Physical Fields or Relativistic Optics are pointed out. Much more information can be found in the books or papers from References [1–10].

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1 Introduction

A generalized Lagrangian mechanical system

$$\Sigma_{GL} = (M, \mathcal{E}, F_e)$$

has the absolute energy

$$\mathcal{E}(x, y) = g_{ij}(x, y)y^i y^j, \quad (y^i = \frac{dx^i}{dt}),$$

or the kinetic energy

$$T(x, y) = \frac{1}{2} g_{ij}(x, y)y^i y^j,$$

where $g_i j(x, y)$ is a d-tensor field on TM, covariant of order two, symmetric, nonsingular and of the constant signature.

The function $F_e(x, y)$ is the external force a priori given as a vertical vector field on TM:

$$F_e(x, y) = F^i(x, y)\frac{\partial}{\partial y^i}.$$

The Lagrange equations of the system Σ_{GL} are:

$$E_i(\mathcal{E}) = F_i(x, y),\ F_i(x, y) = g_{ij}(x, y)F^j(x, y),$$

where

$$E_i(\mathcal{E}) = \frac{d}{dt}\frac{\partial \mathcal{E}}{\partial y^i} - \frac{\partial \mathcal{E}}{\partial x^i}, \quad y^i = \frac{dx^i}{dt}.$$

Thus, M is called the configuration space, TM is the velocity space, $g_{ij}(x, y)$ is the fundamental tensor and the pair $GL^n = (M, g_{ij}(x, y))$ is a generalized Lagrange space.

In some conditions we can associate to the mechanical system Σ_{GL} a canonical semi-spray $\breve{S}$ depending on the system Σ_{GL} only. The semi-spray $\breve{S}$ is a dynamical system on the velocity space TM, having the integral curves given by the evolution curves $E_i(\mathcal{E}) = F_i$.

Consequently, the pair $(TM, \breve{S})$ determines the geometrical theory of the mechanical system Σ_{GL}. The Riemannian, Finslerian and Lagrangian mechanical systems are pointed out as the particular cases of the mechanical systems Σ_{GL}. The fundamental geometric object field on the velocity spaces TM of the systems Σ_{GL} and its applications are studied too.

The geometry of Lagrange and Hamilton spaces was studied in [6, 7, 9]. The works [1–6, 8, 10] contains distinct applications of the theory of Lagrange spaces. We use the terminology from [7].

2 Generalized Lagrange space

The notion of generalized Lagrange space GL^n can be introduced by the following definition:

Definition 2.1. *A generalized Lagrange space (or a GL^n-space) is a pair $GL^n = (M, g(x, y))$, where M is a real n-dimensional manifold and $g_{ij}(x, y)$ is a d-tensor on the tangent manifold TM, covariant of order 2, symmetric, non-singular and of constant signature. This tensor is called fundamental for the GL^n-space.*

We mentioned that a Lagrange space $L^n = (M, L(x, y))$ is a generalized Lagrange space with the fundamental tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L(x, y))}{\partial y^i \partial y^j}, \quad (x, y) \in \pi^{-1}(U) \in TM. \tag{2.1}$$

But not any GL^n-space is an L^n-space. Indeed, if GL^n-space is given by its fundamental tensor $g_{ij}(x, y)$, it may happen that the system of the partial differential equations (2.1) does not admit solutions with respect to $L(x, y)$.

In this context the following assertions are important which contain large necessary and sufficient conditions of the existence of solutions for the partial differential equations (2.1):

Proposition 2.1. 1^0. *If the system* (2.1) *admits a solution $L(x, y)$, then the d-tensor field $\frac{\partial g_{ij}}{\partial y^k}$ is completely symmetric.*

2^0. *If the condition 1^0 is verified, i.e. the d-tensor field $\frac{\partial g_{ij}}{\partial y^k}$ is completely symmetric, and the functions $g_{ij}(x, y)$ are 0-homogeneous with respect to y^i, then the* *function*

$$L(x,y) = g_{ij}(x,y)y^i y^j + A_i(x)y^i + U(x) \tag{2.2}$$

is a solution of the system of equations (2.1) *for any arbitrary d-co-vector field* $A_i(x)$ *and any arbitrary function* $U(x)$ *on the manifold* M.

The proof of Proposition 2.1 is not complicated. Of course, the Lagrange space $L^n = (M, L(x,y))$ with the fundamental function (2.2) gives us an important class of Lagrange spaces which describe the gravitational and electromagnetic phenomena.

