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On partial inverse operations in the lattice of submodules

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Abstract. In the present work two partial operations in the lattice of submodules $\boldsymbol{L}({}_RM)$ are defined and investigated. They are the inverse operations for ω-product and α-coproduct studied in [6]. This is the continuation of the article [7], in which the similar questions for the operations of α-product and ω-coproduct are investigated.

The partial inverse operation of *left quotient* $N /_{\odot} K$ of N by K with respect to ω-product is introduced and similarly the *right quotient* $N \backslash K$ of K by N with respect to α-coproduct is defined, where $N, K \in \boldsymbol{L}({}_RM)$. The criteria of existence of such quotients are indicated, as well as the different forms of representation, the main properties, the relations with lattice operations in $\boldsymbol{L}({}_RM)$, the conditions of cancellation and other related questions are elucidated.

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1 Introduction and preliminaries

The purpose of this work is the investigation of two partial inverse operations in the lattice of submodules $\boldsymbol{L}({}_RM)$ of an arbitrary left R-module ${}_RM$.

Using the preradicals of standard types α_N^M and ω_N^M, in the article [6] four operations in $\boldsymbol{L}({}_RM)$ were introduced and studied: α-product, ω-product, α-coproduct and ω-coproduct (see also [5]). For two of these operations (for α-product and ω-coproduct) in the work [7] the inverse operations were introduced: left quotient with respect to α-product and right quotient with respect to ω-coproduct. These quotients exist for every pair of submodules of ${}_RM$.

In the present article in a similar manner the inverse operations are defined for the other two operations (for ω-product and α-coproduct). In contrast to the preceding cases these operations are *partial*, since they exist only in some special cases. We will show the criteria of existence of these quotients, their main properties, relations with the lattice operations of $\boldsymbol{L}({}_RM)$ and other results.

Let R-Mod be the category of unitary left R-modules, where R is an associative ring with unity. For an arbitrary left R-module ${}_RM$ we denote by $\boldsymbol{L}({}_RM)$ the lattice of submodules of ${}_RM$ and by $\boldsymbol{L}^{ch}({}_RM)$ the lattice of *characteristic* (fully invariant) submodules of ${}_RM$ (i.e. of submodules $N \subseteq M$ with $f(N) \subseteq N$ for every R-endomorphism $f\colon {}_RM \to {}_RM$).

A *preradical* r of the category R-Mod is by definition a subfunctor of identity functor of this category $\big($i.e. $r(M) \subseteq M$ for every $M \in R$-Mod and $f\,(r(M)) \subseteq M'$ for every R-morphism $f\colon {}_RM \to {}_RM'\big)$. The family of all preradicals of R-Mod will be denoted by R-pr, and it is a "big" lattice with respect to the operations $\wedge$ and $\vee$, where the preradicals $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha$ and $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha$ for every $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq R$-pr are defined by the rules:

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(X), \qquad \big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(X),$$

for every left R-module ${}_RX$ [1,3,4].

An important role in the theory of preradicals is played by the following two operations in R-pr:

1) the *product* $r \cdot s$ of preradicals $r, s \in R$-pr

$$(r \cdot s)(X) = r\big(s(X)\big), \quad X \in R\text{-Mod};$$

2) the *coproduct* $r : s$ of preradicals $r, s \in R$-pr

$$[(r : s)(X)]\,/\,r(X) = s\,\big(X\,/\,r(X)\big)\,, \quad X \in R\text{-Mod}.$$

In the study of various questions on preradicals the following two types of preradicals turn out to be very useful. Every fixed pair $N \subseteq M$, where $N \in \boldsymbol{L}({}_RM)$, defines the preradicals α_N^M and ω_N^M in R-pr by the rules:

$$\alpha_N^M(X) = \sum_{f\colon M \to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f\colon X \to M} f^{-1}(N),$$

for every $X \in R$-Mod [1,4]. We will call α_N^M and ω_N^M the *standard* preradicals, associated to the pair $N \subseteq M$.

Two special cases are important:

a) the *idempotent preradical* r^M defined by the module ${}_RM$:

$$r^M(X) = \sum_{f\colon M \to X} Im\, f \text{ -- the } \textit{trace} \text{ of } M \text{ in } X \text{ (i.e. } r^M = \alpha_M^M);$$

b) the *radical* r_M defined by ${}_RM$:

$$r_M(X) = \bigcap_{f\colon X \to M} Ker\, f \text{ -- the } \textit{reject} \text{ of } M \text{ in } X \text{ (i.e. } r_M = \omega_0^M).$$

From the definition of preradical it is obvious that for every $r \in R$-pr and every $M \in R$-Mod we have $r(M) \in \boldsymbol{L}^{ch}({}_RM)$. Moreover, the submodule $N \subseteq M$ is characteristic in M if and only if there exists a preradical $r \in R$-pr such that $N = r(M)$. An important property of standard preradicals is that for every $N \in \boldsymbol{L}^{ch}({}_RM)$ we have

$$r(M) = N \;\;\Leftrightarrow\;\; \alpha_N^M \;\leq\; r \;\leq\; \omega_N^M,$$

i.e. α_N^M is the least preradical of R-pr with $r(M) = N$, and ω_N^M is the greatest preradical with this property.

Using the standard preradicals four operations in $\boldsymbol{L}({}_RM)$ can be defined by the following rules [2,5,6]:

1) *α-product* of $K, N \in \boldsymbol{L}({}_RM)$:
$$K \cdot N = \alpha_K^M(N) = \sum_{f:\ M \to N} f(K);$$

2) *ω-product* of $K, N \in \boldsymbol{L}({}_RM)$:
$$K \odot N = \omega_K^M(N) = \bigcap_{f:\ N \to M} f^{-1}(K);$$

3) *α-coproduct* of $N, K \in \boldsymbol{L}({}_RM)$:
$$(N : K) \,/\, N = \alpha_K^M(M/\, N) = \sum_{f:\ M \to M/N} f(K);$$

4) *ω-coproduct* of $N, K \in \boldsymbol{L}({}_RM)$:
$$(N \odot K) \,/\, N = \omega_K^M(M/\, N) = \bigcap_{f:\ M/N \to M} f^{-1}(K).$$

Some properties of these operations are indicated in the works [2, 5, 6], as well as their relations with the lattice operations of $\boldsymbol{L}({}_RM)$. Moreover, in the article [7] the inverse operations for α-product and ω-coproduct are defined and investigated. We remind the basic definitions.

