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Generators of the algebras of invariants for differential system with homogeneous nonlinearities of odd degree

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Abstract. Common Hilbert series and generalized Hilbert series for the algebras of invariants and comitants of polynomial differential systems with homogeneous nonlinearities of the fifth degree are constructed, with their help the Krull dimensions of these algebras are obtained. The relations between Hilbert series of various algebras of invariants and comitants of polynomial differential systems with homogeneous nonlinearities of odd degree are given. With their help we give a method of the construction of generators of the algebras of invariants for corresponding systems.

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Consider the system of differential equations

$$\frac{dx^j}{dt} = \sum_{\gamma\in\Gamma} a^j_{j_1 j_2 \ldots j_\gamma} x^{j_1} x^{j_2} \cdots x^{j_\gamma} \ (j, j_1, j_2, ..., j_\gamma = 1, 2), \tag{1}$$

where Γ is a finite set of different non-negative integers, and the tensor $a^j_{j_1 j_2 \ldots j_\gamma}$ ($\gamma \in \Gamma$) is symmetrical in lower indices, in which the complete convolution takes place. If $0 \in \Gamma$, then the system (1) contains constant terms a^j. We observe that the system (1) is completely determined by the set Γ. Further for the notation of the system (1) we will use $s(\Gamma)$.

Consider the group of center-affine transformations $GL(2, \mathbb{R}) = \{q\}$:

$$y^r = q^r_j x^j, \ \Delta_q = det(q^r_j) \neq 0 \ (r, j = 1, 2). \tag{2}$$

If in (2) $\Delta_q = 1$ for all q, then the group is called unimodular and it is denoted by $SL(2, \mathbb{R})$. Denote by a the totality of all coefficients of the system (1), and by b the totality of all coefficients of the system obtained from system (1) by a transformation $q \in GL(2, \mathbb{R})$. All coefficients and variables of (1) and (2) are real.

Definition 1. *The polynomial $K(a, x)$ of the coefficients of the system* (1) *and the coordinates of the vector $x = (x^1, x^2)$ is called a center-affine comitant of the system* (1) *if there exists a function $\lambda(q)$ such that the equality*

$$K(b, y) = \lambda(q)K(a, x)$$

holds for all $q \in GL(2,\mathbb{R})$ *and all coefficients of the system* (1) *and coordinates of the vector* x.

If the comitant K does not depend on the coordinates of x, then it is called an invariant of the system (1) for the group $GL(2,\mathbb{R})$. As follows from the general theory of algebraic invariants of differential systems with polynomial right-hand sides [1-3], $\lambda(q) = \Delta_q^{-\chi}$, where the integer χ is the weight of comitant.

The problem of studying a polynomial basis for differential systems was first posed in the works of K. S. Sibirsky [1–3] and was further developed in the monograph of N. I. Vulpe [4] and by their disciples. In particular, in work [5] it was shown how to use the minimal bases of comitants (MBC) of the systems $s(0)$ and $s(\Gamma)$, $0 \notin \Gamma$, in order to construct a similar basis of the system $s(0,\Gamma)$. However, the question of the construction of MBC for the system $s(m,n)$ $(m,n \neq 0,\ m \neq n)$ if the MBC for the system $s(m)$ and $s(n)$ are known still remains open.

In this article, using graded algebras and their generators, with the help of Hilbert series a method of solving this problem for $m = 1$ and $n = 2k+1$ $(k \geq 1)$ will be shown.

1 Graded algebras of invariants and comitants, and their Hilbert series for the system $s(1, 2k+1)$ $(k \geq 1)$

Consider system of differential equations with odd nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha_1\alpha_2\ldots\alpha_{2k+1}} x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_{2k+1}},\ (j,\alpha,\alpha_1,\alpha_2,...,\alpha_{2k+1} = 1,2), \quad (3)$$

where $k = 1,2,...$ is a fixed natural number, and the coefficient tensor $a^j_{\alpha_1\alpha_2\ldots\alpha_{2k+1}}$ is symmetrical in lower indices, in which the complete convolution takes place.

Following [6, 7], we recall the definition of Hilbert series for algebras of invariants and comitants of the system (3).

Denote by $S^{(d)}_{1,2k+1}$ the linear space of center-affine (unimodular) comitants of the system (3) of the type

$$(d) = (\delta, d_1, d_2), \quad (4)$$

where δ is the degree of homogeneity of a comitant with respect to phase variables x^1, x^2, and d_1 (d_2) is the degree of homogeneity of this comitant with respect to the coordinates of the tensor a^j_α $(a^j_{\alpha_1\alpha_2\ldots\alpha_{2k+1}})$. The subscripts in $S^{(d)}_{1,2k+1}$ show that this space is associated with the differential system (3), the right side of which is the sum of homogeneities of degree one and $2k+1$ with respect to x^1, x^2. From [6, 7] it is known that the space $S^{(d)}_{1,2k+1}$ is finite for any fixed (d).

Consider the linear space

$$S_{1,2k+1} = \sum_{(d)} S^{(d)}_{1,2k+1}, \quad (5)$$

which is a graded algebra of invariant polynomials with respect to the unimodular group $SL(2,\mathbb{R})$, in which the following inclusion $S^{(d)}_{1,2k+1} S^{(e)}_{1,2k+1} \subseteq S^{(d+e)}_{1,2k+1}$ is satisfied.

Following [6, 7], by a generalized Hilbert series of the algebra (5) we understand the sum

$$H(S_{1,2k+1}, u, z_1, z_2) = \sum_{(d)} dim_{\mathbb{R}} S_{1,2k+1}^{(d)} u^{\delta} z_1^{d_1} z_2^{d_2}, \tag{6}$$

and then we obtain the common Hilbert series from (6) as follows

$$H_{S_{1,2k+1}}(u) = H(S_{1,2k+1}, u, u, u). \tag{7}$$

If we denote by $SI_{1,2k+1}$ the algebra of invariants of the unimodular group $SL(2,\mathbb{R})$ for the system (3), then for the generalized Hilbert series of this algebra we have

$$H(SI_{1,2k+1}, z_1, z_2) = H(S_{1,2k+1}, 0, z_1, z_2), \tag{8}$$

and for a common Hilbert series we obtain

$$H_{SI_{1,2k+1}}(z) = H(SI_{1,2k+1}, z, z). \tag{9}$$

If in the system (3) there are no linear terms (a_{α}^{j}, $j, \alpha = 1, 2$), then we obtain the system $s(2k+1)$, we will denote the algebra of comitants and invariants by S_{2k+1} and SI_{2k+1} respectively. Note that the above-listed algebras are finitely defined [6, 7].

Remark 1. It is known (see, for example, [8]), that the maximum number of algebraically independent homogeneous elements of a graded algebra of polynomials is called its Krull dimension, and it is equal to the order of the pole and the common Hilbert series of this algebra in the unit.

2 Relations between the generalized Hilbert series of algebras SI_1, S_{2k+1} and $SI_{1,2k+1}$

From [6, 7] the generalized Hilbert series for algebras SI_1, S_3 and $SI_{1,3}$ are known. They have the forms

$$H(SI_1, b) = \frac{1}{(1-b)(1-b^2)}, \tag{10}$$

$$H(S_3, u, d) = \frac{N_3(u,d)}{D_3(u,d)}, \tag{11}$$

where

$$\begin{gathered} D_3(u,d) = (1-d^2)^3(1-d^3)^2(1-u^2d)^2(1-u^4d), \\ N_3(u,d) = 1 - d^2 + d^4 + u^2(-d + d^2 + 3d^3 - 2d^5) + \\ +u^4(2d^2 - 3d^4 - d^5 + d^6) + u^6(-d^3 + d^5 - d^7) \end{gathered} \tag{12}$$

and

$$H(SI_{1,3}, b, d) = \frac{NI_{1,3}(b,d)}{DI_{1,3}(b,d)}. \tag{13}$$

Here

$$DI_{1,3}(b,d) = (1-b)(1-b^2)(1-bd)^2(1-b^2d)(1-d^2)^3(1-d^3)^2,$$
$$NI_{1,3}(b,d) = 1-d^2+d^4+b(-d+d^2+3d^3-2d^5)+ \tag{14}$$
$$+b^2(2d^2-3d^4-d^5+d^6)+b^3(-d^3+d^5-d^7).$$

