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# Method of construction of topologies on any finite set

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**Abstract.** Let a topology $\tau$ be defined on a finite set. We give the definition of quasiatoms in the lattice $(\tau, \subseteq)$ and study their properties. For any splitting of a finite set $X$ into $k$ subsets we give a method of constructing any topology on the set $X$ for which this splitting is the set of all quasiatoms and the weight of this topological space is equal to $k$.

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## 1 Introduction

In this article we study the properties of topologies which are defined on a finite set. This article is a continuation of the articles [1] and [2]. For any topology which is defined on a finite set we give the definition of quasiatoms in the lattice of all open sets in this topology [1] and we study their properties (see Theorem 1).

Moreover, for any splitting of a finite set $X$ into $k$ subsets we give a method of constructing any topology on $X$ for which this splitting is the set of all quasiatoms and the weight of this topological space is equal to $k$ (see Theorem 3).

## 2 Quasiatoms and their properties

1. Construction of quasiatoms.

Let $\tau$ be a topology on a finite set $X$. We construct by induction:

- The sequences $X_1(\tau), \ldots, X_t(\tau)$ and $X'_1(\tau), \ldots, X'_t(\tau)$ of subsets of the set $X$;

- The sequence of natural numbers $s_1, \ldots, s_t$;

- The sequence $\widetilde{X}_1(\tau), \ldots, \widetilde{X}_t(\tau)$, where $\widetilde{X}_i(\tau)$ is a subset of the set $X$ for any $1 \leq i \leq t$;

- The set $\widetilde{X}(\tau)$ of subsets of the set $X$ as follows:

**1.1.** We take:

$X_1(\tau) = X$;

The set $\widetilde{X}_1(\tau)$ is equal to the set of all atoms in the lattice $(\tau, \subseteq)$;

$X'_1(\tau) = \bigcup\limits_{U \in \widetilde{X}_1(\tau)} U.$



[1] It is known that $(\tau, \subseteq)$ is a lattice for any topology $\tau$. Necessary concepts from lattice theory can be found in [3] and [4].

**1.2.** Suppose that for natural number $t$ have already been defined:

- The sequences $X_1(\tau), \ldots, X_t(\tau)$ and $X'_1(\tau), \ldots, X'_t(\tau)$ of the subsets $X_i(\tau)$ and $X'_1(\tau)$ of the set $X$;
- The sequence of natural numbers $s_1, \ldots, s_t$;
- The sequence $\widetilde{X}_1(\tau), \ldots, \widetilde{X}_t(\tau)$, where $\widetilde{X}_i(\tau)$ is some subset of the set $X$, for $1 \le i \le t$.

**1.3**. We take:

$X_{t+1}(\tau) = X \setminus \bigcup\limits_{j=1}^{t} X'_i(\tau)$ and $\tau_{t+1} = \tau|_{X_{t+1}(\tau)} = \{U \bigcap X_{t+1}(\tau) | U \in \tau\}$.

Let's consider the set $\widetilde{X}_{t+1}(\tau)$ of all atoms in the lattice $(\tau_{t+1}, \subseteq)$ and also we take $X'_{t+1}(\tau) = \bigcup\limits_{U \in \widetilde{X}_{t+1}(\tau)} U$.

**1.4**. As $X_{i+1}(\tau) \subset X_i(\tau)$ for any $i$, then from the finiteness of the set $X$ it follows that there exists such natural number $k$ that $X_{k+1} = \emptyset$. Then we take $\widetilde{X}(\tau) = \bigcup\limits_{i=1}^{k} \widetilde{X}_i(\tau)$.

*Remark* 1. It is easy to notice that $X'_i(\tau) \subseteq X_i(\tau)$ for any $1 \le i \le k$.

**Definition 1.** If $\tau$ is a topology on a finite set $X$, then any nonempty subset $U \in \widetilde{X}_i(\tau)$ of the set $X$ is called an atom of the level $i$ in the lattice $(\tau, \subseteq)$.

Every atom of some level is called a quasiatom if there is no necessity to specify its level.

**Theorem 1.** *(Necessary designations see above in the construction of quasiatoms.) Let:*

*$\tau$ be a topology on a finite set $X$;*
*$U \in \widetilde{X}(\tau)$ and $V(U) = \bigcap\limits_{W \in \tau, U \subseteq W} W$;*
*$S_i(U, \tau) = V(U) \bigcap X'_i(\tau)$ for $1 \le i \le k$.*
*Then the following statements are true:*

***Statement 1.*** *The set $\{U \mid U \in \widetilde{X}(\tau)\}$ is a splitting of the set $X$.*

***Statement 2.*** *The set $\{X'_1(\tau), \ldots, X'_k(\tau)\}$ is a splitting of the set $X$.*

***Statement 3.*** *The set $\{\widetilde{X}_i(\tau) \mid 1 \le i \le k\}$ is a splitting of the set $\widetilde{X}(\tau)$.*

***Statement 4.*** *If $U \in \widetilde{X}(\tau)$ and $i(U)$ is a natural number such that $U \in \widetilde{X}_{i(U)}(\tau)$ (the existence and uniqueness of the number $i(U)$ follow from the fact that $\{\widetilde{X}_i(\tau) \mid 1 \le i \le k\}$ is a splitting of the set $\widetilde{X}(\tau)$ (see Statement 3)) then:*
*$U = S_t(U, \tau)$ for $t = i(U)$;*
*$S_t(U, \tau) \ne \emptyset$ for $1 \le t < i(U)$;*
*$S_t(U, \tau) = \emptyset$ for $i(U) < t \le k$.*

***Statement 5.*** *If $U, U' \in \widetilde{X}(\tau)$ and $U' \bigcap S_t(U,\tau) \neq \emptyset$ for some $1 < t \leq k$, then $S_q(U',\tau) \subseteq S_q(U,\tau)$ for any $1 \leq q \leq i(U')$.*

***Statement 6.*** *If $U \in \widetilde{X}(\tau)$, then $V(U) = \bigcup\limits_{t=1}^{i(U)} S_t(U,\tau)$.*

***Statement 7.*** *If $U \in \widetilde{X}(\tau)$, then $U = V(U) \bigcap X'_{i(U)} = V(U) \bigcap X_{i(U)}$.*

***Statement 8.*** *The set $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ is the minimal base of the topological space $(X,\tau)$ and the cardinal of the set $\widetilde{X}(\tau)$ is equal to the weight of the topological space $(X,\tau)$.*

*Proof.* **Statement 1.** Let $x \in X$ and $m = \max\{i \mid x \in X_i(\tau)\}$. Then $x \notin X_{m+1}(\tau) = X \setminus \bigcup\limits_{i=1}^{m} X'_i(\tau)$ and $x \in X_m(\tau) = X \setminus \bigcup\limits_{i=1}^{m-1} X'_i(\tau)$, and hence, $x \in X'_m(\tau)$. As $X'_m = \bigcup\limits_{U \in \widetilde{X}_m} U$, then $x \in U$ for some $U \in \widetilde{X}_m(\tau) \subseteq \widetilde{X}(\tau)$.