Let $K, N \in \boldsymbol{L}({}_RM)$. The *left quotient* of N by K with respect to α-product (denoted by $N \,/_{\!\centerdot}\, K$) is the greatest among submodules $L \in \boldsymbol{L}({}_RM)$ with the condition $L \cdot K \subseteq N$. Using the properties of α-product the left quotient $N \,/_{\!\centerdot}\, K$ can be represented in the form:
$$N \,/_{\!\centerdot}\, K = \sum_{\alpha \in \mathfrak{A}} \{L_\alpha \in \boldsymbol{L}({}_RM) \mid L_\alpha \cdot K \subseteq N\}.$$

Dually, for $K, N \in \boldsymbol{L}({}_RM)$ the *right quotient* of K by N with respect to ω-coproduct is defined as the least submodule $L \in \boldsymbol{L}({}_RM)$ with the property $N \odot L \supseteq K$. It is denoted by $N {}_{\odot}\!\backslash\, K$ and can be represented as
$$N {}_{\odot}\!\backslash\, K = \bigcap_{\alpha \in \mathfrak{A}} \{L_\alpha \in \boldsymbol{L}({}_RM) \mid N \odot L_\alpha \supseteq K\}.$$

In the work [7] some "arithmetic" of these operations is developed, indicating the basic properties and relations (in particular, their form in the case ${}_RM = {}_RR$).

In this work our task is to try to develop the similar theory for the other two operations: for ω-product and α-coproduct. These cases have some distinguishing features, which show the necessity to study them separately.

2 The left quotient with respect to ω-product

By definition, for $K, N \in \boldsymbol{L}({}_RM)$ the *ω-product* of K and N is the following submodule of M:
$$K \odot N = \omega_K^M(N) = \bigcap_{f:\ N \to M} f^{-1}(N) \quad \text{(see Section 1)}.$$

Now we remind some properties of this operation [6].

Proposition 2.1. 1) $K \odot N \subseteq K \bigcap N$ *(so* $0 \odot N = 0$ *and* $K \odot 0 = 0$*).*

2) $K \odot M = \omega_K^M(M) = \bigcap_{f:\ M \to M} f^{-1}(K)$ *is the largest characteristic submodule of* M *which is contained in* K*; therefore, if* $K \in \boldsymbol{L}^{ch}({}_RM)$*, then* $K \odot M = K$*.*

3) $M \odot N = \omega_M^M(N) = \bigcap_{f:\ N \to M} f^{-1}(M) = N$ *for every* $N \in \boldsymbol{L}({}_RM)$*.*

4) *The operation of* ω*-product is monotone in both variables.*

5) $(K \odot N) \odot L \supseteq K \odot (N \odot L)$ *for every* $K, N, L \in \boldsymbol{L}({}_RM)$*.*

6) $\big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\big) \odot N = \bigcap_{\alpha \in \mathfrak{A}} (K_\alpha \odot N)$ *for every* $K_\alpha, N \in \boldsymbol{L}({}_RM)$*.*

7) *If* ${}_RM = {}_RR$ *and* $J, I \in \boldsymbol{L}({}_RR)$*, then*
$J \odot I = \omega_J^R({}_RI) = \bigcap_{f:\ {}_RJ \to {}_RR} f^{-1}(J) = \{i \in I \mid f(i) \in J \ \ \forall f:\ {}_RJ \to {}_RR\}.$ □

By analogy with the preceding cases (see [7]) we introduce the following definition of the inverse operation for ω-product.

Definition 2.1. Let $N, K \in \boldsymbol{L}({}_RM)$. The *left quotient* of N by K with respect to ω-product is defined as the least submodule $L \in \boldsymbol{L}({}_RM)$ with the property $L \odot K \supseteq N$. We denote this submodule by $N /_{\odot} K$. It is defined by the conditions:

a) $(N /_{\odot} K) \odot K \supseteq N$;

b) if $L \odot K \supseteq N$ for some $L \in \boldsymbol{L}({}_RM)$, then $L \supseteq N /_{\odot} K$.

Another form of this definition is the

Proposition 2.2. $N \subseteq L \odot K \ \Leftrightarrow \ N /_{\odot} K \subseteq L$.

Proof. $(\Rightarrow)$ is the condition *b*) of Definition 2.1.

$(\Leftarrow)$ Let $N /_{\odot} K \subseteq L$. Then using *a*) and the monotony of ω-product we have $N \subseteq (N /_{\odot} K) \odot K \subseteq L \odot K$, so $N \subseteq L \odot K$. □

The following statement is the answer to the question on the *existence* of the left quotient $N /_{\odot} K$ for a pair of submodules $N, K \in \boldsymbol{L}({}_RM)$.

Proposition 2.3. *Let* $N, K \in \boldsymbol{L}({}_RM)$*. The left quotient* $N /_{\odot} K$ *of* N *by* K *with respect to* ω*-product exists if and only if* $N \subseteq K$*. In this case it can be represented in the form:*

$$N /_{\odot} K = \bigcap_{\alpha \in \mathfrak{A}} \{L_\alpha \in \boldsymbol{L}({}_RM) \mid L_\alpha \odot K \supseteq N\}.$$

Proof. $(\Rightarrow)$ If there exists the left quotient $N /_{\odot} K$, then

$$N \subseteq (N /_{\odot} K) \odot K \subseteq (N /_{\odot} K) \cap K \quad \big(\text{Proposition 2.1, 1)}\big),$$

therefore $N \subseteq N /_{\odot} K$ and $N \subseteq K$.