With the help of equalities (10)–(14) was proved

Lemma 1. *Between the generalized Hilbert series of the algebras* SI_1, S_3 *and* $SI_{1,3}$ *the following relation exists*

$$H(SI_{1,3},b,d) = H(SI_1,b)H(S_3,u,d)|_{u^2=b}. \tag{15}$$

In [6, 7] we have found the generalized Hilbert series for the algebra S_5 of unimodular comitants of the system $s(5)$

$$H(S_5,u,f) = \frac{N_5(u,f)}{D_5(u,f)}, \tag{16}$$

where

$$D_5(u,f) = (1+f)(1-f^2)^2(1-f^3)^3(1-f^4)^2(1-f^5)^2(1-u^2f)^2\times$$
$$\times(1-u^4f)^2(1-u^6f),$$
$$N_5(u,f) = \sum_{k=0}^{8} R_{2k}(f)u^{2k},$$

and

$$R_0(f) = 1+f-f^3+f^4+4f^5+11f^6+16f^7+17f^8+13f^9+13f^{10}+13f^{11}+$$
$$+17f^{12}+16f^{13}+11f^{14}+4f^{15}+f^{16}-f^{17}+f^{19}+f^{20},$$
$$R_2(f) = -2f-f^2+5f^3+12f^4+15f^5+15f^6+11f^7+10f^8+17f^9+27f^{10}+$$
$$+25f^{11}+16f^{12}-f^{13}-9f^{14}-5f^{15}+2f^{16}+3f^{17}+3f^{18}-2f^{20}-2f^{21},$$
$$R_4(f) = -f+3f^2+6f^3+5f^4-3f^6-10f^7-13f^8-15f^9-18f^{10}-32f^{11}-$$
$$-41f^{12}-51f^{13}-43f^{14}-29f^{15}-18f^{16}-11f^{17}-3f^{18}-2f^{19}-2f^{20}-f^{21}+f^{22},$$
$$R_6(f) = 3f^2+f^3-5f^4-11f^5-11f^6-15f^7-20f^8-35f^9-58f^{10}-85f^{11}-$$
$$-81f^{12}-61f^{13}-21f^{14}-f^{15}-7f^{17}-7f^{18}-8f^{19}-2f^{20}+3f^{21}+4f^{22},$$
$$R_8(f) = 2f^2-f^3-4f^4-6f^5-9f^6-18f^7-21f^8-21f^9-23f^{10}-22f^{11}-$$
$$-6f^{12}+6f^{13}+22f^{14}+23f^{15}+21f^{16}+21f^{17}+18f^{18}+9f^{19}+6f^{20}+$$
$$+4f^{21}+f^{22}-2f^{23},$$
$$R_{18-2k}(f) = -f^{25}R_{2k}(f^{-1})\ (k=\overline{0,4}). \tag{17}$$

According to [6, 7] the initial form of the generating function for the center-affine comitants of the system $s(1,5)$ has the form

$$\varphi_{1,5}^{(0)} = (1-u^{-2})\psi_1^{(0)}(u)\psi_5^{(0)}(u), \tag{18}$$

where

$$\psi_1^{(0)}(u) = \frac{1}{(1-u^2b)(1-b)^2(1-u^{-2}b)},$$

$$\psi_5^{(0)}(u) = \frac{1}{(1-u^6f)(1-u^4f)^2(1-u^2f)^2(1-f)^2(1-u^{-2}f)^2}\cdot \tag{19}$$
$$\cdot\frac{1}{(1-u^{-4}f)^2(1-u^{-6}f)}.$$

Following [6, 7], after solving the functional Cayley's equation

$$H(S_{1,5},u,b,f) - u^{-2}H(S_{1,5},u^{-1},b,f) = \varphi_{1,5}^{(0)}(u),$$

where $\varphi_{1,5}^{(0)}(u)$ is from (18)–(19), we obtain

Theorem 1. *The generalized Hilbert series for the graded algebra of comitants $S_{1,5}$ of the system* (3) $(k=2)$ *is a rational function of* u, b, f *and has the form*

$$H(S_{1,5},u,b,f) = \frac{N_{1,5}(u,b,f)}{D_{1,5}(u,b,f)}, \tag{20}$$

where

$$D_{1,5}(u,b,f) = (1-b)(1-b^2)(1-bu^2)(1-bf)^2(1-b^2f)^2(1-b^3f)(1+f)\times$$
$$\times(1-f^2)^2(1-f^3)^3(1-f^4)^2(1-f^5)^2(1-fu^2)^2(1-fu^4)^2(1-fu^6), \tag{21}$$

$$N_{1,5}(u,b,f) = \sum_{k=0}^{8} R_{2k}(b,f)u^{2k}, \tag{22}$$

and

$$R_0(b,f) = 1+f-f^3+f^4+4f^5+11f^6+16f^7+17f^8+13f^9+13f^{10}+$$
$$+13f^{11}+17f^{12}+16f^{13}+11f^{14}+4f^{15}+f^{16}-f^{17}+f^{19}+f^{20}+b(-2f-$$
$$-f^2+5f^3+12f^4+15f^5+15f^6+11f^7+10f^8+17f^9+27f^{10}+25f^{11}+$$
$$+16f^{12}-f^{13}-9f^{14}-5f^{15}+2f^{16}+3f^{17}+3f^{18}-2f^{20}-2f^{21})+b^2(-f+$$
$$+3f^2+6f^3+5f^4-3f^6-10f^7-13f^8-15f^9-18f^{10}-32f^{11}-41f^{12}-$$
$$-51f^{13}-43f^{14}-29f^{15}-18f^{16}-11f^{17}-3f^{18}-2f^{19}-2f^{20}-f^{21}+f^{22})+$$
$$+b^3(3f^2+f^3-5f^4-11f^5-11f^6-15f^7-20f^8-35f^9-58f^{10}-85f^{11}-$$
$$-81f^{12}-61f^{13}-21f^{14}-f^{15}-7f^{17}-7f^{18}-8f^{19}-2f^{20}+3f^{21}+4f^{22})+$$
$$+b^4(2f^2-f^3-4f^4-6f^5-9f^6-18f^7-21f^8-21f^9-23f^{10}-22f^{11}-$$
$$-6f^{12}+6f^{13}+22f^{14}+23f^{15}+21f^{16}+21f^{17}+18f^{18}+9f^{19}+6f^{20}+4f^{21}+$$
$$+f^{22}-2f^{23})+b^5(-4f^3-3f^4+2f^5+8f^6+7f^7+7f^8+f^{10}+21f^{11}+61f^{12}+$$
$$+81f^{13}+85f^{14}+58f^{15}+35f^{16}+20f^{17}+15f^{18}+11f^{19}+11f^{20}+5f^{21}-$$
$$-f^{22}-3f^{23})+b^6(-f^3+f^4+2f^5+2f^6+3f^7+11f^8+18f^9+29f^{10}+43f^{11}+$$
$$+51f^{12}+41f^{13}+32f^{14}+18f^{15}+15f^{16}+13f^{17}+10f^{18}+3f^{19}-5f^{21}-6f^{22}-$$
$$-3f^{23}+f^{24})+b^7(2f^4+2f^5-3f^7-3f^8-2f^9+5f^{10}+9f^{11}+f^{12}-16f^{13}-$$
$$-25f^{14}-27f^{15}-17f^{16}-10f^{17}-11f^{18}-15f^{19}-15f^{20}-12f^{21}-5f^{22}+$$
$$+f^{23}+2f^{24})+b^8(-f^5-f^6+f^8-f^9-4f^{10}-11f^{11}-16f^{12}-17f^{13}-13f^{14}-$$
$$-13f^{15}-13f^{16}-17f^{17}-16f^{18}-11f^{19}-4f^{20}-f^{21}+f^{22}-f^{24}-f^{25}),$$

$R_2(b,f) = -2f - f^2 + 5f^3 + 12f^4 + 15f^5 + 15f^6 + 11f^7 + 10f^8 + 17f^9+$
$+27f^{10} + 25f^{11} + 16f^{12} - f^{13} - 9f^{14} - 5f^{15} + 2f^{16} + 3f^{17} + 3f^{18} - 2f^{20}-$
$-2f^{21} + b(f + 9f^2 + 12f^3 + 3f^4 - 4f^5 - 3f^6 + 11f^7 + 28f^8 + 34f^9 + 10f^{10}-$
$-22f^{11} - 36f^{12} - 29f^{13} - 3f^{14} + 12f^{15} - f^{16} - 12f^{17} - 11f^{18} - 7f^{19} + 4f^{21}+$
$+4f^{22}) + b^2(2f + 6f^2 - 4f^3 - 15f^4 - 13f^5 - 7f^6 - 6f^7 - 20f^8 - 53f^9 - 92f^{10}-$
$-107f^{11} - 84f^{12} - 44f^{13} + f^{14} + 9f^{15} - 3f^{16} + f^{17} + 4f^{18} + f^{19} + 3f^{20} + 4f^{21}+$
$+2f^{22} - 2f^{23}) + b^3(f - f^2 - 11f^3 - 10f^4 + f^5 - 3f^6 - 26f^7 - 64f^8 - 89f^9-$
$-85f^{10} - 35f^{11} + 31f^{12} + 59f^{13} + 46f^{14} - 3f^{15} - 18f^{16} + 9f^{17} + 25f^{18}+$
$+24f^{19} + 19f^{20} + 5f^{21} - 6f^{22} - 8f^{23}) + b^4(-3f^2 - 9f^3 + 9f^5 + f^6 - 8f^7-$
$-13f^8 - 4f^9 + 15f^{10} + 65f^{11} + 99f^{12} + 99f^{13} + 75f^{14} + 42f^{15} + 35f^{16} + 34f^{17}+$
$+6f^{18} - 6f^{19} - 6f^{20} - 8f^{21} - 8f^{22} - 2f^{23} + 4f^{24}) + b^5(-2f^2 + 12f^4 + 11f^5-$
$-3f^6 - 6f^7 + 9f^8 + 45f^9 + 97f^{10} + 139f^{11} + 118f^{12} + 52f^{13} - f^{14} - 20f^{15}+$
$+7f^{16} + 16f^{17} - 11f^{19} - 19f^{20} - 23f^{21} - 12f^{22} + 2f^{23} + 6f^{24}) + b^6(4f^3+$
$+7f^4 - f^5 - 7f^6 + 4f^7 + 19f^8 + 37f^9 + 45f^{10} + 31f^{11} - 20f^{12} - 51f^{13}-$
$-49f^{14} - 27f^{15} - 9f^{16} - 20f^{17} - 40f^{18} - 39f^{19} - 29f^{20} - 14f^{21} + 4f^{22}+$
$+12f^{23} + 6f^{24} - 2f^{25}) + b^7(f^3 - f^4 - 7f^5 - 6f^6 + 2f^7 + 4f^8 - 2f^9 - 23f^{10}-$
$-59f^{11} - 88f^{12} - 74f^{13} - 45f^{14} - 25f^{15} - 28f^{16} - 46f^{17} - 47f^{18} - 19f^{19}+$
$+6f^{20} + 21f^{21} + 21f^{22} + 8f^{23} - 5f^{24} - 5f^{25}) + b^8(-2f^4 - 2f^5 + 2f^6 + 4f^7-$
$-2f^8 - 10f^9 - 20f^{10} - 24f^{11} - 12f^{12} + 6f^{13} + 8f^{14} - 8f^{16} - 6f^{17} + 12f^{18}+$
$+24f^{19} + 20f^{20} + 10f^{21} + 2f^{22} - 4f^{23} - 2f^{24} + 2f^{25} + 2f^{26}) + b^9(f^5 + f^6-$
$-f^8 + f^9 + 4f^{10} + 11f^{11} + 16f^{12} + 17f^{13} + 13f^{14} + 13f^{15} + 13f^{16} + 17f^{17}+$
$+16f^{18} + 11f^{19} + 4f^{20} + f^{21} - f^{22} + f^{24} + f^{25}),$

