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Basic cohomology attached to a basic function of foliated manifolds

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Abstract. In this paper we study a basic cohomology attached to a basic function of foliated manifolds and in particular, of transversely holomorphic foliations. We also explain how this cohomology depends on the basic function and we study a relative cohomology and a Mayer-Vietoris sequence related to this cohomology. Also, a basic Lichnerowicz cohomology attached to a basic function of a foliated manifold is studied.

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1 Introduction and preliminaries

1.1 Introduction

The of basic cohomology both in Riemannian and in Kählerian foliations was intensively studied by A. El Kacimi-Alaoui (see for instance [2,3] and other papers). On the other hand, P. Monnier in [10] introduced a new cohomology of smooth manifolds, so called *cohomology attached to a function*. This cohomology was considered for the first time in [9] in the context of Poisson geometry, and more generally, Nambu-Poisson geometry. The main goal of this note is to give a similar cohomology for basic forms of foliated manifolds. In this sense, firstly we briefly recall some preliminary notions about the basic cohomology of foliated manifolds and in particular, of transversely holomorphic foliations (see [3]). Next, we define a basic cohomology attached to a basic function for basic forms, we define an associated basic Bott-Chern cohomology, we explain how this cohomology depends on the basic function and we study a relative cohomology and a Mayer-Vietoris sequence related to this cohomology. In particular, we show that if the function does not vanish, then our cohomology coincides with the basic de Rham (Dolbeault) cohomology of a foliated manifold. In the end of paper we make a connection between the basic cohomology attached to a basic function and basic Lichnerowicz cohomology defining a basic Lichnerowicz type cohomology attached to a basic function of foliated manifolds. Also, some classical properties adapted to this cohomology are investigated. The methods used here are similar to those used by [10] and are closely related to those used by [1].

1.2 Preliminaries

Let us consider $\mathcal{M}$ an $(n+m)$-dimensional manifold which will be assumed to be connected and orientable. Differential forms (and in particular functions) will take their values in the field of complex numbers $\mathbb{C}$. If φ is a form, then $\overline{\varphi}$ denotes its complex conjugate and we say that φ is *real* if $\overline{\varphi} = \varphi$.

Definition 1. A codimension n foliation $\mathcal{F}$ on $\mathcal{M}$ is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of $\mathcal{M}$;

(ii) For every $i \in I$, $\varphi_i : U_i \to M$ are submersions, where M is an n-dimensional manifold, called transversal manifold;

(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i \tag{1}$$

for every $(i,j) \in I \times I$ such that $U_i \cap U_j \neq \phi$.

Every fibre of φ_i is called a *plaque* of the foliation. Condition (1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by φ_i and φ_j are the "same". The manifold $\mathcal{M}$ is decomposed into a family of disjoint immersed connected submanifolds of dimension m; each of these submanifolds is called a *leaf* of $\mathcal{F}$.

By $T\mathcal{F}$ we denote the tangent bundle to $\mathcal{F}$, and $\Gamma(\mathcal{F})$ is the space of its global sections, i.e. vector fields tangent to $\mathcal{F}$. We say that a differential form φ is *basic* if it satisfies $i_X\varphi = \mathcal{L}_X\varphi = 0$ for every $X \in \Gamma(\mathcal{F})$. A *basic function* is a function constant on the leaves; such functions form an algebra denoted by $\mathcal{F}_b(\mathcal{M})$. The quotient $Q\mathcal{F} = T\mathcal{M}/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. A vector field $Y \in \mathcal{X}(\mathcal{M})$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$ we have $[X, Y] \in \Gamma(\mathcal{F})$; $\mathcal{X}(\mathcal{M}, \mathcal{F})$ denotes the algebra of foliated vector fields on $\mathcal{M}$. The quotient $\mathcal{X}(\mathcal{M}/\mathcal{F}) = \mathcal{X}(\mathcal{M}, \mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on $\mathcal{M}$.

Throughout this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(z^1, \ldots, z^n, y^1, \ldots, y^m)$ on an open set U on which the foliation is trivial and defined by the equations $dz^i = 0$, $i = 1, \ldots, n$. If $\mathcal{F}$ is transversely holomorphic (see Defnition 1.2.2. below) then the coordinates $z^1, \ldots, z^n$ will be complex.

Definition 2. A transverse structure to $\mathcal{F}$ is a geometric structure on M invariant by all the local diffeomorphisms $f_{i,j}$.

A transverse structure can be considered as a geometric structure on the leaf space $\mathcal{M}/\mathcal{F}$ (which is not a manifold in general).

1.2.1. If M is a Riemannian manifold and all the $f_{i,j}$ are isometries then $\mathcal{F}$ is said to be Riemannian. This means that the normal bundle $Q\mathcal{F}$ is equipped with a Riemannian metric which is "invariant along the leaves".

1.2.2. If M is a complex manifold and all the $f_{i,j}$ are biholomorphic maps then we say that $\mathcal{F}$ is transversely holomorphic. In that case, any transversal to $\mathcal{F}$ inherits a complex structure.

1.2.3. If M is a Hermitian manifold and all the $f_{i,j}$ preserve the Hermitian structure then we say that $\mathcal{F}$ is transversely Hermitian. (The $f_{i,j}$ are in particular biholomorphic maps and isometries.) The complexifed normal bundle $Q = Q\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ is equipped with a Hermitian metric "invariant along the leaves".

1.2.4. If M is a Kählerian manifold and all the $f_{i,j}$ preserve the Kähler structure we say that $\mathcal{F}$ is transversely Kählerian. In particular such a foliation is Hermitian. This is equivalent to the existence of a Hermitian metric g on the normal bundle $Q\mathcal{F}$ which can be writen in a transverse local system of coordinates $(z^1, \dots, z^n)$ in the form $g = g_{k\overline{j}}(z, \overline{z}) dz^k \otimes d\overline{z}^j$ such that its skew-symmetric part $\omega = \frac{i}{2} g_{k\overline{j}} dz^k \wedge d\overline{z}^j$ is closed (ω is a basic 2-form called the *basic Kähler form* of $\mathcal{F}$).