$(\Leftarrow)$ Let $K \supseteq N$. Then the set of submodules

$$\{L_\alpha \in \boldsymbol{L}(_R M) \mid L_\alpha \odot K \supseteq N\}$$

is not empty, since it contains M $\big($by Proposition 2.1, 3) we have $M \odot K = K \supseteq N\big)$. Therefore we can consider the submodule

$$\bigcap_{\alpha \in \mathfrak{A}} \{L_\alpha \in \boldsymbol{L}(_R M) \mid L_\alpha \odot K \supseteq N\},$$

for which by Proposition 2.1, 6) we obtain:

$$\big(\bigcap_{\alpha \in \mathfrak{A}} L_\alpha\big) \odot K = \bigcap_{\alpha \in \mathfrak{A}} (L_\alpha \odot K) \supseteq N.$$

Moreover, by the construction it is clear that $\bigcap_{\alpha \in \mathfrak{A}} L_\alpha$ is the least submodule of $_R M$ with the property $L \odot K \supseteq N$. Therefore we have $\bigcap_{\alpha \in \mathfrak{A}} L_\alpha = N /_{\odot} K$. □

Taking into account this result we can say that the operation of left quotient of Definition 2.1 is a *partial operation* and considering the left quotient $N /_{\odot} K$ we will suppose in continuation that $N \subseteq K$.

Remark. In the case $N \subseteq K$ from the definition of ω-product we have: $L \odot K = \omega_L^M(K) = \bigcap_{f:\, K \to M} f^{-1}(L)$, therefore

$$L \odot K \supseteq N \quad \Leftrightarrow \quad f^{-1}(L) \supseteq N \;\; \forall\, f\colon K \to M \quad \Leftrightarrow \quad f(N) \subseteq L \;\; \forall\, f\colon K \to M.$$

This remark help us to represent the left quotient $N /_{\odot} K$ in other form.

Proposition 2.4. *Let* $K, N \in \boldsymbol{L}(_R M)$ *and* $N \subseteq K$. *Then*

$$N /_{\odot} K = \sum_{f:\, K \to M} f(N) \quad \big(= \alpha_N^K(M)\big).$$

In particular, $N /_{\odot} K$ *is a characteristic submodule of* M.

Proof. Denote $L = \sum_{f:\, K \to M} f(N)$. From the preceding remark we have:

$$f(N) \subseteq L \;\; \forall\, f\colon K \to M \quad \Leftrightarrow \quad N \subseteq f^{-1}(L) \;\; \forall\, f\colon K \to M \quad \Leftrightarrow \quad L \odot K \supseteq N.$$

It remains to verify that L is the least submodule of M with the property $L \odot K \supseteq N$. Let $L' \odot K \supseteq N$. Then $f(N) \subseteq L'$ for every $f\colon K \to M$, therefore $\sum_{f:\, K \to M} f(N) \subseteq L'$, i.e. $L \subseteq L'$. So $L = N /_{\odot} K$. The last statement of proposition is obvious, because $r(M) \in \boldsymbol{L}^{ch}(_R M)$ for every $r \in R$-pr. □

The following statement shows the concordance of left quotient $N /_{\odot} K$ with the order relation $(\subseteq)$ of $\boldsymbol{L}(_R M)$.

Proposition 2.5. 1) (The monotony in the numerator). *If* $N_1 \subseteq N_2$, *then* $N_1 /_{\odot} K \subseteq N_2 /_{\odot} K$ *for every* $K \supseteq N_2$.

2) (The antimonotony in the denominator). *If* $K_1 \subseteq K_2$, *then* $N /_{\odot} K_1 \supseteq N /_{\odot} K_2$ *for every* $N \subseteq K_1$.

Proof. 1) Let $N_1 \subseteq N_2$ and $K \supseteq N_2$. Then by the definition of left quotient we have $(N_2 /_{\odot} K) \odot K \supseteq N_2 \supseteq N_1$, therefore by Proposition 2.2 we obtain $N_2 /_{\odot} K \supseteq N_1 /_{\odot} K$.

2) Let $K_1 \subseteq K_2$ and $N \subseteq K_1$. Then by definition $(N /_{\odot} K_1) \odot K_1 \supseteq N$ and $(N /_{\odot} K_2) \odot K_2 \supseteq N$. Now the condition $K_1 \subseteq K_2$ and the monotony of ω-product imply

$$N \subseteq (N /_{\odot} K_1) \odot K_1 \subseteq (N /_{\odot} K_1) \odot K_2,$$

therefore $N /_{\odot} K_2 \subseteq N /_{\odot} K_1$. □

In continuation we will consider some particular cases, showing the values of the left quotient $N /_{\odot} K$ in dependence on the values of its terms N and K.

Proposition 2.6. 1) *If* $N = K$ *then*

$$N /_{\odot} K = N /_{\odot} N = \sum_{f:\ N \to M} f(N) = r^N(M) \ \ (\textit{the } \underline{\textit{trace}} \textit{ of } N \textit{ in } M).$$

2) *If* $N = 0$, *then for every* $K \subseteq M$ *we have:*

$$N /_{\odot} K = 0 /_{\odot} K = \sum_{f:\ K \to M} f(0) = 0.$$

3) *If* $K = M$, *then for every* $N \subseteq M$ *we have:*

$$N /_{\odot} M = \sum_{f:\ M \to M} f(N) = \alpha_N^M(M),$$

which is the least characteristic submodule of M, *containing* N; *therefore, if* $N \in \boldsymbol{L}^{ch}({}_RM)$, *then* $N /_{\odot} M = N$.

4) *If* $N = M$, *then for the existence of* $N /_{\odot} K$ *we must have* $K = M$, *so* $M /_{\odot} M = \sum_{f:\ M \to M} f(M) = M$.

5) *If* $K = 0$, *then for the existence of* $N /_{\odot} K$ *we must have* $N = 0$, *so* $0 /_{\odot} 0 = 0$. □

It is interesting to clarify the relations between the left quotient $N /_{\odot} K$ and the lattice operations of $\boldsymbol{L}({}_RM)$ (the sum and intersection of submodules).