$R_4(b,f) = -f + 3f^2 + 6f^3 + 5f^4 - 3f^6 - 10f^7 - 13f^8 - 15f^9 - 18f^{10} - 32f^{11}-$
$-41f^{12} - 51f^{13} - 43f^{14} - 29f^{15} - 18f^{16} - 11f^{17} - 3f^{18} - 2f^{19} - 2f^{20} - f^{21}+$
$+f^{22} + b(2f + 6f^2 - 4f^3 - 15f^4 - 13f^5 - 7f^6 - 6f^7 - 20f^8 - 53f^9 - 92f^{10}-$
$-107f^{11} - 84f^{12} - 44f^{13} + f^{14} + 9f^{15} - 3f^{16} + f^{17} + 4f^{18} + f^{19} + 3f^{20} + 4f^{21}+$
$+2f^{22} - 2f^{23}) + b^2(f - 3f^2 - 17f^3 - 11f^4 - 2f^5 - 13f^6 - 44f^7 - 69f^8 - 72f^9-$
$-37f^{10} + 18f^{11} + 81f^{12} + 114f^{13} + 126f^{14} + 96f^{15} + 85f^{16} + 83f^{17} + 50f^{18}+$
$+19f^{19} + 9f^{20} + 5f^{21} - 3f^{23} + f^{24}) + b^3(-5f^2 - 8f^3 + 7f^4 + 5f^5 - 19f^6-$
$-41f^7 - 28f^8 + 24f^9 + 104f^{10} + 190f^{11} + 251f^{12} + 233f^{13} + 175f^{14} + 95f^{15}+$
$+74f^{16} + 52f^{17} + 13f^{18} + f^{19} + 4f^{20} - 4f^{21} - 10f^{22} - 5f^{23} + 4f^{24}) + b^4(-2f^2+$
$+2f^3 + 13f^4 - 4f^5 - 18f^6 + 9f^7 + 72f^8 + 131f^9 + 165f^{10} + 175f^{11} + 133f^{12}+$
$+64f^{13} + 14f^{14} - 17f^{15} - 23f^{16} - 65f^{17} - 98f^{18} - 67f^{19} - 37f^{20} - 23f^{21}-$
$-12f^{22} + 2f^{23} + 5f^{24} - 2f^{25}) + b^5(8f^3 + 11f^4 - 8f^5 - 6f^6 + 23f^7 + 62f^8+$
$+84f^9 + 94f^{10} + 52f^{11} - 48f^{12} - 152f^{13} - 196f^{14} - 197f^{15} - 161f^{16} - 154f^{17}-$
$-115f^{18} - 66f^{19} - 45f^{20} - 29f^{21} - 4f^{22} + 11f^{23} + 5f^{24} - 3f^{25}) + b^6(f^3 - 7f^4-$
$-11f^5 + 15f^6 + 41f^7 + 33f^8 - 13f^9 - 74f^{10} - 140f^{11} - 176f^{12} - 151f^{13}-$
$-114f^{14} - 107f^{15} - 121f^{16} - 130f^{17} - 82f^{18} - 21f^{19} + 13f^{20} + 30f^{21} + 32f^{22}+$
$+17f^{23} - 3f^{24} - 6f^{25} + f^{26}) + b^7(-4f^4 - 3f^5 + 5f^6 - 2f^7 - 23f^8 - 46f^9-$
$-64f^{10} - 80f^{11} - 66f^{12} - 30f^{13} - 10f^{14} + 10f^{16} + 30f^{17} + 66f^{18} + 80f^{19}+$
$+64f^{20} + 46f^{21} + 23f^{22} + 2f^{23} - 5f^{24} + 3f^{25} + 4f^{26}) + b^8(5f^5 + 5f^6 - 8f^7-$
$-21f^8 - 21f^9 - 6f^{10} + 19f^{11} + 47f^{12} + 46f^{13} + 28f^{14} + 25f^{15} + 45f^{16} + 74f^{17}+$
$+88f^{18} + 59f^{19} + 23f^{20} + 2f^{21} - 4f^{22} - 2f^{23} + 6f^{24} + 7f^{25} + f^{26} - f^{27})+$
$+b^9(-2f^6 - f^7 + 5f^8 + 12f^9 + 15f^{10} + 15f^{11} + 11f^{12} + 10f^{13} + 17f^{14} + 27f^{15}+$

$+25f^{16}+16f^{17}-f^{18}-9f^{19}-5f^{20}+2f^{21}+3f^{22}+3f^{23}-2f^{25}-2f^{26}),$

$$R_6(b,f)=3f^2+f^3-5f^4-11f^5-11f^6-15f^7-20f^8-35f^9-58f^{10}-$$
$$-85f^{11}-81f^{12}-61f^{13}-21f^{14}-f^{15}-7f^{17}-7f^{18}-8f^{19}-2f^{20}+3f^{21}+$$
$$+4f^{22}+b(f-f^2-11f^3-10f^4+f^5-3f^6-26f^7-64f^8-89f^9-85f^{10}-$$
$$-35f^{11}+31f^{12}+59f^{13}+46f^{14}-3f^{15}-18f^{16}+9f^{17}+25f^{18}+24f^{19}+$$
$$+19f^{20}+5f^{21}-6f^{22}-8f^{23})+b^2(-5f^2-8f^3+7f^4+5f^5-19f^6-41f^7-$$
$$-28f^8+24f^9+104f^{10}+190f^{11}+251f^{12}+233f^{13}+175f^{14}+95f^{15}+74f^{16}+$$
$$+52f^{17}+13f^{18}+f^{19}+4f^{20}-4f^{21}-10f^{22}-5f^{23}+4f^{24})+b^3(-3f^2+3f^3+$$
$$+8f^4-15f^5-33f^6-6f^7+73f^8+167f^9+226f^{10}+235f^{11}+162f^{12}+$$
$$+29f^{13}-36f^{14}-27f^{15}+45f^{16}+29f^{17}-20f^{18}-43f^{19}-51f^{20}-51f^{21}-$$
$$-22f^{22}+9f^{23}+16f^{24})+b^4(7f^3+4f^4-16f^5-3f^6+45f^7+86f^8+92f^9+$$
$$+67f^{10}+26f^{11}-82f^{12}-174f^{13}-199f^{14}-183f^{15}-160f^{16}-172f^{17}-$$
$$-128f^{18}-50f^{19}-22f^{20}-7f^{21}+14f^{22}+23f^{23}+6f^{24}-8f^{25})+b^5(3f^3-$$
$$-8f^4-12f^5+23f^6+56f^7+53f^8-80f^{10}-194f^{11}-290f^{12}-276f^{13}-$$
$$-176f^{14}-110f^{15}-100f^{16}-137f^{17}-91f^{18}-41f^{19}+2f^{20}+46f^{21}+65f^{22}+$$
$$+35f^{23}-6f^{24}-13f^{25})+b^6(-6f^4+7f^5+25f^6+9f^7-37f^8-79f^9-104f^{10}-$$
$$-112f^{11}-86f^{12}-13f^{13}+16f^{14}-16f^{16}+13f^{17}+86f^{18}+112f^{19}+104f^{20}+$$
$$+79f^{21}+37f^{22}-9f^{23}-25f^{24}-7f^{25}+6f^{26})+b^7(-f^4+6f^5+3f^6-17f^7-$$
$$-32f^8-30f^9-13f^{10}+21f^{11}+82f^{12}+130f^{13}+121f^{14}+107f^{15}+114f^{16}+$$
$$+151f^{17}+176f^{18}+140f^{19}+74f^{20}+13f^{21}-33f^{22}-41f^{23}-15f^{24}+11f^{25}+$$
$$+7f^{26}-f^{27})+b^8(2f^5-6f^6-12f^7-4f^8+14f^9+29f^{10}+39f^{11}+40f^{12}+$$
$$+20f^{13}+9f^{14}+27f^{15}+49f^{16}+51f^{17}+20f^{18}-31f^{19}-45f^{20}-37f^{21}-$$
$$-19f^{22}-4f^{23}+7f^{24}+f^{25}-7f^{26}-4f^{27})+b^9(-f^6+3f^7+6f^8+5f^9-$$
$$-3f^{11}-10f^{12}-13f^{13}-15f^{14}-18f^{15}-32f^{16}-41f^{17}-51f^{18}-43f^{19}-$$
$$-29f^{20}-18f^{21}-11f^{22}-3f^{23}-2f^{24}-2f^{25}-f^{26}+f^{27}),$$