From the randomness of the element $x \in X$ it follows that $X = \bigcup\limits_{U \in \widetilde{X}(\tau)} U$.

Let now $U_1, U_2 \in \widetilde{X}(\tau)$ and $U_1 \neq U_2$. Then there are natural numbers $1 \leq i, j \leq k$ such that $U_1 \in \widetilde{X}_i(\tau)$ and $U_2 \in \widetilde{X}_j(\tau)$.

If $i \neq j$ and $i < j$ (see 1.3 and Remark 1), then $U_2 \subseteq X'_j(\tau) \subseteq X_j \subseteq X \setminus \bigcup\limits_{l<j} X'_l(\tau)$ and $U_1 \subseteq X'_i(\tau)$, and hence,

$$U_1 \bigcap U_2 \subseteq X'_i(\tau) \bigcap \Big( X \setminus \bigcap_{l<j} X'_l(\tau) \Big) = \emptyset,$$

i.e. in this case $U_1 \bigcap U_2 = \emptyset$.

For the case when $j < i$ the equation $U_1 \bigcap U_2 = \emptyset$ is proved analogously.

Let now $i = j$. Then $U_1$ and $U_2$ are atoms in a lattice $(\tau_i, \subseteq)$. As $U_1 \bigcap U_2 \in \tau_i$ and $U_1 \bigcap U_2 \subset U_1$, then $U_1 \bigcap U_2 = \emptyset$ in this case, too.

Statement 1 is proved.

**Statement 2.** We prove this statement by induction on the number $k$.

If $k = 1$, then $X_2(\tau) = \emptyset$, and hence, the set $X$ is the union of all atoms of the lattice $(\tau, \subseteq)$, i.e. $X = \bigcup\limits_{U \in \widetilde{X_1}(\tau)} U = X'_1(\tau)$. Then $\{X'_1(\tau)\}$ is a splitting of the set $X$.

Let now for $k = t$ and for any finite topological space Statement 2 be proved and let $k = t + 1$.

As the set of all atoms of the level $i - 1$ in the lattice $(\tau|_{X_2(\tau)}, \subseteq)$ of topological space

$$(X_2(\tau), \tau|_{X_2(\tau)}) = (X \setminus X'_1(\tau), \tau|_{X \setminus X'_1(\tau)})$$

coincides with the set of all atoms of a level $i$ in the lattice $(\tau, \subseteq)$ of topological space $(X, \tau)$ for any $2 \leq i \leq t+1$, then by the inductive assumption,

$$\{X'_2(\tau), \dots, X'_{t+1}(\tau)\} = \{X'_1(\tau|_{X \setminus X'_1(\tau)}), \dots, X'_t(\tau|_{X \setminus X'_1(\tau)}\})$$

is a splitting of the set $X \setminus X'_1(\tau)$. Then $\{X'_1(\tau), X'_2(\tau), \dots, X'_{t+1}(\tau)\}$ will be a splitting of the set $X$.

Statement 2 is proved.

**Statement 3**. As, according to 1.4, $\widetilde{X}(\tau) = \bigcup\limits_{i=1}^{k} \widetilde{X}_i(\tau)$, then it should be checked up that $\widetilde{X}_i \bigcap \widetilde{X}_j = \emptyset$ for $i \neq j$.

We assume the contrary, i.e. that $\widetilde{X}_i(\tau) \bigcap \widetilde{X}_j(\tau) \neq \emptyset$ for some numbers $i \neq j$.

We can assume that $i < j$. Then $U \subseteq X'_i(\tau)$ and

$$U \subseteq X'_j(\tau) = X \setminus \big(\bigcup_{l=1}^{j-1} X'_l(\tau)\big) \subseteq X \setminus X'_i(\tau).$$

We have received the contradiction, hence $\widetilde{X}_i(\tau) \bigcap \widetilde{X}_j(\tau) = \emptyset$.

Statement 3 is proved.

**Statement 4**. Let $U \in \widetilde{X}(\tau)$. Then (see the definition of the number i(U)) $U \in \widetilde{X}_{i(U)}(\tau)$, and hence, $U \subseteq X'_{i(U)}(\tau)$. As $U$ is an atom of the level $i(U)$ then there exists such $V_0 \in \tau$ that $U = V_0 \bigcap X_{i(U)}(\tau)$, and as (see Remark 2) $X'_{i(U)}(\tau) \subseteq X_{i(U)}(\tau)$, then $U \subseteq X'_{i(U)}(\tau) \bigcap V_0 \subseteq X_{i(U)}(\tau) \bigcap V_0 = U$, and hence, $U = V_0 \bigcap X'_{i(U)}(\tau)$.

If $t = i(U)$, then $U \in \widetilde{X}_t(\tau)$, and hence, $U \subseteq X'_t(\tau)$. As $U \subseteq \bigcap\limits_{W \in \tau, U \subseteq W} W = V(U)$ then $U \subseteq V(U) \bigcap X'_t(\tau) = S_t(U, \tau)$.