Proposition 2.7. 1) *For every submodules* $N_1, N_2, K \in \boldsymbol{L}({}_RM)$ *with* $N_1 + N_2 \subseteq K$ *the following relation holds:*

$$(N_1 + N_2) /_{\odot} K = (N_1 /_{\odot} K) + (N_2 /_{\odot} K).$$

2) *For every submodules $N_1, N_2, K \in \boldsymbol{L}({}_RM)$ with $N_1 + N_2 \subseteq K$ the following relation holds:*

$$(N_1 \cap N_2) /_{\odot} K \subseteq (N_1 /_{\odot} K) \cap (N_2 /_{\odot} K).$$

3) *For every submodules $N, K_1, K_2 \in \boldsymbol{L}({}_RM)$ with $N \subseteq K_1 \cap K_2$ the following relation holds:*

$$N /_{\odot} (K_1 + K_2) \subseteq (N /_{\odot} K_1) \cap (N /_{\odot} K_2).$$

4) *For every submodules $N, K_1, K_2 \in \boldsymbol{L}({}_RM)$ with $N \subseteq K_1 \cap K_2$ the following relation holds:*

$$N /_{\odot} (K_1 \cap K_2) \supseteq (N /_{\odot} K_1) + (N /_{\odot} K_2).$$

Proof. 1) The inclusion ($\supseteq$) follows from the monotony in the numerator of the left quotient (Proposition 2.5, 1)).

($\subseteq$) Denote $L = (N_1 /_{\odot} K) + (N_2 /_{\odot} K)$. From the monotony of ω-product and the relations $L \supseteq N_1 /_{\odot} K$ and $L \supseteq N_2 /_{\odot} K$ we obtain:

$$L \odot K \supseteq (N_1 /_{\odot} K) \odot K \supseteq N_1,$$
$$L \odot K \supseteq (N_2 /_{\odot} K) \odot K \supseteq N_2.$$

Therefore $N_1 + N_2 \subseteq L \odot K$ and by Proposition 2.2 $(N_1 + N_2) /_{\odot} K \subseteq L$.

The statements 2), 3) and 4) follow from the monotony or antimonotony of the left quotient (Proposition 2.5). □

Remark. The relations of Proposition 2.7 can be generalized for the infinite case, for example:

$$\Big(\sum_{\alpha \in \mathfrak{A}} N_\alpha\Big) /_{\odot} K = \sum_{\alpha \in \mathfrak{A}} (N_\alpha /_{\odot} K),$$

where $N_\alpha \subseteq K$ for every $\alpha \in \mathfrak{A}$.

In the following two statements some new properties of left quotient $N /_{\odot} K$ are indicated.

Proposition 2.8. *For every submodules $N, L, K \in \boldsymbol{L}({}_RM)$ such that $N \subseteq L \subseteq K$ the following relation holds:*

$$(N /_{\odot} K) /_{\odot} (L /_{\odot} K) \subseteq N /_{\odot} L.$$

Proof. By definition $L \subseteq (N /_{\odot} K) \odot K$. From the monotony of ω-product and "semiassociativity" of Proposition 2.1, 5) we obtain:

$$N \subseteq (N /_{\odot} L) \odot L \subseteq (N /_{\odot} L) \odot [(L /_{\odot} K) \odot K] \subseteq [(N /_{\odot} L) \odot (L /_{\odot} K)] \odot K.$$

Therefore $N /_{\odot} K \subseteq (N /_{\odot} L) \odot (L /_{\odot} K)$ and so $(N /_{\odot} K) /_{\odot} (L /_{\odot} K) \subseteq N /_{\odot} L$. □

Proposition 2.9. *For every submodules $N, K, L \in \boldsymbol{L}({}_RM)$ such that $K \subseteq L$ the following relation holds:*

$$(N \odot K) /_{\odot} L \subseteq N \odot (K /_{\odot} L).$$

Proof. By definition $K \subseteq (K /_{\odot} L) \odot L$. Using the monotony of ω-product and Proposition 2.1, 5) we have:

$$N \odot K \subseteq N \odot [(K /_{\odot} L) \odot L] \subseteq [N \odot (K /_{\odot} L)] \odot L.$$

Therefore $(N \odot K) /_{\odot} L \subseteq N \odot [K /_{\odot} L]$. □

Some properties of the left quotient $N /_{\odot} K$ are true in the additional assumption that the operation of ω-product in $\boldsymbol{L}({}_R M)$ is associative, i.e. $(K \odot L) \odot L = K \odot (N \odot L)$ for every $K, N, L \in \boldsymbol{L}({}_R M)$ (see Proposition 2.1, 5)).

Proposition 2.10. *Let ${}_R M$ be a left R-module with the property that the operation of ω-product in $\boldsymbol{L}({}_R M)$ is associative. Then:*

1) *for every submodules $N, K, L \in \boldsymbol{L}({}_R M)$ such that $N \subseteq L \odot K$ the following relation holds:*
$$(N /_{\odot} K) /_{\odot} L = N /_{\odot} (L \odot K);$$

2) *for every submodules $N, K, L \in \boldsymbol{L}({}_R M)$ such that $N \subseteq L$ the following relation holds:*
$$(N \odot K) /_{\odot} (L \odot K) \subseteq N /_{\odot} L.$$

Proof. 1) $(\supseteq)$ By definition $N \subseteq (N /_{\odot} K) \odot K$ and $N /_{\odot} K \subseteq [(N /_{\odot} K) /_{\odot} L] \odot L$. Multiplying the last relation on the right by K, by the monotony and associativity of ω-product we obtain:

$$N \subseteq (N /_{\odot} K) \odot K \subseteq \{[(N /_{\odot} K) /_{\odot} L] \odot L\} \odot K = [(N /_{\odot} K) /_{\odot} L] \odot (L \odot K).$$

Therefore $N /_{\odot} (L \odot K) \subseteq (N /_{\odot} K) /_{\odot} L$.