$$R_8(b,f)=2f^2-f^3-4f^4-6f^5-9f^6-18f^7-21f^8-21f^9-23f^{10}-22f^{11}-$$
$$-6f^{12}+6f^{13}+22f^{14}+23f^{15}+21f^{16}+21f^{17}+18f^{18}+9f^{19}+6f^{20}+4f^{21}+$$
$$+f^{22}-2f^{23}+b(-3f^2-9f^3+9f^5+f^6-8f^7-13f^8-4f^9+15f^{10}+65f^{11}+$$
$$+99f^{12}+99f^{13}+75f^{14}+42f^{15}+35f^{16}+34f^{17}+6f^{18}-6f^{19}-6f^{20}-8f^{21}-$$
$$-8f^{22}-2f^{23}+4f^{24})+b^2(-2f^2+2f^3+13f^4-4f^5-18f^6+9f^7+72f^8+$$
$$+131f^9+165f^{10}+175f^{11}+133f^{12}+64f^{13}+14f^{14}-17f^{15}-23f^{16}-65f^{17}-$$
$$-98f^{18}-67f^{19}-37f^{20}-23f^{21}-12f^{22}+2f^{23}+5f^{24}-2f^{25})+b^3(7f^3+4f^4-$$
$$-16f^5-3f^6+45f^7+86f^8+92f^9+67f^{10}+26f^{11}-82f^{12}-174f^{13}-199f^{14}-$$
$$-183f^{15}-160f^{16}-172f^{17}-128f^{18}-50f^{19}-22f^{20}-7f^{21}+14f^{22}+23f^{23}+$$
$$+6f^{24}-8f^{25})+b^4(f^3-8f^4-9f^5+33f^6+61f^7+20f^8-67f^9-145f^{10}-$$
$$-201f^{11}-272f^{12}-243f^{13}-200f^{14}-178f^{15}-181f^{16}-141f^{17}-16f^{18}+$$
$$+81f^{19}+77f^{20}+72f^{21}+55f^{22}+25f^{23}-10f^{24}-9f^{25}+4f^{26})+b^5(-8f^4+$$
$$+f^5+24f^6+7f^7-42f^8-89f^9-123f^{10}-182f^{11}-209f^{12}-130f^{13}-55f^{14}+$$
$$+55f^{16}+130f^{17}+209f^{18}+182f^{19}+123f^{20}+89f^{21}+42f^{22}-7f^{23}-24f^{24}-$$
$$-f^{25}+8f^{26})+b^6(13f^5+6f^6-35f^7-65f^8-46f^9-2f^{10}+41f^{11}+91f^{12}+$$
$$+137f^{13}+100f^{14}+110f^{15}+176f^{16}+276f^{17}+290f^{18}+194f^{19}+80f^{20}-$$
$$-53f^{22}-56f^{23}-23f^{24}+12f^{25}+8f^{26}-3f^{27})+b^7(3f^5-5f^6-11f^7+4f^8+$$

$+29f^9+45f^{10}+66f^{11}+115f^{12}+154f^{13}+161f^{14}+197f^{15}+196f^{16}+152f^{17}+$
$+48f^{18}-52f^{19}-94f^{20}-84f^{21}-62f^{22}-23f^{23}+6f^{24}+8f^{25}-11f^{26}-8f^{27})+$
$+b^8(-6f^6-2f^7+12f^8+23f^9+19f^{10}+11f^{11}-16f^{13}-7f^{14}+20f^{15}+f^{16}-$
$-52f^{17}-118f^{18}-139f^{19}-97f^{20}-45f^{21}-9f^{22}+6f^{23}+3f^{24}-11f^{25}-12f^{26}+$
$+2f^{28})+b^9(3f^7+f^8-5f^9-11f^{10}-11f^{11}-15f^{12}-20f^{13}-35f^{14}-58f^{15}-$
$-85f^{16}-81f^{17}-61f^{18}-21f^{19}-f^{20}-7f^{22}-7f^{23}-8f^{24}-2f^{25}+3f^{26}+4f^{27}),$

$$R_{18-2k}(b,f)=-b^9f^{30}R_{2k}(b^{-1},f^{-1}),\ (k=\overline{0,4}). \qquad (23)$$

In view of equality (7), from Theorem 1 follows

Corollary 1. *The common Hilbert series for the graded algebra of comitants $S_{1,5}$ of the system* (3) ($k=2$) *has the form*

$$H(S_{1,5},u)=\frac{N_{1,5}(u)}{D_{1,5}(u)},$$

where

$$D_{1,5}(u)=(1-u)^6(1+u)^7(1+u^2)^2(1-u^3)^5(1-u^5)^3(1-u^7),$$

$$N_{1,5}(u)=1+2u+u^2+12u^4+47u^5+119u^6+234u^7+415u^8+697u^9+$$
$$+1145u^{10}+1773u^{11}+2577u^{12}+3481u^{13}+4411u^{14}+5268u^{15}+6000u^{16}+$$
$$+6487u^{17}+6667u^{18}+6487u^{19}+6000u^{20}+5268u^{21}+4411u^{22}+3481u^{23}+$$
$$+2577u^{24}+1773u^{25}+1145u^{26}+697u^{27}+415u^{28}+234u^{29}+119u^{30}+$$
$$+47u^{31}+12u^{32}+u^{34}+2u^{35}+u^{36}.$$

It is known that the point $u=1$ is a pole of function $H_{S_{1,5}}(u)$ of multiplicity k ($k\geq 1$) if and only if this point is a zero of multiplicity k for the function $\dfrac{1}{H_{S_{1,5}}(u)}$. It is easy to see that

$$\frac{1}{H_{S_{1,5}}(u)}=\frac{(1-u)^{15}}{N_{1,5}(u)(1+u)^{-7}(1+u^2)^{-2}(1+u+u^2)^{-5}}\times$$
$$\times\frac{1}{(1+u+u^2+u^3+u^4)^{-3}(1+u+u^2+u^3+u^4+u^5+u^6)^{-1}}$$

Thence we find that

$$\lim_{u\to 1}(1-u)^{15}\cdot H_{S_{1,5}}(u)\neq 0,$$

which means that at the point $u=1$ the function $H_{S_{1,5}}(u)$ has a pole of multiplicity 15.

With the help of Remark 1 and Corollary 1 we obtain

Theorem 2. *The Krull dimension of a graded algebra $S_{1,5}$ is* 15.

Considering the equality (8) from Theorem 1 follows

Corollary 2. *The generalized Hilbert series for the graded algebra of invariants $SI_{1,5}$ of the system* (3) $(k = 2)$ *is a rational function of* b, f *and has the form*

$$H(SI_{1,5}, b, f) = \frac{NI_{1,5}(b, f)}{DI_{1,5}(b, f)}, \tag{24}$$

where

$$\begin{gathered} DI_{1,5}(b, f) = (1 - b)(1 - b^2)(1 - bf)^2(1 - b^2f)^2(1 - b^3f)(1 + f)(1 - f^2)^2\times \\ \times(1 - f^3)^3(1 - f^4)^2(1 - f^5)^2, \\ NI_{1,5}(b, f) = R_0(b, f), \end{gathered} \tag{25}$$

and $R_0(b, f)$ *is from* (23).

With the help of equalities (8)–(9) and Corollary 2 we obtain

Corollary 3. *The common Hilbert series for the graded algebra of invariants $SI_{1,5}$ of the system* (3) $(k = 2)$ *has the form*

$$H_{SI_{1,5}}(z) = \frac{NI_{1,5}(z)}{DI_{1,5}(z)},$$

where

$$\begin{gathered} DI_{1,5} = (1 - z^2)^4(1 - z^3)^4(1 - z^4)^3(1 - z^5)^2, \\ NI_{1,5}(z) = 1 + z + 9z^4 + 22z^5 + 50z^6 + 79z^7 + 120z^8 + 160z^9 + \\ +221z^{10} + 269z^{11} + 325z^{12} + 339z^{13} + 325z^{14} + 269z^{15} + 221z^{16} + \\ +160z^{17} + 120z^{18} + 79z^{19} + 50z^{20} + 22z^{21} + 9z^{22} + z^{25} + z^{26}. \end{gathered}$$

With the help of Remark 1 and Corollary 3 we have the following result.