If now $x \in S_t(U, \tau) = V(U) \bigcap X'_t(\tau)$ then $x \in X'_t(\tau) = \bigcup\limits_{W \in \widetilde{X}_t(\tau)} W$, and hence, $x \in W_0$ for some $W_0 \in \widetilde{X}_t(\tau)$ and $x \in V(U)$. As $\emptyset \neq W_0 \bigcap V(U) \in \tau|_{X_t(\tau)}$ and $W_0$ is an atom in the lattice $(\tau|_{X_t(\tau)}, \subseteq)$, then $W_0 \subseteq X'_t(\tau)$ and $W_0 \subseteq V(U)$. Then

$$x \in W_0 \subseteq V(U) \bigcap X'_t(\tau) = \Big(\bigcap_{W \in \tau, U \subseteq W} W\Big) \bigcap X'_t(\tau) \subseteq V_0 \bigcap X'_t(\tau) = U.$$

From the randomness of the element $x$ it follows that $S_t(U, \tau) = V(U) \bigcap X'_t(\tau) \subseteq U$, and hence, $S_t(U, \tau) = V(U) \bigcap X'_t(\tau) = U$.

We prove Statement 4 for the case when $t = i(U)$.

Let now $1 \leq t < i(U) \leq k$. As (see 1.3) $X_{i(U)}(\tau) \subseteq X_t(\tau)$, then $U \subseteq X_{i(U)}(\tau) \bigcap V(U) \subseteq V(U) \bigcap X_t(\tau) \in \tau_t$ (the definition of $\tau_t$ see in 1.3). From the finiteness of the set $\tau_t$ it follows that there exists an atom $W$ in the lattice $(\tau_t, \subseteq)$ such that $W \subseteq V(U) \bigcap X_t(\tau)$. Then $W \in \widetilde{X}_t$, and hence, $\emptyset \neq W \subseteq X'_t(\tau) \bigcap V(U) = S_t(U, \tau)$.

We have proved Statement 4 for the case when $t < i(U)$.

Let now $i(U) < t \leq k$. Then (see the construction of the sets $X_j(\tau)$ in 1.3)

$$X_t(\tau) = X \setminus \bigcup_{l=1}^{t-1} X'_l(\tau) \subseteq X \setminus X'_{i(U)}(\tau),$$

and hence,

$$S_t(U,\tau) = V(U) \bigcap X'_t(\tau) \subseteq V_0 \bigcap X'_t \subseteq V_0 \bigcap \Big( X_{i(U)}(\tau) \bigcap X'_t \Big) =$$

$$\Big( V_0 \bigcap X_{i(U)}(\tau) \Big) \bigcap X'_t = U \bigcap X'_t(\tau) \subseteq X'_{i(U)}(\tau) \bigcap X'_t(\tau) = \emptyset$$

(the definition of the set $V_0$ see above), i.e. $S_t(U,\tau) = \emptyset$.

Statement 4 is proved.

**Statement 5**. Let $U, U' \in \widetilde{X}(\tau)$ and $U' \bigcap S_t(U,\tau) \neq \emptyset$ for some $1 < t \leq k$.

If $x \in U' \bigcap S_t(U,\tau)$ then $x \in U' = S_{i(U')}(U',\tau) \subseteq X'_{i(U')}(\tau)$ and $x \in S_t(U,\tau) \subseteq X'_t(\tau)$. As (see Statement 2) $X'_i(\tau) \bigcap X'_j(\tau) = \emptyset$ for $i \neq j$ then $t = i(U')$.

From the definition of the number $i(U')$ (see Statement 4) it follows that $U'$ is an atom in the lattice $(\tau_{i(U')}, \subseteq)$. As $V(U) \bigcap X_{i(U')}(\tau) \in \tau_{i(U')}$ and

$$U' \bigcap \Big( V(U) \bigcap X_{i(U')}(\tau) \Big) = U' \bigcap (S_{i(U')}(U,\tau) \neq \emptyset$$

then $U' \bigcap S_{i(U')}(U,\tau) = U'$, and hence, $U' \subseteq S_{i(U')}(U,\tau) \subseteq V(U)$. Then $V(U') = \bigcap_{W \in \tau, U' \subseteq W} W \subseteq V(U)$, and hence,

$$S_q(U',\tau) = X'_q(\tau) \bigcap V(U') \subseteq X'_q(\tau) \bigcap V(U) = S_q(U,\tau)$$

for any $q \leq i(U')$.

Statement 5 is proved.

**Statement 6**. Let $U \in \widetilde{X}(\tau)$.

From the definition of the sets $S_t(U,\tau)$ (see the formulation of this theorem) it follows that

$$\bigcup_{t=1}^{i(U)} S_t(U,\tau) = \bigcup_{t=1}^{i(U)} \Big( V(U) \bigcap X'_t(U,\tau) \Big) \subseteq V(U).$$

Let now $z \in V(U)$. As (see Statement 2) $\{X'_1(\tau), \ldots, X'_k(\tau)\}$ is a splitting of the set $X$ then $z \in X'_q(\tau)$ for some $1 \leq q \leq k$, and hence, $z \in V(U) \bigcap X'_q(\tau) = S_q(U,\tau)$. Then (see Statement 4) $q \leq i(U)$, and hence, $z \in \bigcup_{t=1}^{i(U)} S_t(U,\tau)$.

From the randomness of the element $z \in V(U)$ it follows that $V(U) \subseteq \bigcup_{t=1}^{i(U)} S_t(U,\tau)$, and hence, $V(U) = \bigcup_{t=1}^{i(U)} S_t(U,\tau)$.

Statement 6 is proved.

**Statement 7**. If we apply successively Remark 2, Statement 6, Remark 2, and Statement 4 then we obtain

$$U \subseteq V(U) \bigcap X'_{i(U)}(\tau) \subseteq V(U) \bigcap X_{i(U)}(\tau) = \Big(\bigcup_{i=1}^{i(U)} S_i(U,\tau)\Big) \bigcap X_{i(U)}(\tau) \subseteq$$

$$\Big(\Big(\bigcup_{i=1}^{i(U)-1} X'_i(\tau)\Big) \bigcup S_{i(U)}(U,\tau)\Big) \bigcap X_{i(U)}(\tau) =$$

$$\Big(\Big(\bigcup_{i=1}^{i(U)-1} X'_i(\tau)\Big) \bigcap X_{i(U)}(\tau)\Big) \bigcup \Big(S_{i(U)}(U,\tau) \bigcap X_{i(U)}(\tau)\Big) =$$

$$S_{i(U)}(U,\tau) \bigcap X_{i(U)}(\tau) = S_{i(U)}(U,\tau) = U,$$

and hence, $U = V(U) \bigcap X'_{i(U)}(\tau) = V(U) \bigcap X_{i(U)}(\tau)$.

Statement 7 is proved.