For every $r \in \{0, 1, ..., n\}$, let $\Omega^r(\mathcal{M}/\mathcal{F})$ be the space of all basic forms of degree r. The exterior differential $d_b : \Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^{r+1}(\mathcal{M}/\mathcal{F})$ is the restriction of the classical exterior differential to basic forms and it is a basic differential operator of order 1 (in the sense of basic differential operators of order k on Hermitian $\mathcal{F}$-bundles, see its definition, [3] p. 328). The differential complex

$$0 \longrightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \Omega^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \dots \xrightarrow{d_b} \Omega^n(\mathcal{M}/\mathcal{F}) \longrightarrow 0 \tag{2}$$

is called the *basic de Rham complex* of $\mathcal{F}$; its cohomology $H^\bullet(\mathcal{M}/\mathcal{F})$ is the *basic de Rham cohomology* of $\mathcal{F}$.

Now, let us suppose that $\mathcal{F}$ is transversely holomorphic. We consider Q the complexified normal bundle of $Q\mathcal{F}$. Let J be the automorphism of Q associated to the complex structure; J satisfies $J^2 = -\mathrm{Id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles, respectively, denoted by $Q^{1,0}$ and $Q^{0,1} = \overline{Q^{1,0}}$. We have a splitting $Q = Q^{1,0} \oplus Q^{0,1}$ which gives rise to the decomposition

$$\Lambda^r Q^* = \bigoplus_{p+q=r} \Lambda^{p,q},$$

where $\Lambda^{p,q} = \Lambda^p Q^{*1,0} \otimes \Lambda^q Q^{*0,1}$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type* (p, q). They form a vector space denoted by $\Omega^{p,q}(\mathcal{M}/\mathcal{F})$. We have

$$\Omega^r(\mathcal{M}/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(\mathcal{M}/\mathcal{F}). \tag{3}$$

As in the classical case of a complex manifold, [11], the basic exterior differential decomposes into a sum of two operators

$$\partial_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p+1,q}(\mathcal{M}/\mathcal{F})\,;\; \overline{\partial}_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}).$$

We have $\partial_b^2 = \overline{\partial}_b^2 = 0$ and $\partial_b\overline{\partial}_b + \overline{\partial}_b\partial_b = 0$. The differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \Omega^{p,1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} \Omega^{p,n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0 \tag{4}$$

is called the *basic Dolbeault complex* of $\mathcal{F}$; its cohomology $H^{p,\bullet}(\mathcal{M}/\mathcal{F})$ is the *basic Dolbeault cohomology* of foliation $\mathcal{F}$.

2 Basic cohomology attached to a basic function

The cohomology attached to a function of smooth manifolds was defined and intensively studied in [10]. Similarly, in this section, we consider a basic cohomology attached to a basic function on the foliated manifold $(\mathcal{M}, \mathcal{F})$. This cohomology is also defined in terms of basic forms. More precisely, if $(\mathcal{M}, \mathcal{F})$ is a foliated manifold and f is a basic function on $\mathcal{M}$, we define a basic coboundary operator

$$d_{b,f} : \Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^{r+1}(\mathcal{M}/\mathcal{F}) \ , \ d_{b,f}\varphi = f d_b\varphi - r d_b f \wedge \varphi. \tag{5}$$

It is easy to check that $d_{b,f}^2 = 0$, and we denote by $H_f^\bullet(\mathcal{M}/\mathcal{F})$ the cohomology associated with the complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,f})$, called the *basic de Rham cohomology attached to the basic function* f of $\mathcal{F}$.

More generally, for any integer k, we define a basic operator

$$d_{b,f}^k : \Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^{r+1}(\mathcal{M}/\mathcal{F}) \ , \ d_{b,f}^k\varphi = f d_b\varphi - (r-k) d_b f \wedge \varphi. \tag{6}$$

We still have $(d_{b,f}^k)^2 = 0$ and we denote by $H_{f,k}^\bullet(\mathcal{M}/\mathcal{F})$ the cohomology of this complex. We shall restrict our attention to the cohomology $H_f^\bullet(\mathcal{M}/\mathcal{F})$ but most results readily generalize to the cohomology $H_{f,k}^\bullet(\mathcal{M}/\mathcal{F})$.

Using (5), by direct calculus we obtain

Proposition 1. *If $f, g \in \mathcal{F}_b(\mathcal{M})$ then*

(i) $d_{b,f+g} = d_{b,f} + d_{b,g}$, $d_{b,0} = 0$, $d_{b,-f} = -d_{b,f}$;

(ii) $d_{b,fg} = f d_{b,g} + g d_{b,f} - f g d_b$, $d_{b,1} = d_b$, $d_b = \frac{1}{2}(f d_{b,\frac{1}{f}} + \frac{1}{f} d_{b,f})$;

(iii) $d_{b,f}(\varphi \wedge \psi) = d_{b,f}\varphi \wedge \psi + (-1)^{\deg \varphi}\varphi \wedge d_{b,f}\psi$.

Now, if $\mathcal{F}$ is transversely holomorphic taking into account the decomposition (3) and $d_b = \partial_b + \overline{\partial}_b$ we obtain $d_{b,f} = \partial_{b,f} + \overline{\partial}_{b,f}$, where

$$\partial_{b,f} : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}) \ , \ \partial_{b,f}\varphi = f\partial_b\varphi - (p+q)\partial_b f \wedge \varphi, \tag{7}$$

$$\overline{\partial}_{b,f} : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \to \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}) \ , \ \overline{\partial}_{b,f}\varphi = f\overline{\partial}_b\varphi - (p+q)\overline{\partial}_b f \wedge \varphi. \tag{8}$$

We have $\partial_{b,f}^2 = \overline{\partial}_{b,f}^2 = 0$ and $\partial_{b,f}\overline{\partial}_{b,f} + \overline{\partial}_{b,f}\partial_{b,f} = 0$. So, we obtain a differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,f}} \Omega^{p,1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,f}} \dots \xrightarrow{\overline{\partial}_{b,f}} \Omega^{p,n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0 \tag{9}$$

called the *basic Dolbeault complex attached to the basic function* f of $\mathcal{F}$; its cohomology $H_f^{p,\bullet}(\mathcal{M}/\mathcal{F})$ is called the *basic Dolbeault cohomology attached to the basic function* f of foliation $\mathcal{F}$.

2.1 Basic Bott-Chern cohomology attached to a basic function

In this subsection $\mathcal{F}$ is considered to be transversely holomorphic.