Theorem 3. *The Krull dimension of the graded algebra $SI_{1,5}$ is* 13.

With the help of equalities (10), (16)–(17) and (24)–(25) we obtain

Lemma 2. *Between the generalized Hilbert series of the algebras SI_1, S_5 and $SI_{1,5}$ the following relation exists*

$$H(SI_{1,5}, b, f) = H(SI_1, b)H(S_5, u, d)|_{u^2=b}. \tag{26}$$

In view of (15) and (26) we can assume that between the generalized Hilbert series of algebras SI_1, S_{2k+1} and $SI_{1,2k+1}$ there exists the relation

$$H(SI_{1,2k+1}, b, z) = H(SI_1, b)H(S_{2k+1}, u, z)|_{u^2=b} \tag{27}$$

for all $k \geq 1$.

3 Obtaining generators of the algebra $SI_{1,2k+1}$ with the help of generators of the algebras SI_1 and S_{2k+1} ($k \geq 1$)

In [3, p. 97] it is shown that the center-affine invariants (comitants) of the system (1) with respect to the transformation $q \in GL(2, \mathbb{R})$ coincide with the center-affine invariants (comitants) of the system of tensors $a^j_{j_1 j_2 \ldots j_\gamma}$ ($\gamma \in \Gamma$).

Consequently, for the system (3), the invariants of the tensor a^j_α form the algebra SI_1, and the comitants and invariants of the tensor $a^j_{\alpha_1\alpha_2\ldots\alpha_{2k+1}}$ form the algebra S_{2k+1}. Note that the associate invariants of the tensors a^j_α and $a^j_{\alpha_1\alpha_2\ldots\alpha_{2k+1}}$ are included in the algebra $SI_{1,2k+1}$.

Construction 1. *Since the above algebras are finitely defined,* (*see* [6, 7]), *then they can be written as*

$$SI_1 =< I_1, I_2 >,$$
$$S_{2k+1} =< J_1, J_2, ..., J_\lambda, P_1, P_2, ..., P_\mu | f_1, f_2, ..., f_\nu >, \tag{28}$$
$$SI_{1,2k+1} =< I_1, I_2, J_1, J_2, ..., J_\lambda, J_{\lambda+1}, ..., J_{\lambda+\tau} | \varphi_1, \varphi_2, ..., \varphi_\omega >,$$

where

$$I_1 = a^\alpha_\alpha,\ I_2 = a^\alpha_\beta a^\beta_\alpha, \tag{29}$$

$J_1, J_2, ..., J_\lambda, P_1, P_2, ..., P_\mu$ *are generators and* $f_1, f_2, ..., f_\nu$ *are defining relations of the algebra* S_{2k+1}. *We observe that* P_i ($i = \overline{1,\mu}$) *has even degree with respect to* x^1, x^2 [5, 6]. *The algebra* $SI_{1,2k+1}$ *has generators* $I_1, I_2, J_1, J_2, ..., J_\lambda, J_{\lambda+1}, ..., J_{\lambda+\tau}$ *and defining relations* $\varphi_1, \varphi_2, ..., \varphi_\omega$. *From equality* (27) *and proprieties of the generalized Hilbert series and generating functions* [5, 6] *it follows that* $\tau = \mu$ *and if the comitants* $P_1, P_2, ..., P_\mu$ *are written in simple tensor form* [4], *then replacing in them the product* $x^- x^-$ *by* $a^-_\ominus \varepsilon^{\ominus -}$ *we obtain respectively generators* $J_{\lambda+1}, J_{\lambda+2}, ..., J_{\lambda+\mu}$, *where* $\ominus$ *means the operation of convolution of given indices.*

The validity of Construction 1 is confirmed by the following arguments: Let in (28) $\tau > \mu$. For clarity, assume that in (28) $\tau = \mu + 1$ and the invariant $J_{\lambda+\mu+1}$ is given in a simple tensor form [4]. This means that the simple tensor form of this invariant consists of a product of tensor's coordinates a^-_- and $a^-_{--\ldots--}$. From this invariant, after replacing all a^-_- by $x^\alpha x^- \varepsilon_{\alpha -}$, avoiding the mistakes in the index notation, we obtain the comitant $P_{\lambda+\mu+1}$ from S_{2k+1}. If this comitant is different from P_j ($j = \overline{1,\mu}$), it should be expressed polynomially via generators J_i ($i = \overline{1,\lambda}$), P_j ($j = \overline{1,\mu}$) of algebra S_{2k+1}. Then, making the inverse transition from obtained polynomial expression $P_{\lambda+\mu+1}$ according to the Construction 1, we conclude that the expression $J_{\lambda+\mu+1}$ is polynomially expressed through $I_1, I_2, J_1, ..., J_\lambda, J_{\lambda+1}, ..., J_{\lambda+\mu}$. Therefore it follows that $J_{\lambda+\mu+1}$ can not be found among the generators of algebra $SI_{1,2k+1}$. This contradiction shows that $\tau = \mu$.

The process described above will be illustrated on algebras SI_1, S_3 and $SI_{1,3}$. From [4] follows

Proposition 1. *The following comitants and invariants are generators of the algebra* S_3

$$
\begin{gathered}
J_1=a^\alpha_{\alpha pr}a^\beta_{\beta qs}\varepsilon^{pq}\varepsilon^{rs},\ J_2=a^\alpha_{\beta pr}a^\beta_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs},\ J_3=a^\alpha_{pru}a^\beta_{\alpha qs}a^\gamma_{\beta\gamma v}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv},\\
J_4=a^\alpha_{pru}a^\beta_{\gamma qs}a^\gamma_{\alpha\beta v}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv},\ J_5=a^\alpha_{pru}a^\beta_{qsk}a^\gamma_{\alpha vl}a^\delta_{\beta\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{kl},\\
J_6=a^\alpha_{prk}a^\beta_{qsm}a^\gamma_{luh}a^\delta_{\alpha nv}a^\mu_{\beta\gamma j}a^\nu_{\delta u\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}\varepsilon^{mn}\varepsilon^{uv}\varepsilon^{hj},\ P_1=a^\alpha_{\alpha\beta\gamma}x^\beta x^\gamma,\\
P_2=a^p_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma x^q\varepsilon_{pq},\ P_3=a^\alpha_{p\alpha\beta}a^\beta_{q\gamma\delta}x^\gamma x^\delta\varepsilon^{pq},\ P_4=a^\alpha_{\alpha\beta\gamma}a^\beta_{\delta\mu\theta}x^\gamma x^\delta x^\mu x^\theta,\\
P_5=a^\alpha_{\beta\gamma\delta}a^\beta_{\alpha\mu\theta}x^\gamma x^\delta x^\mu x^\theta,\ P_6=a^\alpha_{\alpha pr}a^\beta_{\gamma\delta q}a^\gamma_{\beta\nu s}x^\delta x^\nu\varepsilon^{pq}\varepsilon^{rs},\\
P_7=a^\alpha_{\delta pr}a^\beta_{\alpha\beta q}a^\gamma_{\gamma\nu s}x^\delta x^\nu\varepsilon^{pq}\varepsilon^{rs},\ P_8=a^\alpha_{prk}a^\beta_{\alpha qs}a^\gamma_{\beta\delta l}a^\delta_{\gamma\mu\theta}x^\mu x^\theta\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},\\
P_9=a^\alpha_{\alpha\beta p}a^\beta_{\gamma q\delta}a^\gamma_{\eta\mu\theta}x^\delta x^\eta x^\mu x^\theta\varepsilon^{pq},\ P_{10}=a^\alpha_{\beta\gamma\delta}a^\beta_{\alpha\eta\mu}a^\gamma_{\nu\theta\chi}x^\delta x^\eta x^\mu x^\nu x^\theta x^\chi,\\
P_{11}=a^\alpha_{pr\gamma}a^\beta_{q\delta\eta}a^\gamma_{s\beta\nu}a^\delta_{\alpha\mu\theta}x^\eta x^\nu x^\mu x^\theta\varepsilon^{pq}\varepsilon^{rs},\\
P_{12}=a^\alpha_{pru}a^\beta_{mgk}a^\gamma_{\mu qn}a^\delta_{\theta sh}a^\eta_{\eta vl}x^\mu x^\theta\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon^{kl},\\
(\varepsilon_{11}=\varepsilon_{22}=0,\ \varepsilon_{12}=-\varepsilon_{21}=1;\ \varepsilon^{11}=\varepsilon^{22}=0,\ \varepsilon^{12}=-\varepsilon^{21}=1).
\end{gathered}
\tag{30}
$$