**Statement 8**. In the beginning we shall show that the set $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ is a base of the topological space $(X,\tau)$.

Let $W \in \tau$ and $x \in W$. Then (see Statement 1) $x \in U_x$ for some $U_x \in \widetilde{X}(\tau)$. As (see Statement 4) $U_x \in \widetilde{X}_{i(U_x)}(\tau)$ then $U_x$ is an atom in the lattice $(\tau_{i(U_x)}, \subseteq)$ and $x \in W \bigcap U_x \in \tau_{i(U_x)}$. Then $W \bigcap U_x = U_x$, and hence, $U_x \subseteq W$. From the definition of the set $V(U)$ for $U \in \widetilde{X}$ it follows that $V(U_x) \subseteq W$. Then $W = \bigcup\limits_{x \in W} \{x\} \subseteq \bigcup\limits_{x \in W} V(U_x) \subseteq W$, and hence, $W = \bigcup\limits_{x \in W} V(U_x)$.

From the randomness of the set $W \in \tau$ it follows that the set $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ is a base of the topological space $(X,\tau)$.

Now let's show that the set $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ is the minimal base of the topological space $(X,\tau)$.

Let $\mathcal{B}$ be the minimal base of the topological space $(X,\tau)$.

If $U_0 \in \widetilde{X}(\tau)$ and $x \in U_0$ then there exists $W_0 \in \mathcal{B}$ such that $x \in W_0 \subseteq V(U_0)$. As $U_0 \in \widetilde{X}_{i(U_0)}(\tau)$ and $x \in U_0 \bigcap W_0$ then

$$\emptyset \neq U_0 \bigcap W_0 = U_0 \bigcap \Big(W_0 \bigcap X_{i(U_0)}(\tau)\Big) \in \tau_{i(U_0)},$$

and as $U_0$ is an atom in the lattice $(\tau_{i(U_0)}, \subseteq)$ then $U_0 \subseteq W_0$.

Then $V(U_0) = \bigcap\limits_{W \in \tau, U_0 \subseteq W} W \subseteq W_0 \subseteq V(U_0)$, and hence, $V(U_0) = W_0 \in \mathcal{B}$.

From the randomness $U_0 \in \widetilde{X}(\tau)$ it follows that $\{V(U) \mid U \in \widetilde{X}(\tau)\} \subseteq \mathcal{B}$, and as $\mathcal{B}$ is the minimal base of the topological space $(X,\tau)$ then $\{V(U) \mid U \in \widetilde{X}(\tau)\} = \mathcal{B}$, and hence, $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ is the minimal base of the topological space $(X,\tau)$.

To complete the proof of this statement it remains to check up that $V(U) \neq V(U')$ for any $U, U' \in \widetilde{X}(\tau)$ such that $U \neq U'$.

We suppose the contrary, i.e. that $V(U) = V(U')$ for some $U, U' \in \widetilde{X}(\tau)$ and $U \neq U'$.

We can assume that $i(U) \leq i(U')$.

If $i(U) = i(U')$ then (see Statement 7) $U = V(U) \bigcap X_{i(U)}(\tau) = V(U') \bigcap X_{i(U')}(\tau) = U'$. We have received the contradiction, and hence, $V(U) \neq V(U')$ for the case when $i(U) = i(U')$.

If $i(U) < i(U')$ (see 1.3) then $X_{i(U)}(\tau) \supset X_{i(U')}(\tau)$. Then (see Statement 7 and Remark 2)

$$X'_{i(U)}(\tau) \supseteq U = V(U) \bigcap X_{i(U)}(\tau) \supseteq V(U') \bigcap X_{i(U')}(\tau) = U'$$

and $U' \subseteq X'_{i(U')}(\tau)$, and hence, $X'_{i(U)}(\tau) \bigcap X'_{i(U')}(\tau) \neq \emptyset$.

We have received the contradiction with the statement 2, and hence, $V(U) \neq V(U')$ and for the case when $i(U) \neq i(U')$.

Statement 8 is proved, and hence, the theorem is completely proved. $\square$

**Theorem 2.** *Let:*

*- $\tau$ and $\tau'$ be topologies on a finite set $X$;*

*- $\widetilde{X}(\tau)$ and $\widetilde{X}(\tau')$ be the sets of all quasiatoms in the lattices $(\tau, \subseteq)$ and $(\tau', \subseteq)$, accordingly;*

*- $\widetilde{X}_i(\tau)$ and $\widetilde{X}_i(\tau')$ be the sets of all atoms of a level $i$ in the lattices $(\tau, \subseteq)$ and $(\tau', \subseteq)$, accordingly, for $i \in \mathbb{N}$;*

*- $k = \max\{i \mid \widetilde{X}'_i(\tau) \neq \emptyset\}$ and $k' = \max\{i \mid \widetilde{X}'_i(\tau') \neq \emptyset\}$;*

*- $X'_i(\tau) = \bigcup\limits_{U \in \widetilde{X}_i(\tau)} U$ and $X'_i(\tau') = \bigcup\limits_{U' \in \widetilde{X}_i(\tau')} U$ for $1 \leq i \leq k$;*

*- $S_i(U, \tau) = \Big(\bigcap\limits_{W \in \tau, U \subseteq W} W\Big) \bigcap X'_i(\tau)$ for any $U \in \widetilde{X}(\tau)$ and any $1 \leq i \leq k$ and $S_i(U', \tau') = \Big(\bigcap\limits_{W' \in \tau', U' \subseteq W'} W'\Big) \bigcap X'_i(\tau')$ for any $U' \in \widetilde{X}(\tau')$ and any $1 \leq i \leq k'$;*

*- $i(U)$ and $i'(U')$ be such natural numbers that $U \in \widetilde{X}'_{i(U)}(\tau)$ and $U' \in \widetilde{X}'_{i(U')}(\tau')$ for $U \in \widetilde{X}(\tau)$ and $U' \in \widetilde{X}(\tau')$.*

*Then $\tau = \tau'$ if and only if the following equalities are true:*

*1. $k = k'$ and $\widetilde{X}_i(\tau) = \widetilde{X}_i(\tau')$ for any $1 \leq i \leq k$;*

*2. ${X'}_i(\tau) = {X'}_i(\tau')$ for any $1 \leq i \leq k$;*

*3. $\widetilde{X}(\tau) = \widetilde{X}(\tau')$;*

*4. $i(U) = i'(U)$ for any $U \in \widetilde{X}(\tau) = \widetilde{X}(\tau')$;*

*5. $S_i(U, \tau) = S_i(U, \tau')$ for any $U \in \widetilde{X}(\tau)$ and any $1 \leq i \leq i(U)$.*

*Proof.* **Necessity**. Let $\tau = \tau'$.