Definition 3. The differential complex

$$\ldots \longrightarrow \Omega^{p-1,q-1} \xrightarrow{\partial_{b,f}\overline{\partial}_{b,f}} \Omega^{p,q} \xrightarrow{\partial_{b,f}\oplus\overline{\partial}_{b,f}} \Omega^{p+1,q} \oplus \Omega^{p,q+1} \longrightarrow \ldots \tag{10}$$

is called the *basic Bott-Chern complex attached to the basic function* f of $\mathcal{F}$ and the corresponding Bott-Chern cohomology groups of bidegree (p,q) are given by

$$H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F}) = \frac{\ker\{\partial_{b,f} : \Omega^{p,q} \to \Omega^{p+1,q}\} \cap \ker\{\overline{\partial}_{b,f} : \Omega^{p,q} \to \Omega^{p,q+1}\}}{\operatorname{im}\{\partial_{b,f}\overline{\partial}_{b,f} : \Omega^{p-1,q-1} \to \Omega^{p,q}\}}. \tag{11}$$

It is easy to see that $\bigoplus_{p,q} H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F})$ inherits a bigraded algebra structure induced by the exterior product of these forms. The above definition implies the canonical maps

$$H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F}) \to H^{p+q}_{f}(\mathcal{M}/\mathcal{F})\,;\; H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F}) \to H^{p,q}_{f}(\mathcal{M}/\mathcal{F}). \tag{12}$$

Now, let us consider the dual of the Bott-Chern cohomology groups attached to the basic function f, given by

$$H^{p,q}_{f,A}(\mathcal{M}/\mathcal{F}) = \frac{\ker\{\partial_{b,f}\overline{\partial}_{b,f} : \Omega^{p,q} \to \Omega^{p+1,q+1}\}}{\operatorname{im}\{\partial_{b,f} : \Omega^{p-1,q} \to \Omega^{p,q}\} + \operatorname{im}\{\overline{\partial}_{b,f} : \Omega^{p,q-1} \to \Omega^{p,q}\}}$$

called the *basic Aeppli cohomology groups attached to the basic function* f of $\mathcal{F}$.

Similarly to Lemma 2.5. from [12], we have

Proposition 2. *The exterior product induces a bilinear map*

$$\wedge : H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F}) \times H^{r,s}_{f,A}(\mathcal{M}/\mathcal{F}) \to H^{p+r,q+s}_{f,A}(\mathcal{M}/\mathcal{F}). \tag{13}$$

Proof. Let $\varphi, \psi \in \Omega^{p,q}(\mathcal{M}/\mathcal{F})$. If φ is $d_{b,f}$-closed and ψ is $\partial_{b,f}\overline{\partial}_{b,f}$-closed then $\varphi\wedge\psi$ is $\partial_{b,f}\overline{\partial}_{b,f}$-closed. Also, if φ is $d_{b,f}$-closed and ψ is $d_{b,f}$-exact then $\varphi\wedge\psi$ is $d_{b,f}$-exact and if φ is $\partial_{b,f}\overline{\partial}_{b,f}$-exact and ψ is $\partial_{b,f}\overline{\partial}_{b,f}$-closed then $\varphi\wedge\psi$ is $d_{b,f}$-exact. For the last assertion, we have

$$\begin{aligned} \varphi\wedge\psi &= \partial_{b,f}\overline{\partial}_{b,f}\theta\wedge\psi \\ &= \frac{1}{2}d_{b,f}[(\overline{\partial}_{b,f} - \partial_{b,f})\theta\wedge\psi + (-1)^{p+q}\theta\wedge(\partial_{b,f} - \overline{\partial}_{b,f})\psi]. \end{aligned}$$

□

In particular, one gets $H^{p,q}_{f,BC}(\mathcal{M}/\mathcal{F}) \times H^{n-p,n-q}_{f,A}(\mathcal{M}/\mathcal{F}) \to H^{n,n}_{f,A}(\mathcal{M}/\mathcal{F})$.

Remark 1. If $(\mathcal{M},\mathcal{F},\omega)$ is transversely Kählerian with the basic Kähler form ω, then taking into account $\partial_b\omega = \overline{\partial}_b\omega = 0$ by direct calculations it follows that if the basic function $f \in \mathcal{F}_b(\mathcal{M})$ satisfies the equation

$$3\partial_b f \wedge \overline{\partial}_b f - f\partial_b\overline{\partial}_b f = 0, \tag{14}$$

then ω is $\partial_{b,f}\overline{\partial}_{b,f}$-closed, so it defines a cohomology class $[\omega]_{f,A} \in H^{1,1}_{f,A}(\mathcal{M}/\mathcal{F})$ called the *basic Aeppli class attached to the basic function* f of ω.

2.2 Dependence on the basic function

A natural question to ask about the cohomology $H^r_f(\mathcal{M}/\mathcal{F})$ is how it depends on the basic function f. Similar to the Proposition 3.2. from [10], we explain this fact for our basic cohomology. We have

Proposition 3. *If $h \in \mathcal{F}_b(\mathcal{M})$ does not vanish, then cohomologies $H^\bullet_f(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{fh}(\mathcal{M}/\mathcal{F})$ are isomorphic.*

Proof. For each $r \in \mathbb{N}$, consider the linear isomorphism

$$\Phi^r : \Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^r(\mathcal{M}/\mathcal{F}) \ , \ \Phi^r(\varphi) = \frac{\varphi}{h^r}. \tag{15}$$

If $\varphi \in \Omega^r(\mathcal{M}/\mathcal{F})$, one checks easily that

$$\Phi^{r+1}(d_{b,fh}\varphi) = d_{b,f}(\Phi^r(\varphi)). \tag{16}$$

Thus Φ induces an isomorphism between cohomologies $H^\bullet_f(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{fh}(\mathcal{M}/\mathcal{F})$. □

Corollary 1. *If the basic function f does not vanish, then $H^\bullet_f(\mathcal{M}/\mathcal{F})$ is isomorphic to the basic de Rham cohomology $H^\bullet(\mathcal{M}/\mathcal{F})$.*

Proof. We take $h = \frac{1}{f}$ in the above proposition. □

If $\mathcal{F}$ is transversely holomorphic, similarly, one gets

Proposition 4. *If $h \in \mathcal{F}_b(\mathcal{M})$ does not vanish, then cohomologies $H^{\bullet,\bullet}_f(\mathcal{M}/\mathcal{F})$ and $H^{\bullet,\bullet}_{fh}(\mathcal{M}/\mathcal{F})$ are isomorphic.*

Corollary 2. *If the basic function f does not vanish, then $H^{\bullet,\bullet}_f(\mathcal{M}/\mathcal{F})$ is isomorphic to the basic Dolbeault cohomology $H^{\bullet,\bullet}(\mathcal{M}/\mathcal{F})$.*

2.3 A relative cohomology

The relative de Rham cohomology was first defined in [1] p. 78. Also, a relative vertical cohomology of real foliated manifolds can be found in [13]. In this subsection we construct a similar version for our basic cohomology.