Example 2.1. The pair $GL^n = (M, g_{ij})$ with $g_{ij}(x,y) = e^{2\sigma}(x,y)\gamma_{ij}(x)$, where $\gamma_{ij}(x)$ is a semi-defined Riemannian tensor on the base manifold M and $\sigma(x,y)$ is a function on TM with the property $\frac{\partial\sigma}{\partial y^i}$ is a non-vanishing co-vector field, is a generalized semi-defined space.

Example 2.2. The pair $GL^n = (M, g_{ij}(x,y))$ with

$$g_{ij} = \gamma_{ij}(x) + (1 - \frac{1}{n^2(x,y)})y_i y_j, \quad y_i = g_{ij}y^j,$$

where $y_{ij}(x)$ is a semi-defined Riemannian tensor on the base manifold M and $n(x,y) > 1$ is a smooth function (n is a refractive index), gives us a generalized semi-defined Lagrange space. This metric is caled by R.G Bell (see [5, 6, 8]) the Miron's metric of Relativistic Optics.

Example 2.3. Any semi-defined Finsler space $F^n = (M, F(x,y))$ is a generalized semi-defined Lagrange space.

For a generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$ to determine the non-linear connection, obtained from the fundamental tensor $g_ij(x,y)$, is an important problem. In the previous three examples this is possible but, in general, it is not possible. In the book [9] we pointed out a method for determining a non-linear connection in the weakly regular spaces GL^n, where the absolute energy

$$\mathcal{E}(x,y) = g_{ij}(x,y)y^i y^j \tag{2.3}$$

of the GL^n-space is a regular Lagrangian.

We end the section with the following remarks:

Remarks: 2.1. The following sequence of inclusions holds:

$$\{R^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}.$$

2.2. The Lagrange geometry is the geometrical theory of the spaces included in the above sequence of inclusions.

3 The mechanical systems Σ_{GL}

In this section the generalized Lagrangian mechanical systems are defined and studied.

Definitions 3.1. *A generalized Lagrangian mechanical system is a triple*

$$\Sigma_{GL} = (M, \frac{1}{2}\mathcal{E}(x,y), F_e(x,y)),$$

where:

a) M is an n-dimensional real manifold (configuration space);

b) $\frac{1}{2}\mathcal{E}(x,y) = T(x,y)$ is the kinetic energy, $\mathcal{E}(x,y) = g_{ij}(x,y)y^i y^j$ being the absolute energy of a generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$;

c) $F_e = F^i(x,y)\frac{\partial}{\partial y^i}$ is a given vertical vector field on TM (called the external forces).

Of course, this definition has a geometrical meaning. The examples of GL^n-spaces, expressed in Section 2, give us very good examples of generalized Lagrangian mechanical systems if we fix the components $F^i(x,y)$ of the external forces. For instance, we can consider the external forces of the Liouville type: $F^i = a(x,y)y^i$, where $a(x,y)$ is a scalar function.

The covariant components of the external forces F_e are as follows:

$$F_i(x,y) = g_{ij}(x,y)F^j(x,y). \tag{3.1}$$

In the following, one may consider only the generalized Lagrangian mechanical systems for which we can determine a non-linear connection N by means of the fundamental tensor $g_{ij}(x,y)$ of Σ_{GL}. For instance, in the case when $g_{ij}(x,y)$ is weakly regular. Thus the absolute energy $\mathcal{E}(x,y)$ is a regular Lagrangian. So, the kinetic energy $T(x,y) = \frac{1}{2}\mathcal{E}(x,y)$ is a regular Lagrangian. This property holds for the Riemanian mechanical systems Σ_R, for the Finslerian mechanical systems Σ_F and for the Lagrangian mechanical systems Σ_L.

Therefore, further we can assume that the generalized Lagrangian mechanical systems $\Sigma_{GL} = (M, T(x,y), F_e(x,y))$ have a weakly regular fundamental tensor $g_{ij}(x,y)$, i.e. $T(x,y) = \frac{1}{2}g_{ij}(x,y)y^i y^j$ is a regular Lagrangian.

Consequently, we have:

Proposition 3.1. *The following entries are the components of a co-vector fields*

$$E_i(\mathcal{E}) = \frac{d}{dt}\frac{\partial \mathcal{E}}{\partial y^i} - \frac{\partial \mathcal{E}}{\partial x^i}, \quad y^i = \frac{dx^i}{dt}.$$

Thus, we can give now the following Postulate.