$(\subseteq)$ By the definition of left quotient and Proposition 2.1, 5) we have:

$$N \subseteq [N /_{\odot} (L \odot K)] \odot (L \odot K) \subseteq \{[N /_{\odot} (L \odot K)] \odot L\} \odot K.$$

Therefore $N /_{\odot} K \subseteq [N /_{\odot} (L \odot K)] \odot L$ and so $(N /_{\odot} K) /_{\odot} L \subseteq N /_{\odot} (L \odot K)$.

2) By definition $N \subseteq (N /_{\odot} L) \odot L$. Applying the monotony and associativity of ω-product we have:

$$N \odot K \subseteq [(N /_{\odot} L) \odot L] \odot K = (N /_{\odot} L) \odot (L \odot K).$$

Therefore $(N \odot K) /_{\odot} (L \odot K) \subseteq N /_{\odot} L$. □

The following results deal with the property of cancellation for the left quotient with respect to ω-product.

Remark. By Proposition 2.2 (implication $(\Rightarrow)$) from $L \odot N \subseteq L \odot N$ it follows

$$(L \odot N) /_{\odot} N \subseteq L$$

for every $L, N \in \boldsymbol{L}({}_R M)$, where $L \odot N \subseteq L \bigcap N \subseteq N$.

Proposition 2.11. *Let* $N, K \in \boldsymbol{L}({}_R M)$. *The following conditions are equivalent:*

1) $(N \odot K) \big/_{\odot} K = N$;

2) $N = L \big/_{\odot} K$ *for some submodule* $L \in \boldsymbol{L}({}_R M)$.

Proof. 1) ⇒ 2) is obvious.

2) ⇒ 1). Let $N = L \big/_{\odot} K$, where $L \in \boldsymbol{L}({}_R M)$. By definition $(L \big/_{\odot} K) \odot K \supseteq L$, and using the monotony of left quotient in the numerator (Proposition 2.5, 1)) we obtain

$$(N \odot K) \big/_{\odot} K = [(L \big/_{\odot} K) \odot K] \big/_{\odot} K \supseteq L \big/_{\odot} K = N,$$

i.e. $(N \odot K) \big/_{\odot} K \supseteq N$. From the preceding remark we have the inverse inclusion, therefore $(N \odot K) \big/_{\odot} K = N$. □

Proposition 2.12. *Let* $N, K \in \boldsymbol{L}({}_R M)$ *and* $N \subseteq K$. *The following conditions are equivalent:*

1) $(N \big/_{\odot} K) \odot K = N$;

2) $N = L \odot K$ *for some submodule* $L \in \boldsymbol{L}({}_R M)$.

Proof. 1) ⇒ 2) is obvious.

2) ⇒ 1). Let $N = L \odot K$, where $L \in \boldsymbol{L}({}_R M)$. By definition $(N \big/_{\odot} K) \odot K \supseteq N$ and by the last remark $(L \odot K) \big/_{\odot} K \subseteq L$. Now using the monotony of ω-product we obtain:

$$(N \big/_{\odot} K) \odot K = [(L \odot K) \big/_{\odot} K] \odot K \subseteq L \odot K = N,$$

therefore $(N \big/_{\odot} K) \odot K = N$. □

3 The right quotient with respect to α-coproduct

In this section the similar questions are discussed as in the preceding one for the operation of α-coproduct. We remind that the *α-coproduct* in ${}_R M$ of submodules $N, K \in \boldsymbol{L}({}_R M)$ is defined by the rule:

$$N : K = \pi_N^{-1}\big(\alpha_K^M(M \,/\, N)\big) = \{m \in M \mid m + N \in \sum_{f :\, M \to M/N} f(K)\},$$

where $\pi_N : M \to M \,/\, N$ is the natural morphism, i.e.

$$(N : K) \,/\, N = \alpha_K^M(M \,/\, N) \ \ \text{(see Section 1)}.$$

Some properties of operation of α-coproduct are enumerated in the following statement [6].

Proposition 3.1. 1) $N : K \supseteq N + K$ *for every* $N, K \in \boldsymbol{L}({}_R M)$, *so* $M : K = M$ *and* $N : M = M$ *for every* $N, K \in \boldsymbol{L}({}_R M)$.

2) *If* $N = 0$, *then* $0 : K$ *is the least characteristic submodule of* M *containing* K *for every* $K \in \boldsymbol{L}({}_R M)$; *therefore if* $K \in \boldsymbol{L}^{ch}({}_R M)$, *then* $0 : K = K$.

3) *If* $K = 0$*, then* $N : 0 = N$ *for every* $N \in \boldsymbol{L}({}_RM)$*.*

4) *The operation of* α*-coproduct is monotone in both variables.*

5) *If* ${}_RM$ *is a* <u>*projective*</u> *module, then*

$$(N : K) : L \subseteq N : (K : L)$$

for every submodules $N, K, L \in \boldsymbol{L}({}_RM)$*.*

6) $N : \big(\sum\limits_{\alpha \in \mathfrak{A}} K_\alpha \big) = \sum\limits_{\alpha \in \mathfrak{A}} (N : K_\alpha)$ *for every* $N, K_\alpha \in \boldsymbol{L}({}_RM)$*.*

7) *If* ${}_RM = {}_RR$*, then* $N : K = KR + N$ *for every left ideals* $N, K \in \boldsymbol{L}({}_RR)$*.* □

Now we introduce the inverse operation for the α-coproduct which in some sense is dual to the preceding notion (Definition 2.1).

Definition 3.1. Let $K, N \in \boldsymbol{L}({}_RM)$. The *right quotient* of K by N with respect to α-coproduct is defined as the greatest among submodules $L \subseteq M$ with the property $N : L \subseteq K$. We denote this submodule by $N \mathbin{\vdots\backslash} K$. It is determined by the conditions:

a) $N : (N \mathbin{\vdots\backslash} K) \subseteq K$;

b) if $N : L \subseteq K$, then $L \subseteq N \mathbin{\vdots\backslash} K$.

