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# Isohedral tilings on Riemann surfaces of genus 2

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**Abstract.** 3 Riemann surfaces which possess rich metrics are considered. In previous paper [1] the classification of fundamental isohedral tilings for groups of conformal automorphisms of these surfaces was obtained. Here the classification of fundamental isohedral tilings is obtained for groups of conformal and anticonformal automorphisms of the surfaces. The tilings are given by the adjacency symbols of the corresponding tilings on the universal covering hyperbolic plane.

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## 1 Introduction

The present article is a direct continuation of [1], and the work [1] is essentially used here.

Recall some papers where tilings on the hyperbolic plane or Riemann surfaces were researched. Some tilings on Riemann surfaces of genus 2 were investigated in [2] and [3]. A method of obtaining fundamental tilings for any discrete two-dimensional group with compact fundamental domain was discussed in [4, 5]. Some methods of obtaining tilings with given transitivity properties on the hyperbolic plane (as well as on the Euclidean plane and the sphere) were discussed in [6, 7]. The work [8] is devoted to symmetry groups and fundamental tilings of the compact non-orientable surface of genus 3.

## 2 Riemann surfaces

Remind some basic notions.

Definition 1. A map (or a tiling of a compact Riemann surface) is a closed compact Riemann surface divided into simply connected open regions by a finite number of arcs and simple closed curves called edges. Such a region, together with its boundary, is called a tile. Edges meet only at their endpoints called vertices, and each vertex is incident to at least 3 edges, where loops are counted twice (cf. [3]).

Definition 2. Let $W$ be a tiling of a Riemann surface and $G$ be a discrete isometry group. The tiling $W$ is called isohedral with respect to the group $G$ if $G$ maps the tiling $W$ onto itself and acts transitively on the set of all tiles.


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Thus the goal of our investigations is an enumeration of isohedral tilings on closed compact orientable Riemann surfaces of genus 2.

Definitions 3. [9] Let $X$ be a compact Riemann surface of genus $g$. A symmetry of $X$ is an anticonformal involution $T : X \to X$. The fixed point set $F(T)$ consists of $k$ disjoint Jordan curves, where $0 \leq k \leq g+1$. The species of $T$ is defined to be $+k$ if $X - F(T)$ has two components and to be $-k$ if $X - F(T)$ has one component. Define the species of a fixed-point free symmetry to be 0. Finally the symmetry type of $X$ is defined to be the unordered list of the species of the symmetries of $X$, one being listed for each conjugacy class of symmetries in the full automorphism group of $X$.

Every Riemann surface of genus $g \geq 2$ is the quotient space $H^2/K$ where $H^2$ is the hyperbolic plane, $K$ is a fixed-point free discrete isometry group of $H^2$. Lifting elements of the automorphism group $G$ of the Riemann surface onto the universal covering plane we obtain an isometry group $\Gamma$ of the plane such that $K$ is a normal subgroup of $\Gamma$ and the quotient-group $\Gamma/K \cong G$.

In [9] symmetric Riemann surfaces were investigated using some results on algebraic curves and group-theoretic methods. For each symmetric Riemann surface of genus 2, a table lists its automorphism group, universal covering hyperbolic group and symmetry type. The admissible automorphism groups are $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, $D_4$, $D_4 \times C_2$, $D_8$, $D_{10}$, $D_6 \times C_2$, $D_{12}$, $G^*_{24}$ and $G^*_{48}$. In general, the metrics of Riemann surfaces are not given. However, for the automorphism groups $D_{10}$, $G^*_{24}$ and $G^*_{48}$, the Riemann surfaces admit realizations via regular maps with quite determined metrics (the corresponding groups on the universal cover have no parameters). Now turn to these 3 Riemann surfaces.

The first Riemann surface is obtained from a regular 8-gon with angle of $\pi/4$ by the indentification of opposite sides with translations (Fig. 1, $a$). For more detailed description of this Riemann surface, its conformal automorphisms and subgroups of the full group of conformal automorphisms refer to [1, 10].

48 conformal automorphisms are the following. There are 3 'rotations' $r_k$, $k = 1$, 2, 3, of order 8, each of them having 2 invariant points (an $r_k$ lifts to rotations through angle of $\pi/4$ on the universal covering hyperbolic plane). Their powers $r_k^3$, $r_k^5$, $r_k^7$ are also 'rotations' of order 8, the powers $r_k^2$ and $r_k^6$ are of order 4. The 'inversion' $i = r_1^4 = r_2^4 = r_3^4$ of order 2 has 6 invariant points and is a hyperelliptic involution. There are 12 'rotations' $s_k$, $k = 1, 2, \ldots, 12$, of order 2, each of them having 2 invariant points. There are 4 'rotations' $v_k$, $k = 1, 2, 3, 4$, of order 3, each of them having 4 invariant points. Their powers $v_k^2$ are of order 3, too. 4 isometries $t_k = iv_k$, $k = 1, 2, 3, 4$, of order 6 have no invariant points, as well as their powers $t_k^5 = iv_k^2$.

Remark that the 3 'rotations' $r_k$, $k = 1, 2, 3$, are conjugate in the full automorphism group. Moreover, the 6 'rotations' $r_k$, $r_k^7$, $k = 1, 2, 3$, of order 8 are conjugate in the full automorphism group. However we are more interested in geometric nature of those isometries, and further we will omit the conjugacy of isometries when it is not essential. In [1] a repeated misprint occurs, so in the description of automorphisms (pp. 42–43) one should read 'conjugate' instead of 'non-conjugate'.