From the construction of atoms of a level $i$ (see 1.4) it follows that $\widetilde{X}_i(\tau) = \widetilde{X}_i(\tau')$ for any $i \in \mathbb{N}$.

Then $k = k'$ and ${X'}_i(\tau) = {X'}_i(\tau')$ for any $1 \leq i \leq k$.

Moreover, $\widetilde{X}(\tau) = \bigcup\limits_{i=1}^{k} \widetilde{X}_i(\tau) = \bigcup\limits_{i=1}^{k'} \widetilde{X}_i(\tau') = \widetilde{X}(\tau')$ and $i(U) = i'(U)$ for any $U \in \widetilde{X}$.

As $\bigcap\limits_{W\in\tau, U\subseteq W} W = \bigcap\limits_{W\in\tau', U\subseteq W} W$ for any $U \in \widetilde{X}$ then

$$S_i(U,\tau) = \Big(\bigcap_{W\in\tau, U\subseteq W} W\Big)\bigcap X'_i(\tau) = \Big(\bigcap_{W\in\tau', U\subseteq W} W\Big)\bigcap X'_i(\tau') = S_i(U,\tau'),$$

for any $U \in \widetilde{X}$ and any $1 \le i \le i(U)$.

Necessity is proved.

**Sufficiency**. Let topologies $\tau$ and $\tau'$ be defined on a finite set $X$ and let equalities $1-5$ be true.

If $V(U) = \bigcap\limits_{W\in\tau, U\subseteq W} W$ and $V'(U) = \bigcap\limits_{W\in\tau', U\subseteq W} W$, then (see Statement 8 of Theorem 1) $\{V(U) \mid U \in \widetilde{X}(\tau)\}$ and $\{V'(U) \mid U \in \widetilde{X}(\tau')\}$ are bases in topological spaces $(X,\tau)$ and $(X,\tau')$, accordingly.

Then (see Statement 6 of Theorem 1)

$$V(U) = \bigcup_{t=1}^{i(U)} S_t(U,\tau) = \bigcup_{t=1}^{i(U)} S_t(U,\tau') = V'(U),$$

and hence, $\{V(U) \mid U \in \widetilde{X}(\tau)\} = \{V'(U) \mid U \in \widetilde{X}(\tau')\}$.

As any topology is defined unique by any its base then $\tau = \tau'$.

The theorem is completely proved. □

## 2. A method of the construction of topology on any finite set

**Theorem 3.** *Let us have:*

1. *A finite set $X$ which has the cardinality $n$;*
2. *A natural number $k$, $1 \le k \le n$;*
3. *A splitting $\widetilde{X} = \{U_1, \ldots U_k\}$ of the set $X$;*
4. *A splitting $\{\widetilde{X}_1, \ldots, \widetilde{X}_t\}$ of the set $\widetilde{X}$ and let $X'_j = \bigcup\limits_{U\in\widetilde{X}_j} U$ for any $1 \le j \le t$;*

*5. For every $U \in \widetilde{X}$ we shall designate by $i(U)$ such a natural number that $U \in \widetilde{X}_{i(U)}$ (as $\{\widetilde{X}_i \mid 1 \le i \le t\}$ is a splitting of the set $\widetilde{X}$ then the number $i(U)$ exists and is unique);*

*6. For any $U \in \widetilde{X}$ and any $1 \le j \le i(U)$ there exists such a nonempty subset $S_j(U) \subseteq X'_j$ that:*

*- $S_{i(U)}(U) = U$ for any $U \in \widetilde{X}$;*

*- If $U, U' \in \widetilde{X}$ and $S_i(U)\bigcap U' \ne \emptyset$ for some $i \le i(U)$, then $S_l(U') \subseteq S_l(U)$ for any $1 \le l \le i(U')$.*

*Then the following statements are true:*

***Statement 3.1**. $\{X'_j \mid 1 \le j \le t\}$ is a splitting of the set $X$.*

***Statement 3.2**. There exists the unique topology $\tau$ on the set $X$ such that the following statements are true:*

***3.2.1***. $\bigcup\limits_{j=1}^{i(U)} S_j(U) = \bigcap\limits_{W\in\tau, U\subseteq W} W$*;*

***3.2.2***. *The weight of the topological space* $(X,\tau)$ *is equal to* $k$*;*

***3.2.3***. $\widetilde{X}_i = \widetilde{X}_i(\tau)$ *and* $X'_i = X'_i(\tau)$ *for any* $1 \le i \le t$*, and hence,*

$$\widetilde{X} = \bigcup_{i=1}^{t} \widetilde{X}_i = \bigcup_{i=1}^{t} \widetilde{X}_i(\tau) = \widetilde{X}(\tau)$$

*(the definition of the set* $\widetilde{X}(\tau)$*, the set* $\widetilde{X}_i(\tau)$*), and the set* $X'_i(\tau)$ *see in 1.3);*

***3.2.4***. $S_i(U,\tau) = S_i(U)$ *for any* $U \in \widetilde{X}$ *and any* $1 \le i \le i(U)$ *(the definition of the set* $S_i(U,\tau)$ *see in Theorem 1).*

*Proof.* **Statement 3.1**. If $x \in X$ then there exists $U \in \widetilde{X}$ such that $x \in U$ (see the condition 3 of this theorem). Then (see the condition 6 of this theorem) $x \in U = S_{i(U)}(U) \subseteq X'_{i(U)}$.

From the randomness of the element $x \in X$ it follows that $X = \bigcup\limits_{i=1}^{t} X'_i$.

To complete the proof of this statement we need to check up that $X'_i \bigcap X'_j = \emptyset$ for $i \neq j$.

We assume the contrary, i.e. that $X'_i \bigcap X'_j \neq \emptyset$ for some $i \neq j$, and let $z \in X'_i \bigcap X'_j$. Then (see the condition 4 of this theorem) there are $U \in \widetilde{X_i} \subseteq \widetilde{X}$ and $U' \in \widetilde{X_j} \subseteq \widetilde{X_i}(\tau)$ such that $z \in U$ and $z \in U'$.