The following simple remark is useful in applications.

Proposition 3.2. *For every submodules* $K, N, L \in \boldsymbol{L}({}_RM)$ *we have:*

$$N : L \subseteq K \quad \Leftrightarrow \quad L \subseteq N \mathbin{\vdots\backslash} K.$$

Proof. Implication $(\Rightarrow)$ is the condition *b*) of Definition 3.1.

$(\Leftarrow)$ If $L \subseteq N \mathbin{\vdots\backslash} K$, then from the monotony of α-coproduct and condition *a*) we have:

$$N : L \subseteq N : (N \mathbin{\vdots\backslash} K) \subseteq K, \text{ i.e. } N : L \subseteq K. \qquad \square$$

Now we study the question on the *existence* of right quotient with respect to α-coproduct.

Proposition 3.3. *Let* $K, N \in \boldsymbol{L}({}_RM)$*. The right quotient* $N \mathbin{\vdots\backslash} K$ *of* K *by* N *with respect to* α*-coproduct exists if and only if* $N \subseteq K$*. In this case we have:*

$$N \mathbin{\vdots\backslash} K = \sum_{\alpha \in \mathfrak{A}} \{L_\alpha \in \boldsymbol{L}({}_RM) \mid N : L_\alpha \subseteq K\}.$$

Proof. $(\Rightarrow)$ If the right quotient $N \mathbin{\vdots\backslash} K$ exists, then by definition $N : (N \mathbin{\vdots\backslash} K) \subseteq K$ and since $N + (N \mathbin{\vdots\backslash} K) \subseteq N : (N \mathbin{\vdots\backslash} K)$ (Proposition 3.1, 1)), we obtain $N \mathbin{\vdots\backslash} K \subseteq K$ and $N \subseteq K$.

$(\Leftarrow)$ Let $N \subseteq K$. Then the set of submodules of M

$$\{L_\alpha \in \boldsymbol{L}({}_RM) \mid N : L_\alpha \subseteq K\}$$

is not empty, since it contains the submodule 0 (by Proposition 3.1, 3) we have $N:0 = N \subseteq K$). So we can consider the submodule

$$L = \sum_{\alpha\in\mathfrak{A}} \{L_\alpha \in \boldsymbol{L}({}_RM) \mid N : L_\alpha \in K\},$$

for which by Proposition 3.1, 6) we have

$$N : \Big(\sum_{\alpha\in\mathfrak{A}} L_\alpha\Big) = \sum_{\alpha\in\mathfrak{A}} (N : L_\alpha) \subseteq K.$$

Therefore $N : L \subseteq K$ and by construction it is clear that L is the greatest submodule of M with this property, so by definition $L = N \,{:}\!\backslash\, K$. □

In such way we obtain a *partial operation* of right quotient $N \,{:}\!\backslash\, K$, which is considered in the case $N \subseteq K$, the condition which provides its existence. The right quotient $N \,{:}\!\backslash\, K$ in this case can be represented in the following form.

Proposition 3.4. *Let* $N, K \in \boldsymbol{L}({}_RM)$ *and* $N \subseteq K$. *Then:*

$$N \,{:}\!\backslash\, K = \bigcap_{f:\, M \to M/N} f^{-1}(K/N) \quad \big(= \omega^{M/N}_{K/N}(M)\big).$$

Therefore $N \,{:}\!\backslash\, K$ *is a characteristic submodule of* M.

Proof. Let $L = \bigcap_{f:\, M \to M/N} f^{-1}(K/N)$. Then for every $f : M \to M/N$ we have $L \subseteq f^{-1}(K/N)$, i.e. $f(L) \subseteq K/N$. Therefore $\sum_{f:\, M \to M/N} f(L) \subseteq K/N$ and $\pi_N^{-1}\big(\sum_{f:\, M \to M/N} f(L)\big) \subseteq K$, i.e. $\pi_N^{-1}\big(\alpha^M_L(M/N)\big) \subseteq K$ and by the definition of α-coproduct $N : L \subseteq K$.

Moreover, L is the greatest submodule of M with this property. Indeed, if $N : L' \subseteq K$, then $\pi_N^{-1}\big(\sum_{f:\, M \to M/N} f(L')\big) \subseteq K$, therefore $\sum_{f:\, M \to M/N} f(L') \subseteq K/N$, i.e. $f(L') \subseteq K/N$ for every $f: M \to M/N$. This means that $L' \subseteq f^{-1}(K/N)$ for every $f: M \to M/N$, so $L' \subseteq \bigcap_{f:\, M \to M/N} f^{-1}(K/N) = L$. This proves that $N \,{:}\!\backslash\, K = L$. □

Remark. As was mentioned above $N \,{:}\!\backslash\, K \subseteq K$, since by Proposition 3.1, 1) $N + (N \,{:}\!\backslash\, K) \subseteq N : (N \,{:}\!\backslash\, K) \subseteq K$.

Now we show the behaviour of the right quotient $N \,{:}\!\backslash\, K$ relative to the partial order $(\subseteq)$ of the lattice $\boldsymbol{L}({}_RM)$.

Proposition 3.5. 1) (The monotony in the numerator). *If* $K_1 \subseteq K_2$, *then* $N \,{:}\!\backslash\, K_1 \subseteq N \,{:}\!\backslash\, K_2$ *for every* $N \subseteq K_1 \cap K_2$.

2) (The antimonotony in the denominator). *If* $N_1 \subseteq N_2$, *then* $N_1 \,{:}\!\backslash\, K \supseteq N_2 \,{:}\!\backslash\, K$ *for every* $K \in \boldsymbol{L}({}_RM)$ *with* $N_1 + N_2 \subseteq K$.

Proof. 1) By definition $N : (N \,\dot{\backslash}\, K_1) \subseteq K_1 \subseteq K_2$, therefore $N \,\dot{\backslash}\, K_1 \subseteq N \,\dot{\backslash}\, K_2$.