From [9] we obtain

Proposition 2. *The set of the generators of the algebra $SI_{1,3}$ consists of the invariants $I_1,\ I_2$ from* (29)*, J_1-J_6 from* (30)*, and the expressions*

$$
\begin{gathered}
J_7=a^\alpha_p a^\beta_{q\alpha\beta}\varepsilon^{pq},\ J_8=a^\alpha_p a^\beta_\gamma a^\gamma_{q\alpha\beta}\varepsilon^{pq},\ J_9=a^\alpha_p a^\beta_{qr\alpha}a^\gamma_{s\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\\
J_{10}=a^\alpha_p a^\beta_r a^\gamma_{q\alpha\beta}a^\delta_{s\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},J_{11}=a^\alpha_p a^\beta_r a^\gamma_{q\alpha\delta}a^\delta_{s\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\\
J_{12}=a^\alpha_p a^\beta_{qru}a^\gamma_{sv\alpha}a^\delta_{\beta\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv},\ J_{13}=a^\alpha_p a^\beta_{qru}a^\gamma_{sv\beta}a^\delta_{\alpha\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv},\\
J_{14}=a^\alpha_p a^\beta_r a^\gamma_{mqu}a^\delta_{hnv}a^\mu_{sg\mu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma},\\
J_{15}=a^\alpha_p a^\beta_{ru\beta}a^\gamma_{mgs}a^\delta_{qkn}a^\eta_{vhl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon^{kl}\varepsilon_{\alpha\delta}\varepsilon_{\gamma\eta},\\
J_{16}=a^\alpha_p a^\beta_r a^\gamma_u a^\delta_{qsg}a^\eta_{bmh}a^\mu_{vnc}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon^{bc}\varepsilon_{\alpha\eta}\varepsilon_{\beta\mu}\varepsilon_{\gamma\delta},\\
J_{17}=a^\alpha_p a^\beta_r a^\gamma_{qum}a^\delta_{sgb}a^\eta_{vhk}a^\mu_{ncl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon^{bc}\varepsilon^{kl}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\mu}\varepsilon_{\beta\eta},\\
J_{18}=a^\alpha_p a^\beta_{ruk}a^\gamma_{sg\alpha}a^\delta_{qvb}a^\mu_{ce\delta}a^\eta_{ldh}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{gh}\varepsilon^{kl}\varepsilon^{bc}\varepsilon^{de}\varepsilon_{\beta\eta}\varepsilon_{\gamma\mu},\\
(\varepsilon_{11}=\varepsilon_{22}=0,\ \varepsilon_{12}=-\varepsilon_{21}=1;\ \varepsilon^{11}=\varepsilon^{22}=0,\ \varepsilon^{12}=-\varepsilon^{21}=1).
\end{gathered}
\tag{31}
$$

We will show how with the help of comitants P_1-P_{12} from (30), we can obtain the generators (31) of the algebra $SI_{1,3}$ or expressions containing them in the sense indicated below. For this we use the substitutions given in Construction 1.

1) We take as the most typical example the comitant P_2 from (30) for which all possible substitutions $x^- x^- = a^-_\ominus\varepsilon^{\ominus -}$ lead to the equalities:

$$
\begin{gathered}
P_2\Big|_{\substack{x^\alpha x^\beta=a^\alpha_\delta\varepsilon^{\delta\beta}\\ x^\gamma x^q=a^q_r\varepsilon^{r\gamma}}}=a^\alpha_\delta a^q_r a^p_{\alpha\beta\gamma}\varepsilon^{\delta\beta}\varepsilon^{r\gamma}\varepsilon_{pq}=J_8-I_1J_7;\\
P_2\Big|_{\substack{x^\alpha x^\beta=a^\alpha_\delta\varepsilon^{\delta\beta}\\ x^\gamma x^q=a^\gamma_r\varepsilon^{rq}}}=a^\alpha_\delta a^\gamma_r a^p_{\alpha\beta\gamma}\varepsilon^{\delta\beta}\varepsilon^{rq}\varepsilon_{pq}=J_8;
\end{gathered}
\tag{32}
$$

Since J_8 from (31) is contained in all above mentioned equalities, then one can keep J_8 as a generator of algebra $SI_{1,3}$ or take the first expression from (32).

2) Similarly, for P_1 all possible substitutions $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ leads to the equality:

$$P_1\Big|_{x^\beta x^\gamma = a^\beta_p \varepsilon^{p\gamma}} = a^\beta_p a^\alpha_{\alpha\beta\gamma} \varepsilon^{p\gamma} = J_7, \tag{33}$$

which is contained in the set of generators of algebra $SI_{1,3}$ from (31).

3) For P_3 each substitution of the type $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ leads to the equality:

$$P_3\Big|_{x^\gamma x^\delta = a^\gamma_r \varepsilon^{r\delta}} = a^\gamma_r a^\alpha_{p\alpha\beta} a^\beta_{q\gamma\delta} \varepsilon^{pq} \varepsilon^{r\delta} = -J_9, \tag{34}$$

where J_9 is contained in the set of generators of algebra $SI_{1,3}$ from (31).

4) For P_4 and P_5 all possible substitutions $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ lead to one of the equalities:

$$\begin{aligned}
&P_4\Big|_{\substack{x^\gamma x^\delta = a^\gamma_p \varepsilon^{p\delta}\\ x^\mu x^\theta = a^\mu_r \varepsilon^{r\theta}}} = a^\gamma_p a^\mu_r a^\alpha_{\alpha\beta\gamma} a^\beta_{\delta\mu\theta} \varepsilon^{p\delta} \varepsilon^{r\theta} = J_{10} - I_1 J_9;\\
&P_4\Big|_{\substack{x^\gamma x^\delta = a^\delta_p \varepsilon^{p\gamma}\\ x^\mu x^\theta = a^\mu_r \varepsilon^{r\theta}}} = a^\delta_p a^\mu_r a^\alpha_{\alpha\beta\gamma} a^\beta_{\delta\mu\theta} \varepsilon^{p\gamma} \varepsilon^{r\theta} = J_{10};\\
&P_5\Big|_{\substack{x^\gamma x^\delta = a^\gamma_p \varepsilon^{p\delta}\\ x^\mu x^\theta = a^\mu_r \varepsilon^{r\theta}}} = a^\gamma_p a^\mu_r a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\mu\theta} \varepsilon^{p\delta} \varepsilon^{r\theta} = J_{11};\\
&P_5\Big|_{\substack{x^\gamma x^\mu = a^\gamma_p \varepsilon^{p\mu}\\ x^\delta x^\theta = a^\delta_r \varepsilon^{r\theta}}} = a^\gamma_p a^\delta_r a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\mu\theta} \varepsilon^{p\mu} \varepsilon^{r\theta} = J_{11} + \tfrac{1}{2} I_2 J_2;\\
&P_5\Big|_{\substack{x^\gamma x^\mu = a^\gamma_p \varepsilon^{p\mu}\\ x^\delta x^\theta = a^\theta_r \varepsilon^{r\delta}}} = a^\gamma_p a^\theta_r a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\mu\theta} \varepsilon^{p\mu} \varepsilon^{r\delta} = J_{11} + \tfrac{1}{2} J_2 (I_2 - I_1^2).
\end{aligned} \tag{35}$$

We observe that every expression above contains either J_{10} or J_{11} from (31). Therefore, one can keep J_{10} and J_{11} as generators of algebra $SI_{1,3}$ or take one of the above mentioned expressions with J_{10} and another one J_{11} from (35).

5) For P_6, P_7, P_8 all possible substitutions $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ lead to one of the equalities:

$$\begin{aligned}
&P_6\Big|_{x^\delta x^\nu = a^\delta_k \varepsilon^{k\nu}} = a^\delta_k a^\alpha_{\alpha pr} a^\beta_{\gamma\delta q} a^\gamma_{\beta\nu s} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{k\nu} = -J_{12} + I_1 J_3 + \tfrac{1}{2} J_1 J_7 -\\
&\qquad - J_2 J_7 - 2J_{13};\\
&P_7\Big|_{x^\delta x^\nu = a^\delta_k \varepsilon^{k\nu}} = a^\delta_k a^\alpha_{\delta pr} a^\beta_{\alpha\beta q} a^\gamma_{\gamma\nu s} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{k\nu} = J_{13} + I_1 J_3 - \tfrac{1}{2} J_2 J_7;\\
&P_7\Big|_{x^\delta x^\nu = a^\nu_k \varepsilon^{k\delta}} = a^\nu_k a^\alpha_{\delta pr} a^\beta_{\alpha\beta q} a^\gamma_{\gamma\nu s} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{k\delta} = J_{13} - \tfrac{1}{2} I_2 J_7;\\
&P_8\Big|_{x^\mu x^\theta = a^\mu_u \varepsilon^{u\theta}} = a^\mu_u a^\alpha_{prk} a^\beta_{\alpha qs} a^\gamma_{\beta\delta l} a^\delta_{\gamma\mu\theta} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl} \varepsilon^{u\theta} = -J_{15} + I_1 J_5 -\\
&\qquad - \tfrac{1}{2} J_1 J_9 + \tfrac{1}{2} J_2 J_9 + \tfrac{1}{3} J_7 (J_3 + 2J_4),
\end{aligned} \tag{36}$$

where the generators $J_{12},\ J_{13},\ J_{15}$ of algebra $SI_{1,3}$ from (31) are contained in the expressions (36). Therefore, one can keep $J_{12},\ J_{13},\ J_{15}$ as generators of algebra $SI_{1,3}$, or take instead of these one of the above mentioned expressions containing $J_{12},\ J_{13},\ J_{15}$.