As $\widetilde{X}_1, \ldots, \widetilde{X}_t$ is a splitting of the set $\widetilde{X}$ and $i \neq j$ then $U \neq U'$, and as $z \in U \bigcap U'$ then we receive the contradiction whit the condition 3 of this theorem.

Statement 3.1 is proved.

**Statement 3.2**. For any $U \in \widetilde{X}$ we consider the set $W(U) = \bigcup\limits_{i=1}^{i(U)} S_i(U)$, and let $\mathbf{B} = \{W(U) \mid U \in \widetilde{X}\}$.

We designate by $\tau$ the set of all subsets of the set $X$ each of which can be presented as a union of some sets from $\mathbf{B}$.

We show that $\tau$ is the required topology on the set $X$.

As $\emptyset = \bigcup\limits_{U\in\emptyset} U$ then $\emptyset \in \tau$.

Let now $x \in X$. Then (see the condition 3) $x \in U$ for some $U \in \widetilde{X}$, and hence, $x \in U = S_{i(U)}(U) \subseteq \bigcup\limits_{i=1}^{i(U)} S_i(U) = W(U)$.

From the randomness of the element $x \in X$ it follows that $\bigcup\limits_{U\in\widetilde{X}} W(U) = X$, and hence, $X \in \tau$.

Let now $A, C \in \tau$ and $x \in A \bigcap C$.

As $\widetilde{X}$ is a splitting of the set $X$ then there exists $U_x \in \widetilde{X}$ such that $x \in U_x = S_{i(U_x)}(U_x)$.

From the definition of the set $\tau$ it follows that there are $U \in \widetilde{X}$ and $U' \in \widetilde{X}$ such that $x \in W(U) \subseteq A$ and $x \in W(U') \subseteq C$.

As $W(U) = \bigcup\limits_{i=1}^{i(U)} S_i(U)$, then $x \in S_{i_1}(U)$ for some $1 \leq i_1 \leq i(U)$.

So, we have that $U, U_x \in \widetilde{X}$ and $x \in S_{i_1}(U) \bigcap U_x$.

As $x \in U_x = S_{i(U_x)} \subseteq X'_{i(U_x)}$ and $x \in S_{i_1}(U) \subseteq X'_{i_1}$ then (see Statement 3.1) $i(U_x) = i_1 \leq i(U)$. Then (see the condition 6) $S_l(U_x) \subseteq S_l(U)$ for any $1 \leq l \leq i(U_x)$, and hence,

$$W(U_x) = \bigcup_{l=1}^{i(U_x)} S_l(U_x) \subseteq \bigcup_{l=1}^{i(U_x)} S_l(U) \subseteq \bigcup_{l=1}^{i(U)} S_l(U) = W(U) \subseteq A.$$

Similarly, it is proved that $W(U_x) \subseteq W(U') \subseteq C$.

Then $W(U_x) \subseteq A \bigcap C$ for any element $x \in A \bigcap C$, and hence,

$$A \bigcap C = \bigcup_{x \in A \bigcap C} \{x\} \subseteq \bigcup_{x \in A \bigcap C} W(U_x) \subseteq A \bigcap C,$$

i.e. $A \bigcap B = \bigcup\limits_{x \in A \bigcap C} W(U_x)$. From the definition of the set $\tau$ it follows that $A \bigcap C \in \tau$.

As any union of sets each of which is some union of sets from $\mathbf{B}$ is a union of sets from $\mathbf{B}$ then $\tau$ is a topology on the set $X$.

Now let's check up that for the topology $\tau$ Statements 3.2.1 – 3.2.4 are true.

**3.2.1**. Let $U \in \widetilde{X}$ and let $W \in \tau$ be such that $U \subseteq W$.

Let's choose some element $x \in U$. From the definition of the topology $\tau$ it follows that there exists $U' \in \widetilde{X}$ such that $x \in W(U') \subseteq W$. As $W(U') = \bigcup\limits_{j=1}^{i(U')} S_j(U')$ then $x \in S_{j_0}(U')$ for some $j_0 \leq i(U')$.

Then $x \in U \bigcap S_{j_0}(U')$, and hence, (see the condition 6) $S_l(U) \subseteq S_l(U')$ for any $1 \leq l \leq i(U)$.

As $x \in U = S_{i(U)}(U) \subseteq X'_{i(U)}$ and $x \in S_{j_0}(U') \subseteq X'_{j_0}$ then from Statement 3.1 it follows that $i(U) = j_0 \leq i(U')$.

Then $W(U) = \bigcup\limits_{l=1}^{i(U)} S_l(U) \subseteq \bigcup\limits_{l=1}^{i(U')} S_l(U') = W(U') \subseteq W$.

From the randomness of the set $W \in \tau$ it follows that $W(U) \subseteq \bigcap\limits_{W \in \tau, U \subseteq W} W$.

Moreover, as $U = S_{i(U)}(U) \subseteq \bigcup\limits_{i=1}^{i(U)} S_i(U) = W(U) \in \tau$ then $\bigcap\limits_{W \in \tau, U \subseteq W} W \subseteq W(U)$, and hence, $\bigcap\limits_{W \in \tau, U \subseteq W} W = W(U)$.

Statement 3.2.1 is proved.

**3.2.2**. We notice that from the definition of the topology $\tau$ it follows that the set $\mathbf{B}$ is a base of the topological space $(X,\tau)$.

Let's show that $\mathbf{B}$ is the minimal base of the topological space $(X,\tau)$.

Let $\mathbf{B}'$ be the minimal base of the topological space $(X,\tau)$ and let $W \in \mathbf{B}$. Then $W = W(U) = \bigcup\limits_{t=1}^{i(U)} S_t(U)$ for some $U \in \widetilde{X}$.

We choose some element $x \in U$. Then $x \in U = S_{i(U)}(U) \subseteq W(U)$, and as $W(U) \in \tau$ then there exists $W' \in \mathbf{B}'$ such that $x \in W' \subseteq W(U)$.