Let $(\mathcal{M},\mathcal{F})$ and $(\mathcal{M}',\mathcal{F}')$ be two foliated manifolds.

Definition 4. A *morphism* from $(\mathcal{M},\mathcal{F})$ to $(\mathcal{M}',\mathcal{F}')$ is a differentiable mapping $\mu : \mathcal{M} \to \mathcal{M}'$ which sends every leaf F of $\mathcal{F}$ into a leaf F' of $\mathcal{F}'$ such that the restriction map $\mu : F \to F'$ is smooth, and it is holomorphic in transverse coordinates if $\mathcal{F}$ is transversely holomorphic.

We also notice that μ^* preserves the basic forms.

By restricting the standard relation $d\mu^* = \mu^* d'$ to basic forms, (here d' denotes the exterior derivative on $\mathcal{M}'$), we obtain

$$d_b\mu^* = \mu^* d_b'. \tag{17}$$

Now, if $f \in \mathcal{F}_b(\mathcal{M}^{'})$ is a basic function on $(\mathcal{M}^{'}, \mathcal{F}^{'})$, using (17), a straightforward calculus leads to a similar desired relation, namely

$$d_{b,\mu^* f}\mu^* = \mu^* d^{'}_{b,f}. \tag{18}$$

Indeed, for $\varphi \in \Omega^r(\mathcal{M}^{'}/\mathcal{F}^{'})$, we have

$$\begin{aligned} d_{b,\mu^* f}(\mu^*\varphi) &= \mu^* f d_b(\mu^*\varphi) - r d_b(\mu^* f) \wedge \mu^*\varphi \\ &= \mu^* f \mu^*(d^{'}_b\varphi) - r\mu^*(d^{'}_b f) \wedge \mu^*\varphi \\ &= \mu^*(f d^{'}_b\varphi) - \mu^*(r d^{'}_b f \wedge \varphi) \\ &= \mu^*(d^{'}_{b,f}\varphi). \end{aligned}$$

The relation (18) says that we have the homomorphism

$$\mu^* : H^\bullet_f(\mathcal{M}^{'}/\mathcal{F}^{'}) \to H^\bullet_{\mu^* f}(\mathcal{M}/\mathcal{F})\,,\ \mu^*[\varphi] = [\mu^*\varphi].$$

We define the differential complex

$$\ldots \xrightarrow{\widetilde{d}_{b,f}} \Omega^r(\mu) \xrightarrow{\widetilde{d}_{b,f}} \Omega^{r+1}(\mu) \xrightarrow{\widetilde{d}_{b,f}} \ldots$$

where

$$\Omega^r(\mu) = \Omega^r(\mathcal{M}^{'}/\mathcal{F}^{'}) \oplus \Omega^{r-1}(\mathcal{M}/\mathcal{F}), \text{ and } \widetilde{d}_{b,f}(\varphi,\psi) = (d^{'}_{b,f}\varphi, \mu^*\varphi - d_{b,\mu^* f}\psi).$$

Taking into account $d'^2_{b,f} = d^2_{b,\mu^* f} = 0$ and (18) we easily verify that $\widetilde{d}^2_{b,f} = 0$. Denote the cohomology groups of this complex by $H^\bullet_f(\mu)$.

If we regraduate the complex $\Omega^r(\mathcal{M}/\mathcal{F})$ as $\widetilde{\Omega}^r(\mathcal{M}/\mathcal{F}) = \Omega^{r-1}(\mathcal{M}/\mathcal{F})$, then we obtain an exact sequence of differential complexes

$$0 \longrightarrow \widetilde{\Omega}^r(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha} \Omega^r(\mu) \xrightarrow{\beta} \Omega^r(\mathcal{M}^{'}/\mathcal{F}^{'}) \longrightarrow 0 \tag{19}$$

with the obvious mappings α and β given by $\alpha(\psi) = (0,\psi)$ and $\beta(\varphi,\psi) = \varphi$, respectively. From (19) we have an exact sequence in cohomologies

$$\ldots \longrightarrow H^{r-1}_{\mu^* f}(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha^*} H^r_f(\mu) \xrightarrow{\beta^*} H^r_f(\mathcal{M}^{'}/\mathcal{F}^{'}) \xrightarrow{\delta^*} H^r_{\mu^* f}(\mathcal{M}/\mathcal{F}) \longrightarrow \ldots.$$

It is easily seen that $\delta^* = \mu^*$. Here μ^* denotes the corresponding map between cohomology groups. Let $\varphi \in \Omega^r(\mathcal{M}^{'}/\mathcal{F}^{'})$ be a $d^{'}_{b,f}$-closed form, and $(\varphi,\psi) \in \Omega^r(\mu)$. Then $\widetilde{d}_{b,f}(\varphi,\psi) = (0, \mu^*\varphi - d_{b,\mu^* f}\psi)$ and by the definition of the operator δ^* we have

$$\delta^*[\varphi] = [\mu^*\varphi - d_{b,\mu^* f}\psi] = [\mu^*\varphi] = \mu^*[\varphi].$$

Hence we finally get a long exact sequence

$$\ldots \longrightarrow H^{r-1}_{\mu^* f}(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha^*} H^r_f(\mu) \xrightarrow{\beta^*} H^r_f(\mathcal{M}^{'}/\mathcal{F}^{'}) \xrightarrow{\mu^*} H^r_{\mu^* f}(\mathcal{M}/\mathcal{F}) \longrightarrow \ldots. \tag{20}$$

We have

Proposition 5. *If the foliated manifolds $(\mathcal{M},\mathcal{F})$ and $(\mathcal{M}',\mathcal{F}')$ are of the n-th and n'-th codimension, respectively, then*

(i) $\beta^*: H_f^{n+1}(\mu)\to H_f^{n+1}(\mathcal{M}'/\mathcal{F}')$ *is an epimorphism,*

(ii) $\alpha^*: H_{\mu^*f}^{n'}(\mathcal{M}/\mathcal{F})\to H_f^{n'+1}(\mu)$ *is an epimorphism,*

(iii) $\beta^*: H_f^{r}(\mu)\to H_f^{r}(\mathcal{M}'/\mathcal{F}')$ *is an isomorphism for* $r>n+1$,

(iv) $\alpha^*: H_{\mu^*f}^{r}(\mathcal{M}/\mathcal{F})\to H_f^{r+1}(\mu)$ *is an isomorphism for* $r>n'$,

(v) $H_f^r(\mu)=0$ *for* $r>\max\{n+1,n'\}$.