6) For P_9 all possible substitutions $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ lead to one of the equalities:

$$\begin{aligned} P_9\Big|_{\substack{x^\delta x^\eta = a^\delta_r \varepsilon^{r\eta} \\ x^\mu x^\theta = a^\mu_k \varepsilon^{k\theta}}} &= a^\delta_r a^\mu_k a^\alpha_{\alpha\beta p} a^\beta_{\gamma q\delta} a^\gamma_{\eta\mu\theta} \varepsilon^{pq}\varepsilon^{r\eta}\varepsilon^{k\theta} = -J_{14} - \tfrac{1}{2} I_1^2 J_3 + \\ &+ \tfrac{1}{2} I_1 J_1 J_7 + \tfrac{1}{2} I_1 J_2 J_7 - I_1 J_{13} - \tfrac{1}{2}(I_2 J_3 - J_1 J_8) - J_7 J_9; \\ P_9\Big|_{\substack{x^\delta x^\eta = a^\eta_r \varepsilon^{r\delta} \\ x^\mu x^\theta = a^\mu_k \varepsilon^{k\theta}}} &= a^\eta_r a^\mu_k a^\alpha_{\alpha\beta p} a^\beta_{\gamma q\delta} a^\gamma_{\eta\mu\theta} \varepsilon^{pq}\varepsilon^{r\delta}\varepsilon^{k\theta} = -J_{14} + \tfrac{1}{2} I_1^2 J_3 + \\ &+ \tfrac{1}{2} I_1 J_1 J_7 - I_1 (J_{12} - J_{13}) - \tfrac{1}{2}(I_2 J_3 + J_1 J_8) - J_7 J_9, \end{aligned} \tag{37}$$

where the generator J_{14} of algebra $SI_{1,3}$ from (31) is contained in the expressions (37). Therefore, one can either keep J_{14} as a generator of algebra $SI_{1,3}$, or take one of the above mentioned expressions from (37).

7) For P_{10} all possible substitutions $x^- x^- = a^-_\ominus \varepsilon^{\ominus -}$ lead to one of the equalities:

$$\begin{aligned} P_{10}\Big|_{\substack{x^\delta x^\eta = a^\delta_p \varepsilon^{p\eta} \\ x^\mu x^\nu = a^\mu_r \varepsilon^{r\nu} \\ x^\theta x^\chi = a^\theta_u \varepsilon^{u\chi}}} &= a^\delta_p a^\mu_r a^\theta_u a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi} \varepsilon^{p\eta}\varepsilon^{r\nu}\varepsilon^{u\chi} = -\tfrac{1}{2} J_{16} + \tfrac{1}{4}(4 I_1^3 J_3 - I_1^3 J_4 - \\ &- 3 I_1^2 J_1 J_7 - 5 I_1^2 J_2 J_7 + I_1^2 J_{12} + 4 I_1^2 J_{13} + I_1 I_2 J_4 + 2 I_1 J_1 J_8 + \\ &+ 4 I_1 J_7 J_9 + 8 I_1 J_{14} + I_2 J_1 J_7 + 3 I_2 J_2 J_7 - I_2 J_{12} - 4 J_7 J_{10} + \\ &+ 8 J_7 J_{11} - 4 J_8 J_9); \\ P_{10}\Big|_{\substack{x^\delta x^\eta = a^\delta_p \varepsilon^{p\eta} \\ x^\mu x^\nu = a^\nu_r \varepsilon^{r\mu} \\ x^\theta x^\chi = a^\theta_u \varepsilon^{u\chi}}} &= a^\delta_p a^\nu_r a^\theta_u a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi} \varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2} J_{16} + \tfrac{1}{12}(10 I_1^3 J_3 - I_1^3 J_4 - \\ &- 3 I_1^2 J_1 J_7 - 9 I_1^2 J_2 J_7 - 3 I_1^2 J_{12} + 12 I_1^2 J_{13} - 2 I_1 I_2 J_3 - I_1 I_2 J_4 + \\ &+ 12 I_1 J_7 J_9 + 12 I_1 J_{14} + 3 I_2 J_1 J_7 + 9 I_2 J_2 J_7 - 3 I_2 J_{12} - 12 J_7 J_{10} + \\ &+ 24 J_7 J_{11} - 12 J_8 J_9); \\ P_{10}\Big|_{\substack{x^\delta x^\eta = a^\eta_p \varepsilon^{p\delta} \\ x^\mu x^\nu = a^\mu_r \varepsilon^{r\nu} \\ x^\theta x^\chi = a^\theta_u \varepsilon^{u\chi}}} &= a^\eta_p a^\mu_r a^\theta_u a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi} \varepsilon^{p\delta}\varepsilon^{r\nu}\varepsilon^{u\chi} = -\tfrac{1}{2} J_{16} + \tfrac{1}{4}(4 I_1^3 J_3 - I_1^3 J_4 - \\ &- 3 I_1^2 J_1 J_7 - 3 I_1^2 J_2 J_7 + I_1^2 J_{12} + 4 I_1^2 J_{13} + I_1 I_2 J_4 + 2 I_1 J_1 J_8 - \\ &- 2 I_1 J_2 J_8 + 4 I_1 J_7 J_9 + 8 I_1 J_{14} + I_2 J_1 J_7 + 3 I_2 J_2 J_7 - I_2 J_{12} - \\ &- 4 J_7 J_{10} + 8 J_7 J_{11} - 4 J_8 J_9); \\ P_{10}\Big|_{\substack{x^\delta x^\eta = a^\eta_p \varepsilon^{p\delta} \\ x^\mu x^\nu = a^\nu_r \varepsilon^{r\mu} \\ x^\theta x^\chi = a^\theta_u \varepsilon^{u\chi}}} &= a^\eta_p a^\nu_r a^\theta_u a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi} \varepsilon^{p\delta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2} J_{16} + \tfrac{1}{12}(-3 I_1^2 J_{12} - 3 I_2 J_{12} + \\ &+ 12 I_1^2 J_{13} + 12 I_1 J_{14} + 10 I_1^3 J_3 - 2 I_1 I_2 J_3 - I_1^3 J_4 - I_1 I_2 J_4 - \\ &- 3 I_1^2 J_1 J_7 + 3 I_2 J_1 J_7 - 12 J_7 J_{10} + 24 J_7 J_{11} - 9 I_1^2 J_2 J_7 + 9 I_2 J_2 J_7 - \\ &- 6 I_1 J_2 J_8 + 12 I_1 J_7 J_9 - 12 J_8 J_9); \end{aligned}$$

$$\begin{aligned}
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{12}(12I_1^2J_{12} + 12I_1^2J_{13}+\\
&\quad +36I_1J_{14} + 14I_1^3J_3 + 2I_1I_2J_3 - 2I_1^3J_4 - 2I_1I_2J_4 - 15I_1^2J_1J_7+\\
&\quad +3I_2J_1J_7 - 12J_7J_{10} + 24J_7J_{11} - 15I_1^2J_2J_7 + 9I_2J_2J_7 + 12I_1J_1J_8+\\
&\quad +12I_1J_7J_9 - 12J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{4}(4I_1^2J_{13} + 8I_1J_{14}+\\
&\quad +4I_1^3J_3 - 3I_1^2J_1J_7 + I_2J_1J_7 - 4J_7J_{10} + 8J_7J_{11} - 5I_1^2J_2J_7+\\
&\quad +3I_2J_2J_7 + 2I_1J_1J_8 + 4I_1J_7J_9 - 4J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{12}(-6I_1^2J_{12} + 12I_1^2J_{13}+\\
&\quad +12I_1J_{14} + 10I_1^3J_3 - 2I_1I_2J_3 - 4I_1^3J_4 + 2I_1I_2J_4 - 3I_1^2J_1J_7+\\
&\quad +3I_2J_1J_7 - 12J_7J_{10} + 24J_7J_{11} - 9I_1^2J_2J_7 + 9I_2J_2J_7 + 12I_1J_7J_9-\\
&\quad -12J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{4}(4I_1^2J_{13} + 8I_1J_{14}+\\
&\quad +4I_1^3J_3 - 3I_1^2J_1J_7 + I_2J_1J_7 - 4J_7J_{10} + 8J_7J_{11} - 3I_1^2J_2J_7+\\
&\quad +3I_2J_2J_7 + 2I_1J_1J_8 - 2I_1J_2J_8 + 4I_1J_7J_9 - 4J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{12}(-6I_1^2J_{12} + 12I_1^2J_{13}+\\
&\quad +12I_1J_{14} + 10I_1^3J_3 - 2I_1I_2J_3 - 4I_1^3J_4 + 2I_1I_2J_4 - 3I_1^2J_1J_7+\\
&\quad +3I_2J_1J_7 - 12J_7J_{10} + 24J_7J_{11} - 9I_1^2J_2J_7 + 9I_2J_2J_7 - 6I_1J_2J_8+\\
&\quad +12I_1J_7J_9 - 12J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{12}(-6I_1^2J_{12} + 12I_1^2J_{13}+\\
&\quad +8I_1^3J_3 - 4I_1I_2J_3 - 2I_1^3J_4 + 4I_1I_2J_4 + 3I_1^2J_1J_7 + 3I_2J_1J_7-\\
&\quad -12J_7J_{10} + 24J_7J_{11} - 3I_1^2J_2J_7 + 9I_2J_2J_7 - 6I_1J_1J_8 - 6I_1J_2J_8+\\
&\quad +12I_1J_7J_9 - 12J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{4}(I_1^2J_{12} - I_2J_{12}+\\
&\quad +4I_1^2J_{13} + 8I_1J_{14} + 4I_1^3J_3 - I_1^3J_4 + I_1I_2J_4 - 3I_1^2J_1J_7+\\
&\quad +I_2J_1J_7 - 4J_7J_{10} + 8J_7J_{11} - 4I_1^2J_2J_7 + 2I_2J_2J_7 + 2I_1J_1J_8+\\
&\quad +4I_1J_7J_9 - 4J_8J_9);\\
P_{10}\Big|_{\substack{x^\delta x^\eta=a_p^\delta\varepsilon^{p\eta}\\ x^\mu x^\nu=a_r^\nu\varepsilon^{r\mu}\\ x^\theta x^\chi=a_u^\theta\varepsilon^{u\chi}}} &= a_p^\delta a_r^\nu a_u^\theta a^\alpha_{\beta\gamma\delta} a^\beta_{\alpha\eta\mu} a^\gamma_{\nu\theta\chi}\varepsilon^{p\eta}\varepsilon^{r\mu}\varepsilon^{u\chi} = -\tfrac{1}{2}J_{16} + \tfrac{1}{12}(-3I_1^2J_{12} - 3I_2J_{12}+\\
&\quad +12I_1^2J_{13} + 12I_1J_{14} + 10I_1^3J_3 - 2I_1I_2J_3 - I_1^3J_4 - I_1I_2J_4-\\
&\quad -3I_1^2J_1J_7 + 3I_2J_1J_7 - 12J_7J_{10} + 24J_7J_{11} - 6I_1^2J_2J_7 + 6I_2J_2J_7-\\
&\quad -6I_1J_2J_8 + 12I_1J_7J_9 - 12J_8J_9).
\end{aligned} \tag{38}$$