Postulate. *The evolution equations (or the Lagrange equations) of a generalized Lagrangian mechanical system $\Sigma_G L$ are:*

$$E_i(\mathcal{E}) = F_i(x,y), \quad y^i = (dx^i)/dt. \tag{3.2}$$

Of course, the Lagrange equations can be written as follows

$$\frac{d}{dt}\frac{\partial T(x,y)}{\partial y^i} - \frac{\partial T(x,y)}{\partial x^i} = \frac{1}{2}F_i(x,y), \quad y^i = \frac{dx^i}{dt}. \tag{3.2$'$}$$

But (3.2) or (3.2)$'$ has a geometrical meaning. It is convenient to write the Lagrange equations in the following form:

$$\begin{aligned} &\frac{d^2x^i}{dt^2} + 2\breve{G}^i(x, \frac{dx}{dt}) = \frac{1}{2}F^i(x, \frac{dx}{dt}), \\ &2\breve{G}^i(x,y) = \frac{1}{2}g^{is}(\frac{\partial^2 \mathcal{E}}{\partial y^i \partial x^j}y^j - \frac{\partial \mathcal{E}}{\partial x^i}). \end{aligned} \tag{3.3}$$

Therefore, we can apply this method in the case of the mechanical systems Σ_R, Σ_F, Σ_L.

One gets an important result:

Theorem 3.1. *We have:*

1^0. *The operator*

$$\breve{S} = y^i\frac{\partial}{\partial x^i} - 2(\breve{G}^i(x,y) - \frac{1}{2}F^i(x,y)\frac{\partial}{\partial y^i}) \tag{3.4}$$

is a semi-spray on the velocity space $TM = TM\backslash\{\emptyset\}$.

2^0. *The integral curves of the vector field* $\breve{S}$ *are the evolution curves* (3.3) *of the mechanical system* Σ_{GL}.

3^0. $\breve{S}$ *is a dynamical system, determined only by the mechanical system* $\Sigma_G L$.

The proof of this fundamental theorem can be found in the book [7].

Now, we can develop theory of the mechanical systems $\Sigma_{GL} = (M, \mathcal{E}, F_e)$ only by means of the mechanical entries $g_{ij}(x,y)$, $\mathcal{E}(x,y)$, $F_e(x,y)$, $S(x,y)$.

Let us consider the energy of the systems Σ_{GL}:

$$E_{\mathcal{E}} = y^i\frac{\partial \mathcal{E}}{\partial y^i} - \mathcal{E}. \tag{3.5}$$

We can prove:

Theorem 2.2. *The variation of energy* $E_{\mathcal{E}}$ *along the evolution curves (3.3) is given by:*

$$\frac{dE_{\mathcal{E}}}{dt} = \frac{dx^i}{dt}F^i(x, \frac{dx}{dt}). \tag{3.6}$$

The external force $F_e(x,y)$ is called dissipative if

$$g_{ij}F^iy^j \leq 0. \tag{3.7}$$

From the previous theorem, follows

Theorem 3.3. *The energy* $E_{\mathcal{E}}$ *of the mechanical system* Σ_{GL} *is decreasing on the evolution curves (3.3) if and only if the external forces* F_e *are dissipative.*

Moreover, for the mechanical systems Σ_{GL} *the following assertions are true:*
1^0. *The canonical non-linear connection* $\breve{N}$ *has the local components:*

$$\breve{N}^i_j = \frac{\partial \breve{G}^i}{\partial y^j} - \frac{1}{4}\frac{\partial F^i}{\partial y^j}. \tag{3.8}$$

2^0. *The canonical metrical connection* $\breve{N}$ *of the mechanical system* Σ_{GL} *are the generalized Christoffel equations:*

$$\begin{aligned} \breve{L}^i_{jk} &= \frac{1}{2} g^{is}\Big(\frac{\breve{\delta} g_{js}}{\delta x^k} + \frac{\breve{\delta} g_{sk}}{\delta x^j} - \frac{\breve{\delta} g_{jk}}{\delta x^s}\Big), \\ \breve{C}^i_{jk} &= \frac{1}{2} g^{is}\Big(\frac{\partial g_{j}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{sk}}{\partial y^s}\Big), \end{aligned} \tag{3.9}$$

where $\frac{\breve{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i}$ - $\frac{\partial \breve{G}^s}{\partial y^i}\frac{\partial}{\partial y^s}$.

Thus $D\Gamma(\breve{N}) = (\breve{L}_{jk}, \breve{C}_{jk})$ is the canonical $\breve{N}$-linear connection of Σ_{GL} with respect to the fundamental tensor $g_{ij}(x, y)$.

Conclusion. By using the geometrical object fields $\breve{S}$, $\breve{N}$, $D\Gamma(\breve{N})$ we can study the theory of Generalized Lagrangian Mechanical Systems Σ_{GL}.

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