If $\mathcal{F}$ is transversely holomorphic, taking into account that $d_{b,f}=\partial_{b,f}+\overline{\partial}_{b,f}$, from (18), one gets

$$\overline{\partial}_{b,\mu^*f}\mu^*=\mu^*\overline{\partial}'_{b,f}. \tag{21}$$

Similarly, we define the differential complex

$$\ldots\xrightarrow{\widetilde{\overline{\partial}}_{b,f}}\Omega^{p,q}(\mu)\xrightarrow{\widetilde{\overline{\partial}}_{b,f}}\Omega^{p,q+1}(\mu)\xrightarrow{\widetilde{\overline{\partial}}_{b,f}}\ldots$$

where

$$\Omega^{p,q}(\mu)=\Omega^{p,q}(\mathcal{M}'/\mathcal{F}')\oplus\Omega^{p,q-1}(\mathcal{M}/\mathcal{F}),\ \text{and}\ \widetilde{\overline{\partial}}_{b,f}(\varphi,\psi)=(\overline{\partial}'_{b,f}\varphi,\mu^*\varphi-\overline{\partial}_{b,\mu^*f}\psi).$$

Taking into account $\overline{\partial}'^2_{b,f}=\overline{\partial}^2_{b,\mu^*f}=0$ and (21) we easily verify that $\widetilde{\overline{\partial}}^2_{b,f}=0$. Denote the cohomology groups of this complex by $H_f^{p,\bullet}(\mu)$. We also notice that by using the same tehnique as above, we get a long exact sequence in cohomologies

$$\ldots\longrightarrow H_{\mu^*f}^{p,q-1}(\mathcal{M}/\mathcal{F})\xrightarrow{\alpha^*}H_f^{p,q}(\mu)\xrightarrow{\beta^*}H_f^{p,q}(\mathcal{M}'/\mathcal{F}')\xrightarrow{\mu^*}H_{\mu^*f}^{p,q}(\mathcal{M}/\mathcal{F})\longrightarrow\ldots, \tag{22}$$

and we have

Proposition 6. *If the transversely holomorphic $(\mathcal{M},\mathcal{F})$ and $(\mathcal{M}',\mathcal{F}')$ are of the n-th and n'-th complex codimension, respectively, then*

(i) $\beta^*: H_f^{p,n+1}(\mu)\to H_f^{p,n+1}(\mathcal{M}'/\mathcal{F}')$ *is an epimorphism,*

(ii) $\alpha^*: H_{\mu^*f}^{p,n'}(\mathcal{M}/\mathcal{F})\to H_f^{p,n'+1}(\mu)$ *is an epimorphism,*

(iii) $\beta^*: H_f^{p,q}(\mu)\to H_f^{p,q}(\mathcal{M}'/\mathcal{F}')$ *is an isomorphism for* $q>n+1$,

(iv) $\alpha^*: H_{\mu^*f}^{p,q}(\mathcal{M}/\mathcal{F})\to H_f^{p,q+1}(\mu)$ *is an isomorphism for* $q>n'$,

(v) $H_f^{p,q}(\mu)=0$ *for* $q>\max\{n+1,n'\}$.

2.4 A Mayer-Vietoris sequence and a homotopy morphism

Since the differentials $d_{b,f}$ and $\overline{\partial}_{b,f}$ commutes with the restrictions to open subsets, one can construct, in the same way as for the de Rham cohomology (see [1, 8, 14]), some Mayer-Vietoris exact sequences, namely:

Theorem 1. *If* $\mathcal{U} = \{U, V\}$ *is an open cover of* $\mathcal{M}$*, we have the long exact sequence*

$$\ldots \to H_f^{r-1}((U \cap V)/\mathcal{F}) \to H_f^r(\mathcal{M}/\mathcal{F}) \xrightarrow{A} H_f^r(U/\mathcal{F}) \oplus H_f^r(V/\mathcal{F}) \xrightarrow{B}$$
$$\xrightarrow{B} H_f^r((U \cap V)/\mathcal{F}) \to \ldots,$$

where for $[\varphi] \in H_f^r(\mathcal{M}/\mathcal{F})$ *and* $([\sigma_U], [\tau_V]) \in H_f^r(U/\mathcal{F}) \oplus H_f^r(V/\mathcal{F})$*, we define*

$$A([\varphi]) = ([\sigma_U], [\tau_V]) \text{ and } B([\sigma_U], [\tau_V]) = [\sigma|_{U \cap V} - \tau|_{U \cap V}].$$

Theorem 2. *If* $\mathcal{U} = \{U, V\}$ *is an open cover of* $\mathcal{M}$*, we have the long exact sequence*

$$\ldots \to H_f^{p,q-1}((U \cap V)/\mathcal{F}) \to H_f^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{A} H_f^{p,q}(U/\mathcal{F}) \oplus H_f^{p,q}(V/\mathcal{F}) \xrightarrow{B}$$
$$\xrightarrow{B} H_f^{p,q}((U \cap V)/\mathcal{F}) \to \ldots,$$

where for $[\varphi] \in H_f^{p,q}(\mathcal{M}/\mathcal{F})$ *and* $([\sigma_U], [\tau_V]) \in H_f^{p,q}(U/\mathcal{F}) \oplus H_f^{p,q}(V/\mathcal{F})$*, we define*

$$A([\varphi]) = ([\sigma_U], [\tau_V]) \text{ and } B([\sigma_U], [\tau_V]) = [\sigma|_{U \cap V} - \tau|_{U \cap V}].$$

Following [10], we give

Definition 5. Let $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ be two foliated manifolds and $f \in \mathcal{F}_b(\mathcal{M})$ and $f' \in \mathcal{F}_b(\mathcal{M}')$. A *morphism* from the pair $(\mathcal{M}, \mathcal{F}, f)$ to the pair $(\mathcal{M}', \mathcal{F}', f')$ is a pair (μ, α) formed by a morphism $\mu : (\mathcal{M}, \mathcal{F}) \to (\mathcal{M}', \mathcal{F}')$ and a real valued function $\alpha : \mathcal{M} \to \mathbb{R}$, such that α does not vanish on $\mathcal{M}$ and $f' \circ \mu = \alpha f$.