So one can either keep J_{16} as a generator of algebra $SI_{1,3}$, or take one of the above mentioned expressions from (38) containing J_{16}.

8) For P_{11} all possible substitutions $x^- x^- = a^-_{\ominus}\varepsilon^{\ominus -}$ lead to one of the equalities:

$$\begin{aligned}
P_{11}\Big|_{\substack{x^\eta x^\nu = a^\eta_b \varepsilon^{b\nu}\\ x^\mu x^\theta = a^\mu_k \varepsilon^{k\theta}}} &= a^\eta_b a^\mu_k a^\alpha_{pr\gamma} a^\beta_{q\delta\eta} a^\gamma_{s\beta\nu} a^\delta_{\alpha\mu\theta} \varepsilon^{b\nu}\varepsilon^{k\theta}\varepsilon^{pq}\varepsilon^{rs} = -4J_{17} + \tfrac{1}{3}(10I_1^2 J_5 + 2I_1J_1J_9 - \\
&-4I_1J_3J_7 + 2I_1J_4J_7 - 6I_1J_{15} - 14I_2J_5 - 4J_1J_{10} + 4J_1J_{11} + \\
&+4J_2J_7^2 + 4J_4J_8 + 2J_7J_{12} + 4J_9^2);\\
P_{11}\Big|_{\substack{x^\eta x^\mu = a^\eta_b \varepsilon^{b\mu}\\ x^\nu x^\theta = a^\nu_k \varepsilon^{k\theta}}} &= a^\eta_b a^\nu_k a^\alpha_{pr\gamma} a^\beta_{q\delta\eta} a^\gamma_{s\beta\nu} a^\delta_{\alpha\mu\theta} \varepsilon^{b\mu}\varepsilon^{k\theta}\varepsilon^{pq}\varepsilon^{rs} = -4J_{17} + \tfrac{1}{3}(10I_1^2 J_5 + 2I_1J_1J_9 + \\
&+4I_1J_3J_7 - 6I_1J_4J_7 - 6I_1J_{15} + 3I_2J_1J_2 - 2I_2J_5 - 4J_1J_{10} + \\
&+4J_1J_{11} + 4J_2J_7^2 + 4J_4J_8 + 2J_7J_{12} + 4J_9^2);\\
P_{11}\Big|_{\substack{x^\eta x^\mu = a^\mu_b \varepsilon^{b\eta}\\ x^\nu x^\theta = a^\nu_k \varepsilon^{k\theta}}} &= a^\mu_b a^\nu_k a^\alpha_{pr\gamma} a^\beta_{q\delta\eta} a^\gamma_{s\beta\nu} a^\delta_{\alpha\mu\theta} \varepsilon^{b\eta}\varepsilon^{k\theta}\varepsilon^{pq}\varepsilon^{rs} = -4J_{17} + \tfrac{1}{3}(-3I_1^2 J_1J_2 - \\
&-2I_1^2J_5 + 2I_1J_1J_9 + 4I_1J_3J_7 - 6I_1J_4J_7 - 6I_1J_{15} + 3I_2J_1J_2 - \\
&-2I_2J_5 - 4J_1J_{10} + 4J_1J_{11} + 4J_2J_7^2 + 4J_4J_8 + 2J_7J_{12} + 4J_9^2);\\
P_{11}\Big|_{\substack{x^\eta x^\mu = a^\eta_b \varepsilon^{b\mu}\\ x^\nu x^\theta = a^\theta_k \varepsilon^{k\nu}}} &= a^\eta_b a^\theta_k a^\alpha_{pr\gamma} a^\beta_{q\delta\eta} a^\gamma_{s\beta\nu} a^\delta_{\alpha\mu\theta} \varepsilon^{b\mu}\varepsilon^{k\nu}\varepsilon^{pq}\varepsilon^{rs} = 4J_{17} - \tfrac{1}{3}(3I_1^2 J_1J_2 + \\
&+2I_1^2J_5 - 2I_1J_1J_9 + 4I_1J_3J_7 - 2I_1J_4J_7 + 6I_1J_{15} - 3I_2J_1J_2 + \\
&+2I_2J_5 + 4J_1J_{10} - 4J_1J_{11} - 4J_2J_7^2 - 4J_4J_8 - 2J_7J_{12} - 4J_9^2);\\
P_{11}\Big|_{\substack{x^\eta x^\mu = a^\mu_b \varepsilon^{b\eta}\\ x^\nu x^\theta = a^\theta_k \varepsilon^{k\nu}}} &= a^\mu_b a^\theta_k a^\alpha_{pr\gamma} a^\beta_{q\delta\eta} a^\gamma_{s\beta\nu} a^\delta_{\alpha\mu\theta} \varepsilon^{b\eta}\varepsilon^{k\nu}\varepsilon^{pq}\varepsilon^{rs} = -4J_{17} + \tfrac{1}{3}(10I_1^2 J_5 + \\
&+2I_1J_1J_9 - 4I_1J_3J_7 + 2I_1J_4J_7 - 6I_1J_{15} + 3I_2J_1J_2 - 2I_2J_5 - \\
&-4J_1J_{10} + 4J_1J_{11} + 4J_2J_7^2 + 4J_4J_8 + 2J_7J_{12} + 4J_9^2).
\end{aligned} \tag{39}$$

Similarly to previous cases one can either keep J_{17} as a generator of algebra $SI_{1,3}$, or take one of the above mentioned expressions from (39) containing J_{17}.

9) For P_{12} all possible substitutions $x^- x^- = a^-_{\ominus}\varepsilon^{\ominus -}$ leads to the equality:

$$\begin{aligned}
P_{12}\Big|_{x^\mu x^\theta = a^\theta_b \varepsilon^{b\mu}} &= a^\alpha_{pru} a^\beta_{mgk} a^\gamma_{\mu qn} a^\delta_{\theta sh} a^\eta_{\eta vl} a^\theta_b \varepsilon^{b\mu} \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{uv}\varepsilon^{mn}\varepsilon^{gh}\varepsilon^{kl} = \\
&= 2J_{18} + \tfrac{1}{3}(-2I_1J_1J_3 + 2I_1J_1J_4 + 2I_1J_2J_3 - 2I_1J_2J_4 + \\
&+3J_1^2J_7 - 6J_1J_2J_7 + 3J_2^2J_7).
\end{aligned} \tag{40}$$

In this case one can either keep J_{18} as a generator of algebra $SI_{1,3}$, or take above mentioned expression from (40) which contains this invariant.

The main conclusion: *The set of generators of the algebra of invariants $SI_{1,3}$ contains the generators of invariants of the algebras SI_1 and SI_3 and also any one invariant of expressions* (32)–(40)*, containing respectively* $J_7, J_8, ..., J_{18}$.

Remark 2. The obtaining of the generators $J_7 - J_{18}$ from (31) of the algebra $SI_{1,3}$ with the help of comitants $P_1 - P_{12}$ of algebra S_3 via the Construction 1 is much easier in comparison with the method presented in [9].

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