As $W' \in \tau$ then (see the definition of the topology $\tau$) there exists $U' \in \widetilde{X}$ such that $x \in W(U') = \bigcup\limits_{i=1}^{i(U')} S_i(U') \subseteq W'$, and hence, $x \in S_{j_0}(U')$ for some $1 \le j_0 \le i(U')$.

Then $x \in U \bigcap S_{j_0}(U')$, and according to the condition 6 $S_l(U) \subseteq S_l(U')$ for any $l \le i(U)$.

Moreover, as in the proof of Statement 3.2.1 it can be proved that $i(U) = j_0 \le i(U')$. Then

$$W(U) = \bigcup_{j=1}^{i(U)} S_j(U) \subseteq \bigcup_{j=1}^{i(U)} S_j(U') \subseteq \bigcup_{j=1}^{i(U')} S_j(U) = W(U') \subseteq W' \subseteq W(U),$$

i.e. $W(U) = W' \in \mathbf{B}'$. From the randomness of $U \in \widetilde{X}$ it follows that $\mathbf{B} \subseteq \mathbf{B}'$.

Then from the minimality of the base $\mathbf{B}'$ it follows that $\mathbf{B} = \mathbf{B}'$.

To complete the proof of Statement 3.2.2 it remains to prove that $W(U) \neq W(U')$ for any $U, U' \in \widetilde{X}$ and $U \neq U'$

We can assume that $i(U) \le i(U')$.

If $i(U) < i(U')$ (see the definition of the set $W(U)$, the property 6, and Statement 3.2.1) then

$$W(U) \bigcap X'_{i(U')} = \Big(\bigcup_{i=1}^{i(U)} S_i(U)\Big) \bigcap X'_{i(U')} \subseteq \Big(\bigcup_{i=1}^{i(U)} X'_i\Big) \bigcap X'_{i(U')} = \emptyset \neq$$

$$S_{i(U')}(U') = \Big(\Big(\bigcup_{i=1}^{i(U')-1} S_i(U)\Big) \bigcap X'_{i(U')}\Big) \bigcup \Big(S_{i(U')} \bigcap X'_{i(u')}\Big) =$$

$$\Big(\bigcup_{i=1}^{i(U')} S_i(U')\Big) \bigcap X'_{i(U')} = W(U') \bigcap X'_{i(U')},$$

and hence, $W(U) \neq W(U')$ for the case $i(U) < i(U')$.

If $i(U) = i(U')$ then (see Statement 3.2.1)

$$W(U) \bigcap X'_{i(U)} = \Big(\bigcup_{i=1}^{i(U)} S_i(U)\Big) \bigcap X'_{i(U)} = S_{i(U)} = U \neq$$

$$U' = S_{i(U')} = \Big(\bigcup_{i=1}^{i(U)} S_i(U')\Big) \bigcap X'_{i(U)} = W(U') \bigcap X'_{i(U)},$$

and hence, $W(U) \neq W(U')$ for the case $i(U) = i(U')$, too.

Statement 3.2.2 is proved.

**3.2.3**. We prove the equalities $\widetilde{X}_i = \widetilde{X}_i(\tau)$ and $X_i = X'{}_i(\tau)$ by induction on $i$.

Let $i = 1$.

In the beginning let's prove that $\widetilde{X}_1 \subseteq \widetilde{X}_1(\tau)$.

We assume the contrary, i.e. that $\widetilde{X}_1 \not\subseteq \widetilde{X}_1(\tau)$, and let $U \in \widetilde{X}_1 \setminus \widetilde{X}_1(\tau)$. Then $i(U) = 1$, and hence, $U = S_1(U) = \bigcup\limits_{i=1}^{1} S_i(U) \in \tau$. As $U \notin \widetilde{X}_1(\tau)$, then $W \subset U$, for some $W \in \tau$. From the definition of the topology $\tau$ it follows that there exists $U' \in \widetilde{X}$ such that $\bigcup\limits_{i=1}^{i(U')} S_i(U') \subseteq W \subset U$, and hence, $S_1(U') \bigcap U = S_1(U') \neq \emptyset$. Then by the condition 6 $U = S_1(U) \subseteq S_1(U') \subseteq W \subset U$, and hence, $U = W$.

We have received the contradiction (see the choice of $U$), and hence, $\widetilde{X}_1 \subseteq \widetilde{X}_1(\tau)$.

Let now $U \in \widetilde{X}_1(\tau)$. Then $U$ is an atom in the lattice $(\tau, \subseteq)$, and hence, $U \in \tau$. From the definition of the topology $\tau$ it follows that there exists $U' \in \widetilde{X}$ such that $W(U') = \bigcup\limits_{i=1}^{i(U')} S_i(U') \subseteq U$. By the condition 6 of this theorem $X'_1 \supseteq S_1(U') \neq \emptyset$. As $X'_1 = \bigcup\limits_{W \in \widetilde{X}_1}$ then $U_0 \bigcap S_1(U') \neq \emptyset$ for some $U_0 \in \widetilde{X_1}$. Then $i(U_0) = 1$ and by the condition 6, we receive that $U_0 = S_1(U_0) \subseteq S_1(U') \subseteq \bigcup\limits_{i=1}^{i(U')} S_i(U') \subseteq U$. As $U_0 = \bigcup\limits_{i=1}^{1} S_i(U_0) \in \tau$ and $U$ is an atom in the lattice $(\tau, \subseteq)$ then $U = U_0 \in \widetilde{X}_1$.

From the randomness of the set $U \in \widetilde{X}_1(\tau)$ it follows that $\widetilde{X}_1(\tau) \subseteq \widetilde{X}_1$, and hence, $\widetilde{X}_1(\tau) = \widetilde{X}_1$.

Then $X'_1 = \bigcup\limits_{U \in \widetilde{X}_1} U = \bigcup\limits_{U \in \widetilde{X}_1(\tau)} U = X'_1(\tau)$.

Hence, the equalities $\widetilde{X}_i = \widetilde{X}_i(\tau)$ and $X_i = X'{}_i(\tau)$ for $i = 1$ are true.

We suppose that the equalities $\widetilde{X}_i(\tau) = \widetilde{X}_i$ and $X'_i(\tau) = X'_i$ are true for $i \leq s$ and any finite set, any natural number $k$, any set $\widetilde{X}$, the sets $\widetilde{X}_1, \dots, \widetilde{X}_k$, and the subsets $S'_j(U) = S_{s+j}(U)$ for $U \in \widetilde{Y}$ and $j \leq k - s$ for which the conditions of this theorem are satisfied.