If (μ, α) is a morphism from the pair $(\mathcal{M}, \mathcal{F}, f)$ to the pair $(\mathcal{M}', \mathcal{F}', f')$ then the map $\Omega^r(\mathcal{M}'/\mathcal{F}') \mapsto \Omega^r(\mathcal{M}/\mathcal{F})$ defined by $\varphi \mapsto \frac{\mu^*\varphi}{\alpha^r}$ induces an homomorphism in cohomology $H_{f'}^r(\mathcal{M}'/\mathcal{F}') \mapsto H_f^r(\mathcal{M}/\mathcal{F})$. We also notice that if μ is a diffeomorphism then $H_{f'}^r(\mathcal{M}'/\mathcal{F}')$ and $H_f^r(\mathcal{M}/\mathcal{F})$ are isomorphic. Similarly, we obtain the isomorphism of Dolbeault cohomology groups $H_{f'}^{p,q}(\mathcal{M}'/\mathcal{F}')$ and $H_f^{p,q}(\mathcal{M}/\mathcal{F})$.

Definition 6. A *homotopy* from the pair $(\mathcal{M}, \mathcal{F}, f)$ to the pair $(\mathcal{M}', \mathcal{F}', f')$ is given by two smooth maps

$$h : \mathcal{M} \times [0, 1] \to \mathcal{M}', \; a : \mathcal{M} \times [0, 1] \to \mathbb{R},$$

such that for each $t \in [0, 1]$, we have a morphism

$$H_t \equiv (h(\cdot, t), a(\cdot, t)) : (\mathcal{M}, \mathcal{F}, f) \to (\mathcal{M}', \mathcal{F}', f')$$

(i.e., a does not vanish, $f' \circ h(x, t) = a(x, t) f(x)$ and for all $t \in [0, 1]$ the map H_t sends each leaf F of $\mathcal{F}$ into another leaf F' of $\mathcal{F}'$).

If $H = (h, a)$ is a homotopy from $(\mathcal{M}, \mathcal{F}, f)$ to $(\mathcal{M}', \mathcal{F}', f')$, we obtain a map at cohomology level

$$H_t^* : H_{f'}^{\bullet}(\mathcal{M}/\mathcal{F}') \to H_f^{\bullet}(\mathcal{M}/\mathcal{F}).$$

2.5 A basic Lichnerowicz type cohomology attached to a function

In this subsection we make a connection between the studied basic cohomology attached to a basic function and the basic Lichnerowicz cohomology defining a basic Lichnerowicz type cohomology attached to a function. Also, some classical properties adapted to this new cohomology are investigated.

Let $(\mathcal{M},\mathcal{F})$ be a foliated manifold of codimension n and $\theta\in\Omega^1(\mathcal{M}/\mathcal{F})$ be a d_b-closed basic 1-form. Denote by $d_{b,\theta}:\Omega^r(\mathcal{M}/\mathcal{F})\rightarrow\Omega^{r+1}(\mathcal{M}/\mathcal{F})$ the map $d_{b,\theta}=d_b-\theta\wedge$.

Since $d_b\theta=0$, we easily obtain that $d^2_{b,\theta}=0$. The differential complex

$$0\longrightarrow\Omega^0(\mathcal{M}/\mathcal{F})\stackrel{d_{b,\theta}}{\longrightarrow}\Omega^1(\mathcal{M}/\mathcal{F})\stackrel{d_{b,\theta}}{\longrightarrow}\ldots\stackrel{d_{b,\theta}}{\longrightarrow}\Omega^n(\mathcal{M}/\mathcal{F})\longrightarrow 0 \tag{23}$$

is called *the basic Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$; its cohomology groups $H^\bullet_\theta(\mathcal{M}/\mathcal{F})$ are called *the basic Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

This is a basic version of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [7], also known in literature as Morse-Novikov cohomology. We also notice that Vaisman in [15] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds. Some notions concerning to such a basic Lichnerowicz cohomology of real foliations may be found in [4].

In this moment we can not say if the operator $d_{b,f}$ satisfies a Poincaré type Lemma, but if the basic function f does not vanish everywhere we have

Proposition 7. *Let φ be a basic r-form defined on some neighborhood U of $\mathcal{M}$ adapted to the foliation $\mathcal{F}$ such that $d_{b,f}\varphi=0$. Then there exists a basic $(r-1)$-form ψ defined on a neighborhood $U^{'}\subset U$ such that $f\varphi=d_{b,f}\psi-d_bf\wedge\psi$.*

Proof. Let φ be as in hypothesis. From $d_{b,f}\varphi=0$ because $f\neq 0$ we easily obtain

$$d_{b,\theta}\varphi=0\,,\ \ \theta=d_b(\log f^r). \tag{24}$$

But the operator $d_{b,\theta}$ satisfies a Poincaré Lemma, see the proof of Proposition 3.1. from [15], (here we restrict the considerations from [15] to basic forms), and thus there exists a basic $(r-1)$-form defined on $U^{'}\subset U$ such that

$$\begin{aligned}\varphi &= d_{b,\theta}\psi\\ &= d_b\psi-d_b(\log f^r)\wedge\psi\\ &= d_b\psi-r\frac{d_bf}{f}\wedge\psi\\ &= d_b\psi-(r-1)\frac{d_bf}{f}\wedge\psi-\frac{d_bf}{f}\wedge\psi\\ &= \frac{1}{f}\left(fd_b\psi-(r-1)d_bf\wedge\psi\right)-\frac{d_bf}{f}\wedge\psi\\ &= \frac{1}{f}\left(d_{b,f}\psi-d_bf\wedge\psi\right)\end{aligned}$$

which ends the proof. □

Let us return now to announced basic Lichnerowicz type cohomology attached to a basic function. For this purpose we consider again $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ be a d_b-closed basic 1-form. If f is a basic function then we define the following operator:

$$d_{b,f,\theta}\varphi = d_{b,f}\varphi - f\theta \wedge \varphi\,,\ \ \varphi \in \Omega^r(\mathcal{M}/\mathcal{F}). \tag{25}$$