2) The monotony of α-coproduct and the definition of right quotient imply:

$$N_1 : (N_2 \,\dot{\backslash}\, K) \subseteq N_2 : (N_2 \,\dot{\backslash}\, K) \subseteq K\,,$$

therefore $N_2 \,\dot{\backslash}\, K \subseteq N_1 \,\dot{\backslash}\, K$. □

In continuation we consider the right quotient $N \,\dot{\backslash}\, K$ in some particular cases.

Proposition 3.6. 1) *If* $N = K$, *then*

$$N \,\dot{\backslash}\, K = \{m \in M \mid f(m) \in K/N = 0 \;\; \forall f : M \to M/N\} =$$
$$= \bigcap_{f:\, M \to M/N} Ker\, f = r_{M/N}(M),$$

which is the greatest characteristic submodule of M *contained in* N.

2) *If* $N = 0$, *then for every* $K \in \boldsymbol{L}({}_R M)$ *we have:*

$$0 \,\dot{\backslash}\, K = \omega_K^M(M) = \bigcap_{f:\, M \to M} f^{-1}(K),$$

which is the greatest characteristic submodule of M *contained in* K*; therefore if* $K \in \boldsymbol{L}^{ch}({}_R M)$, *then* $0 \,\dot{\backslash}\, K = K$.

3) *If* $K = M$, *then for every* $N \in \boldsymbol{L}({}_R M)$ *we have:*

$$N \,\dot{\backslash}\, M = \{m \in M \mid f(m) \in K/N = M/N \;\; \forall f : M \to M/N\} = M.$$

4) *If* $K = 0$, *then for the existence of right quotient we must have* $N = 0$ *and* $0 \,\dot{\backslash}\, 0 = 0$.

5) *If* $N = M$, *then for the existence of right quotient we must have* $K = M$ *and* $M \,\dot{\backslash}\, M = M$. □

The following statement has to do with the relation between the right quotient and the lattice operations of $\boldsymbol{L}({}_R M)$ (the sum and intersection of submodules).

Proposition 3.7. 1) *For every submodules* $N, K_1, K_2 \in \boldsymbol{L}({}_R M)$ *with* $N \subseteq K_1 \bigcap K_2$ *the following relation holds:*

$$N \,\dot{\backslash}\, (K_1 \bigcap K_2) = (N \,\dot{\backslash}\, K_1) \bigcap (N \,\dot{\backslash}\, K_2).$$

2) $N \,\dot{\backslash}\, (K_1 + K_2) \supseteq (N \,\dot{\backslash}\, K_1) + (N \,\dot{\backslash}\, K_2)$, *where* $N \subseteq K_1 \bigcap K_2$.

3) $(N_1 \bigcap N_2) \,\dot{\backslash}\, K \supseteq (N_1 \,\dot{\backslash}\, K) + (N_2 \,\dot{\backslash}\, K)$, *where* $N_1 + N_2 \subseteq K$.

4) $(N_1 + N_2) \,\dot{\backslash}\, K \subseteq (N_1 \,\dot{\backslash}\, K) \bigcap (N_2 \,\dot{\backslash}\, K)$, *where* $N_1 + N_2 \subseteq K$.

Proof. 1) ($\subseteq$) follows from Proposition 3.5, 1).

($\supseteq$) Denote $L = (N \,\dot{\backslash}\, K_1) \bigcap (N \,\dot{\backslash}\, K_2)$. Then using the monotony of α-coproduct and the relations $L \subseteq N \,\dot{\backslash}\, K_1$ and $L \subseteq N \,\dot{\backslash}\, K_2$, we obtain

$$N : L \subseteq N : (N \,\dot{\backslash}\, K_1) \subseteq K_1,$$
$$N : L \subseteq N : (N \,\dot{\backslash}\, K_2) \subseteq K_2.$$

Therefore $N : L \subseteq K_1 \bigcap K_2$ and by the definition of right quotient we have $L \subseteq N \mathbin{:\!\backslash} (K_1 \bigcap K_2)$.

The statements 2), 3) and 4) follow from the monotony and antimonotony of right quotient (Proposition 3.5). □

Remark. The Proposition 3.7 can be generalized for any set of submodules, in particular:

$$N \mathbin{:\!\backslash} \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha \Big) = \bigcap_{\alpha \in \mathfrak{A}} (N \mathbin{:\!\backslash} K_\alpha).$$

In continuation we indicate some properties of the right quotient $N \mathbin{:\!\backslash} K$ which are true in the cases when the "semiassociativity" of α-coproduct (see Proposition 3.1, 5)) or the associativity of this operation is supposed.

Proposition 3.8. *If ${}_RM$ is a projective module, $L, N, K \in \boldsymbol{L}({}_RM)$ and $N \subseteq K$, then*

$$(L : N) \mathbin{:\!\backslash} (L : K) \supseteq N \mathbin{:\!\backslash} K.$$

Proof. By definition $K \supseteq N : (N \mathbin{:\!\backslash} K)$. Using the monotony of α-coproduct and Proposition 3.1, 5) we have:

$$L : K \supseteq L : [N : (N \mathbin{:\!\backslash} K)] \supseteq (L : N) : (N \mathbin{:\!\backslash} K),$$

therefore by Proposition 3.2 $(L : N) \mathbin{:\!\backslash} (L : K) \supseteq N \mathbin{:\!\backslash} K$. □

Proposition 3.9. *Let ${}_RM$ be a left R-module with the property that in $\boldsymbol{L}({}_RM)$ the operation of α-coproduct is associative. Then the following relations hold:*

1) $L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K) = (N : L) \mathbin{:\!\backslash} K$, *where* $N : L \subseteq K$;

2) $(L \mathbin{:\!\backslash} N) \mathbin{:\!\backslash} (L \mathbin{:\!\backslash} K) \supseteq N \mathbin{:\!\backslash} K$, *where* $L \subseteq N \subseteq K$;

3) $L \mathbin{:\!\backslash} (N : K) \supseteq (L \mathbin{:\!\backslash} N) : K$, *where* $L \subseteq N \bigcap K$.