Let's consider:

- The set $Y = X \setminus \Big(\bigcup\limits_{i=1}^{s} X'_i\Big)$;
- The natural number $k - s$;
- The set $\widetilde{Y} = \bigcup\limits_{i=s+1}^{k} \widetilde{X}_i$;
- The sets $\widetilde{Y}_1 = \widetilde{X}_{s+1}, \dots, \widetilde{Y}_{k-s}1 = \widetilde{X}_k$;
- The subsets $S'_j(U) = S_{s+j}(U)$ for $U \in \widetilde{Y}$ and $j \leq k - s$.

It is easy to notice that the conditions of this theorem are satisfied for them. Then applying Statements 3.2.1, 3.2.2 and 3.2.3 for the case $i = 1$ we can construct a topology $\tau'$ on the set $Y$ such that the set $\{W'(U) = \bigcup_{i=1}^{i(U)-s} S'_i(U) \mid U \in \widetilde{Y}\}$ is a base of the topological space $(Y, \tau')$ and $\widetilde{Y}_1 = \widetilde{Y}_1(\tau')$.

As $S_i \subseteq X'_i$ for $1 \leq i \leq s$ and $X'_i \bigcap X'_j = \emptyset$ for $i \neq j$ then

$$W'(U) = \bigcup_{i=1}^{i'(U)} S'_i(U) = \bigcup_{i=s+1}^{i(U)} S_i(U) = \Big(\bigcup_{i=1}^{i(U)} S_i(U)\Big) \bigcap \Big(X \setminus \Big(\bigcup_{i=1}^{s} X'_i\Big)\Big) =$$

$$\Big(\bigcup_{i=1}^{i(U)} S_i(U)\Big) \bigcap \Big(X \setminus \Big(\bigcup_{i=1}^{s} X'_i(\tau)\Big)\Big) = W(U) \bigcap X_{s+1}(\tau).$$

As the sets $\{W'(U) \mid U \in \widetilde{Y}\}$ and $\{W(U) \mid U \in \widetilde{X}\}$ are bases of the topological spaces $(Y, \tau')$ and $(X, \tau)$, accordingly, and

$$Y = X \setminus \Big(\bigcup_{i=1}^{s} X'_i\Big) = X \setminus \Big(\bigcup_{i=1}^{s} X'_i(\tau)\Big) = X_{s+1}(\tau)$$

then $\tau' = \tau|_{X_{s+1}} = \tau_{s+1}$, and hence, the set $\widetilde{X}_{s+1} = \widetilde{Y}_1$ is the set of all atoms in the lattice $(\tau_{s+1}, \subseteq)$, i.e. $\widetilde{X}_{s+1} = \widetilde{X}_{s+1}(\tau)$.

Then

$$X'_{s+1} = \bigcup_{U \in \widetilde{X}_{s+1}} U = \bigcup_{U \in \widetilde{X}_{s+1}(\tau)} U = X'_{s+1}(\tau).$$

Statement 3.2.3 is proved for case $i = s + 1$.

**3.2.4**. As $X'_i(\tau) = X'_i$ for any $1 \leq i \leq t$ then (see Statements 3.2.1, 3.1, and the condition 6) $S_j(U, \tau) = V(U) \bigcap X'_j(\tau) = \Big(\bigcap_{V \in \tau, U \subseteq V} V\Big) \bigcap X'_j(\tau) = \Big(\bigcup_{i=1}^{i(U)} S_i(U)\Big) \bigcap X'_j = S_j(U)$.

So, we have proved that $S_j(U, \tau) = S_j(x)$ for any $1 \leq j \leq t$ and any $U \in \widetilde{X}$.

Statement 3.2.4 is proved.

To complete the proof of Statement 3.2 it is necessary to check up the uniqueness of the topology for which Statements 3.2.1 - 3.2.4 are true. But it follows from Theorem 2.

Statement 3.2 is proved, and hence, the theorem is completely proved. □

**Example 1.** The method which has been specified in Theorem 3 will be applied now for constructing a topology $\tau$ on a finite set $X$ of cardinality $n$.

1. We fix a positive integer $k \leq n$ (number $k$ be the weight of the topological space $(X, \tau)$).

2. Consider a partition $\widetilde{X}$ of the set $X$ into $k$ subsets (the set $\widetilde{X}$ be the set of all quasiatoms in the topological space $(X, \tau)$).

3. Fix a natural number $t \leq k$ and consider a sequence $k_1, \ldots, k_t$ of positive integers such that $\sum\limits_{i=1}^{t} k_i = k$.

4. Consider a sequence $\widetilde{X}_1, \ldots, \widetilde{X}_t$ such that the set $\{\widetilde{X}_1, \ldots, \widetilde{X}_t\}$ is a partition of the set $\widetilde{X}$ and $|X_i| = k_i$ for $1 \leq i \leq t$. (For every $1 \leq i \leq t$ the set $\widetilde{X}_i$ be the set of all quasiatoms of level $i$ in the topological space $(X, \tau)$.)

5. For any integer $1 \leq i < t$ we consider the set $\overline{X}_i$ of all nonempty subsets of the set $\widetilde{X}_i$.

6. For any integer $2 \leq i \leq t$ we consider a map $f_i : \widetilde{X}_i \to \overline{X}_{i-1}$.

7. For any positive integers $1 < i \leq j \leq t$ and any $U \in \widetilde{X}_j$ we consider the positive integer $i(U) = j$ and the subset $S_i(U)$ of the set $\overline{X_i}$ such that $S_i(U) = \{U\}$ for $i = j$ and $S_i(U) = f_{i+1}(f_{i+2}(\ldots f_j(U)\ldots))$ for $1 < i < j$.

It is easy to see that the natural number $k$, the partition $\widetilde{X}$ of the set $X$, the sequence $\widetilde{X}_1, \ldots, \widetilde{X}_t$, and subsets $S_i(U)$ of the set $\overline{X_i}$ satisfy the conditions of Theorem 2, and hence, on the set $X$ there exists the unique topology $\tau$ for which the set $\{\bigcup\limits_{j=1}^{i(U)} |U \in \widehat{X}\}$ is a base of the topological space $(X, \tau)$.

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