Taking into account that $d_{b,f}(f\theta) = 0$ an easy calculation using Proposition 1 (iii) leads to $d^2_{b,f,\theta} = 0$. Thus, we obtain the differential complex

$$0 \longrightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \overset{d_{b,f,\theta}}{\longrightarrow} \Omega^1(\mathcal{M}/\mathcal{F}) \overset{d_{b,f,\theta}}{\longrightarrow} \ldots \overset{d_{b,f,\theta}}{\longrightarrow} \Omega^n(\mathcal{M}/\mathcal{F}) \longrightarrow 0 \tag{26}$$

which is called *the basic Lichnerowicz complex attached to the basic function* f of $(\mathcal{M},\mathcal{F})$; its cohomology groups $H^\bullet_{f,\theta}(\mathcal{M}/\mathcal{F})$ are called *the basic Lichnerowicz cohomology groups attached to the basic function* f of $(\mathcal{M},\mathcal{F})$.

In the following we prove some classical properties of Lichnerowicz cohomology adapted to our basic Lichnerowicz cohomology attached to a basic function.

Proposition 8. *The basic Lichnerowicz cohomology attached to a basic function f depends only on the basic class of θ. In fact, we have the isomorphism*

$$H^r_{f,\theta - d_b\sigma}(\mathcal{M}/\mathcal{F}) \approx H^r_{f,\theta}(\mathcal{M}/\mathcal{F}).$$

Proof. By direct calculus we easily obtain $d_{b,f,\theta}(e^\sigma\varphi) = e^\sigma d_{b,f,\theta-d_b\sigma}\varphi$, where σ is a basic function and thus the map $[\varphi] \mapsto [e^\sigma\varphi]$ is an isomorphism between cohomologies $H^r_{f,\theta-d_b\sigma}(\mathcal{M}/\mathcal{F})$ and $H^r_{f,\theta}(\mathcal{M}/\mathcal{F})$. □

Proposition 9. *For any basic forms $\varphi\,,\,\psi \in \Omega^\bullet(\mathcal{M}/\mathcal{F})$ we have*

$$d_{b,f,\theta}(\varphi \wedge \psi) = d_{b,f}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge d_{b,f,\theta}\psi. \tag{27}$$

Proof. It follows by direct calculus using the definitions of $d_{b,f}$ and $d_{b,f,\theta}$ and Proposition 1 (iii). □

Also, if θ_1 and θ_2 are two d_b-closed basic 1-forms then

$$d_{f,b,\theta_1+\theta_2}(\varphi \wedge \psi) = d_{b,f,\theta_1}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge d_{b,f,\theta_2}\psi,$$

which says that the wedge product induces the map

$$\wedge : H^{r_1}_{f,\theta_1}(\mathcal{M}/\mathcal{F}) \times H^{r_2}_{f,\theta_2}(\mathcal{M}/\mathcal{F}) \to H^{r_1+r_2}_{f,\theta_1+\theta_2}(\mathcal{M}/\mathcal{F}).$$

Corollary 3. *The wedge product induces the following homomorphism*

$$\wedge : H^r_{f,\theta}(\mathcal{M}/\mathcal{F}) \times H^r_{f,-\theta}(\mathcal{M}/\mathcal{F}) \to H^{2r}_f(\mathcal{M}/\mathcal{F}).$$

If $\mathcal{F}$ is transversely holomorphic then similar basic Lichnerowicz cohomology attached to a basic function of Dolbeault and Bott-Chern type can be defined. Taking into account the decomposition $\theta=\theta^{1,0}+\theta^{0,1}$, we have a decomposition of $d_{b,f,\theta}$ into

$$d_{b,f,\theta}=\partial_{b,f,\theta}+\overline{\partial}_{b,f,\theta}\,,\ \partial_{b,f,\theta}=\partial_{b,f}-f\theta^{1,0}\wedge\,,\ \overline{\partial}_{b,f,\theta}=\overline{\partial}_{b,f}-f\theta^{0,1}\wedge\,. \tag{28}$$

Now, from $d^2_{b,f,\theta}=0$ we obtain

$$\partial^2_{b,f,\theta}=\overline{\partial}^2_{b,f,\theta}=\partial_{b,f,\theta}\overline{\partial}_{b,f,\theta}+\overline{\partial}_{b,f,\theta}\partial_{b,f,\theta}=0. \tag{29}$$

The diferential complex

$$\ldots\stackrel{\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{p,q-1}(\mathcal{M}/\mathcal{F})\stackrel{\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{p,q}(\mathcal{M}/\mathcal{F})\stackrel{\overline{\partial}_{b,f,\theta}}{\longrightarrow}\ldots \tag{30}$$

is called *the basic Dolbeault-Lichnerowicz complex attached to the basic function* f of $(\mathcal{M},\mathcal{F})$; its cohomology groups denoted by $H^{p,\bullet}_{f,\theta}(\mathcal{M}/\mathcal{F})$ are called *the basic Dolbeault-Lichnerowicz cohomology groups attached to the basic function* f of $(\mathcal{M},\mathcal{F})$.