Proof. 1) ($\subseteq$) By definition of right quotient we have:

$$N : (N \mathbin{:\!\backslash} K) \subseteq K,$$
$$L : [L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K)] \subseteq N \mathbin{:\!\backslash} K.$$

Using the monotony of α-coproduct and the associativity of this operation (the "semiassociativity" of Proposition 3.1, 5) is sufficient) we obtain:

$$(N : L) : [L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K)] \subseteq N : [L : (L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K))] \subseteq N : (N \mathbin{:\!\backslash} K) \subseteq K.$$

Therefore $L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K) \subseteq (N : L) \mathbin{:\!\backslash} K$.

($\supseteq$) By the associativity of α-coproduct and by the definition of right quotient we have:

$$N : \{L : [(N : L) \mathbin{:\!\backslash} K]\} = (N : L) : [(N : L) \mathbin{:\!\backslash} K] \subseteq K.$$

So by Proposition 3.2 $L : [(N : L) \mathbin{:\!\backslash} K] \subseteq N \mathbin{:\!\backslash} K$ and $(N : L) \mathbin{:\!\backslash} K \subseteq L \mathbin{:\!\backslash} (N \mathbin{:\!\backslash} K)$.

2) By definition $N \supseteq L : (L \mathbin{:\!\backslash} N)$. Using the monotony and associativity of α-coproduct we obtain:

$$K \supseteq N : (N \mathbin{:\!\backslash} K) \supseteq [L : (L \mathbin{:\!\backslash} N)] : (N \mathbin{:\!\backslash} K) = L : [(L \mathbin{:\!\backslash} N) : (N \mathbin{:\!\backslash} K)].$$

By Proposition 3.2 it follows that $L \mathbin{:\!\backslash} K \supseteq (L \mathbin{:\!\backslash} N) : (N \mathbin{:\!\backslash} K)$ and $(L \mathbin{:\!\backslash} N) \mathbin{:\!\backslash} (L \mathbin{:\!\backslash} K) \supseteq N \mathbin{:\!\backslash} K$.

3) By definition $N \supseteq L : (L \mathbin{:\!\backslash} N)$, therefore by the monotony and associativity of α-coproduct we have:

$$N : K \supseteq [L : (L \mathbin{:\!\backslash} N)] : K = L : [(L \mathbin{:\!\backslash} N) : K]$$

and so $L \mathbin{:\!\backslash} (N : K) \supseteq (L \mathbin{:\!\backslash} N) : K$. □

The following subject of investigation is the cancellation property related to the right quotient $N \mathbin{:\!\backslash} K$. We begin with a simple

Remark. $N \mathbin{:\!\backslash} (N : L) \supseteq L$ for every $N, L \in \boldsymbol{L}({}_RM)$, since denoting $K = N : L$ from the relation $N : L \subseteq K$ we have $L \subseteq N \mathbin{:\!\backslash} K$ (Proposition 3.2).

Proposition 3.10. *Let $K, N \in \boldsymbol{L}({}_RM)$. The following conditions are equivalent:*

1) $N = K \mathbin{:\!\backslash} (K : N)$*;*

2) $N = K \mathbin{:\!\backslash} L$ *for some* $L \in \boldsymbol{L}({}_RM)$ *with* $K \subseteq L$.

Proof. 1) ⇒ 2) is obvious.

2) ⇒ 1). Let $N = K \mathbin{:\!\backslash} L$, where $L \in \boldsymbol{L}({}_RM)$ and $K \subseteq L$. By the definition of right quotient and its monotony in the numerator we have:

$$K : (K \mathbin{:\!\backslash} L) \subseteq L, \quad K \mathbin{:\!\backslash} [K : (K \mathbin{:\!\backslash} L)] \subseteq K \mathbin{:\!\backslash} L.$$

From the preceding remark in the last relation the inverse inclusion holds, therefore

$$N = K \mathbin{:\!\backslash} L = K \mathbin{:\!\backslash} [K : (K \mathbin{:\!\backslash} L)] = K \mathbin{:\!\backslash} (K : N).$$ □

Proposition 3.11. *Let $K, N \in \boldsymbol{L}({}_RM)$ and $K \subseteq N$. The following conditions are equivalent:*

1) $N = K : (K \mathbin{:\!\backslash} N)$*;*

2) $N = K : L$ *for some submodule* $L \in \boldsymbol{L}({}_RM)$.

Proof. 1) ⇒ 2) is obvious.

2) ⇒ 1). Let $N = K : L$, where $L \in \boldsymbol{L}({}_RM)$. By the preceding remark $K \mathbin{:\!\backslash} (K : L) \supseteq L$ and using the monotony of α-coproduct we have:

$$K : [K \mathbin{:\!\backslash} (K : L)] \supseteq K : L.$$

The inverse inclusion follows from the definition of right quotient, therefore:

$$N = K : L = K : [K \mathbin{:\!\backslash} (K : L)] = K : (K \mathbin{:\!\backslash} N).$$ □

Finally, we consider the particular case ${}_RM = {}_RR$. It is known that for every left ideals $N, K \in \boldsymbol{L}({}_RM)$ we have $N : K = KR + N$ (Proposition 3.1, 7)).

Proposition 3.12. *Let $N, K \in \boldsymbol{L}({}_RR)$ and $N \subseteq K$. Then*

$$N \mathbin{\underset{:}{\backslash}} K = \{r \in R \mid r \cdot R \subseteq K\},$$

the annihilator of ${}_R(R/K)$ in R.

Proof. Denoting the right side of indicated relation by L, we have $LR \subseteq K$ and $N : L = LR + N \subseteq K + N = K$, i.e. $N : L \subseteq K$. Moreover, L is the greatest submodule of M with this property. Indeed, if $N : L' \subseteq K$, then $N : L' = L'R + N \subseteq K$ and $L'R \subseteq K$, so by the definition of L it follows that $L' \subseteq L$. Now it is clear that $N \mathbin{\underset{:}{\backslash}} K = L$. □

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