The differential complex

$$\ldots\longrightarrow\Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F})\stackrel{\partial_{b,f,\theta}\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{p,q}(\mathcal{M}/\mathcal{F})\stackrel{\partial_{b,f,\theta}\oplus\overline{\partial}_{b,f,\theta}}{\longrightarrow}$$

$$\stackrel{\partial_{b,f,\theta}\oplus\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{p+1,q}(\mathcal{M}/\mathcal{F})\oplus\Omega^{p,q+1}(\mathcal{M}/\mathcal{F})\longrightarrow\ldots \tag{31}$$

is called *the basic Bott-Chern-Lichnerowicz complex attached to the basic function* f of $(\mathcal{M},\mathcal{F})$ and its cohomology groups

$$H^{\bullet,\bullet}_{f,\theta,BC}(\mathcal{M}/\mathcal{F})=\frac{\mathrm{Ker}\{\Omega^{\bullet,\bullet}\stackrel{\partial_{b,f,\theta}}{\longrightarrow}\Omega^{\bullet+1,\bullet}\}\cap\mathrm{Ker}\{\Omega^{\bullet,\bullet}\stackrel{\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{\bullet,\bullet+1}\}}{\mathrm{Im}\{\Omega^{\bullet-1,\bullet-1}\stackrel{\partial_{b,f,\theta}\overline{\partial}_{b,f,\theta}}{\longrightarrow}\Omega^{\bullet,\bullet}\}}$$

are called *the basic Bott-Chern-Lichnerowicz cohomology groups attached to the basic function* f of $(\mathcal{M},\mathcal{F})$.

Similarly as above, Propositions 10, 11 and Corollary 3 hold for basic Dolbeault-Lichnerowicz cohomology attached to a basic function.

Finally, we notice that a relative cohomology for the basic Lichnerowicz cohomology attached to a basic function may be considered. Let us consider again a morphism of two foliated manifolds $\mu:(\mathcal{M},\mathcal{F})\to(\mathcal{M}',\mathcal{F}')$. Taking into account (18) we obtain

$$d_{b,\mu^*f,\mu^*\theta}\mu^*=\mu^*d'_{b,f,\theta} \tag{32}$$

for any basic function $f\in\mathcal{F}_b(\mathcal{M}')$ and for any d'_b-closed basic 1-form $\theta\in\Omega^1(\mathcal{M}'/\mathcal{F}')$. Indeed, for $\varphi\in\Omega^r(\mathcal{M}'/\mathcal{F}')$, we have

$$d_{b,\mu^*f,\mu^*\theta}(\mu^*\varphi)\ =\ d_{b,\mu^*f}(\mu^*\varphi)-\mu^*f\mu^*\theta\wedge\mu^*\varphi$$

$$\begin{aligned} &= \mu^* d'_{b,f}\varphi - \mu^*(f\theta \wedge \varphi) \\ &= \mu^*(d'_{b,f,\theta}\varphi). \end{aligned}$$

The relation (32) says that we have the homomorphism

$$\mu^* : H^\bullet_{f,\theta}(\mathcal{M}'/\mathcal{F}') \to H^\bullet_{\mu^* f,\mu^*\theta}(\mathcal{M}/\mathcal{F})\,,\ \mu^*[\varphi] = [\mu^*\varphi].$$

We define the differential complex

$$\ldots \xrightarrow{\widetilde{d}_{b,f,\theta}} \Omega^r(\mu) \xrightarrow{\widetilde{d}_{b,f,\theta}} \Omega^{r+1}(\mu) \xrightarrow{\widetilde{d}_{b,f,\theta}} \ldots$$

where

$$\Omega^r(\mu) = \Omega^r(\mathcal{M}'/\mathcal{F}') \oplus \Omega^{r-1}(\mathcal{M}/\mathcal{F}),\ \text{and}\ \widetilde{d}_{b,f,\theta}(\varphi,\psi) = (d'_{b,f,\theta}\varphi, \mu^*\varphi - d_{b,\mu^* f,\mu^*\theta}\psi).$$

Taking into account $d'^2_{b,f,\theta} = d^2_{b,\mu^* f,\mu^*\theta} = 0$ and (32) we easily verify that $\widetilde{d}^2_{b,f,\theta} = 0$. Denote the cohomology groups of this complex by $H^\bullet_{f,\theta}(\mu)$.

Analogouly, we obtain the corresponding cohomology groups $H^{p,\bullet}_{f,\theta}(\mu)$ for transversely holomorphic foliations.

Now, by the same technique as in Subsection 2.3. we can obtain the analog of Propositions 5 and 6 for the basic Lichnerowicz cohomology attached to a basic function $H^\bullet_{f,\theta}(\mathcal{M}/\mathcal{F})$ and $H^{p,\bullet}_{f,\theta}(\mu)$, respectively.

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References

[1] Bott R., Tu L. W. *Differential Forms in Algebraic Topology*, Graduate Text in Math., **82**, Springer-Verlag, Berlin, 1982.

[2] El Kacimi Alaoui A. *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math., 1990, **73**, 57–106.

[3] El Kacimi Alaoui A., Gmira B. *Stabilité du caractère Kählérien transverse*, Israel J. of Math., 1997, **101**, 323–347.

[4] Haddou A. Hassan. *Foliations and Lichnerowicz Basic Cohomology*, International Math. Forum, 2007, **2**(49), 2437–2446.

[5] Ida C. *A note on the relative Lichnerowicz cohomology*, Proceedings of the Conference RIGA 2011, Riemannian Geom. and Appl. Bucharest, Romania, 181–186.

[6] Ida C. *On a Lichnerowicz type cohomology attached to a function*, Preprint.

[7] Lichnerowicz A. *Les variétés de Poisson et leurs algébres de Lie associées*, J. Differential Geom., 1977, **12**(2), 253–300.

[8] Miron R., Pop I. *Topologie Algebrică. Omologie, Omotopie, Spaţii de Acoperire*, Ed. Acad. RSR, Bucureşti, 1974.

[9] Monnier P. *Computations of Nambu-Poisson cohomologies*, Int. J. Math. Math. Sci., 2001, **26**(2), 65–81.

[10] Monnier P. *A cohomology attached to a function*, Diff. Geometry and Applications, 2005, **22**, 49–68.

[11] Morrow J., Kodaira K. *Complex Manifolds*, AMS Chelsea Publ., 1971.

[12] Schweitzer M. *Autour de la cohomologie de Bott-Chern*, Preprint available at http://arxiv.org/abs/07093528.

[13] Tevdoradze Z. *Vertical cohomologies and their application to completely integrable Hamiltonian systems*, Georgian Math. J., 1998, **5**(5), 483–500.

[14] Vaisman I. *Cohomology and differential forms*, M. Dekker Publ. House, New York, 1973.

[15] Vaisman I. *Remarkable operators and commutation formulas on locally conformal Kähler manifolds*, Compositio Math., 1980, **40**, No. 3, 287–299.

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