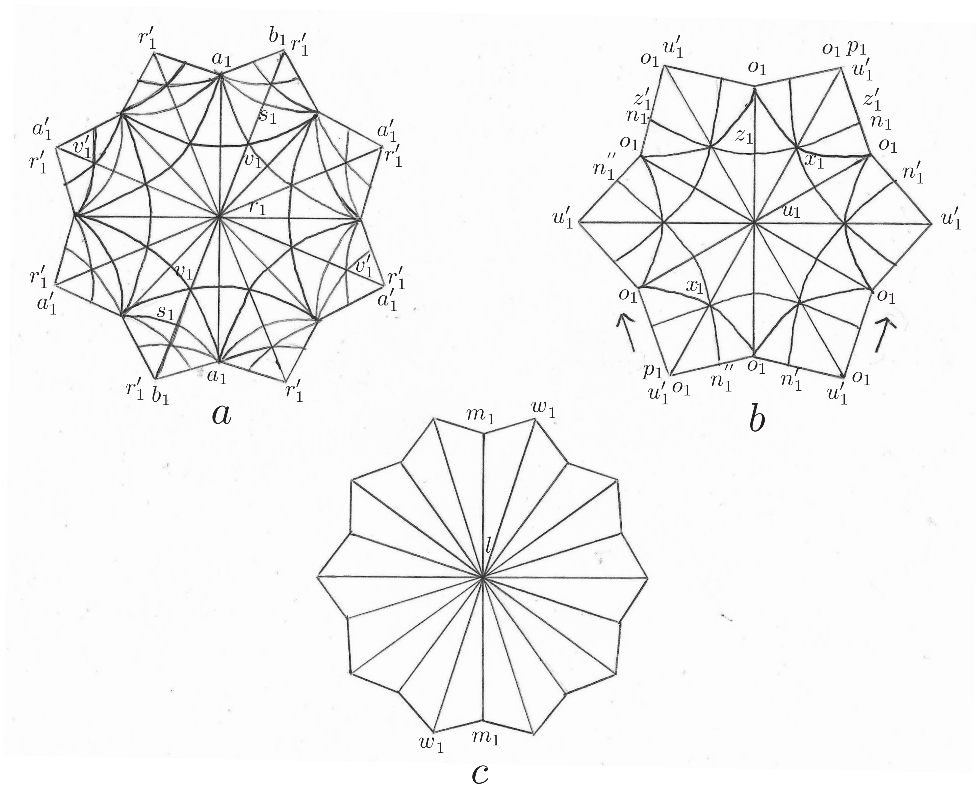



Figure 1. The 3 Riemann surfaces of genus 2 with the known rich metrics

48 anticonformal automorphisms are the following. There are 6 conjugate 'reflections' $a_k$, $k = 1, 2, \ldots, 6$, in pairs of shortest geodesics ($a_k$ and $a'_k$ in Fig. 1, $a$). The species of the symmetry $a_k$ is $-2$. There are 12 conjugate 'reflections' $b_k$, $k = 1, 2, \ldots, 12$, in 'long' geodesics ($b_k$ in Fig. 1, $a$). The species of the symmetry $b_k$ is $-1$. There are 3 isometries $d_k$, $k = 1, 2, 3$, of order 8 without invariant points. Their powers $d_k^3$, $d_k^5$, $d_k^7$ are also of order 8 and without invariant points, $d_k^2 = r_k^2$. There are 4 isometries $f_k$, $k = 1, 2, 3, 4$, of order 12 without invariant points. Their powers $f_k^5$, $f_k^7$, $f_k^{11}$ are also of order 12, the powers $f_k^3$, $f_k^9$, $k = 1, 2, 3, 4$, are two isometries of order 4, all having no invariant points, $f_k^2 = t_k$.

Thus we made sure that the symmetry type of the first Riemann surface is $\{-2, -1\}$.

The second Riemann surface can be obtained from a semiregular equilateral 12-gon with angles of $\pi/3$ and $2\pi/3$ (in Fig. 1, $b$ one pair of sides glued with translations is indicated by arrows, other pairs of sides are glued in a similar way). For more detailed description of this Riemann surface, its automorphisms and subgroups of the full automorphism group refer to [1,11].

24 conformal automorphisms are the following. There are 2 'rotations' $u_k$, $k = 1$, 2, of order 6, each of them having 2 invariant points. Their powers $u_k^5$ are also of order 6, the powers $u_k^3$ are of order 2. The powers $u_1^2 = u_2^4$ and $u_1^4 = u_2^2$ are 'rotations'

of order 3 and have 4 invariant points. There are 3 'rotations' $x_k$, $k = 1, 2, 3$, of order 4, each of them having 2 invariant points. Their powers $x_k^3$ are of order 4, too. The 'inversion' $j = x_1^2 = x_2^2 = x_3^2$ of order 2 has 6 invariant points and is a hyperelliptic involution. There are 6 'rotations' $z_k$, $k = 1, 2, \ldots, 6$, of order 2, each of them having 2 invariant points. The isometry $y = z_1 z_2$ of order 6 has no invariant points, as well as its power $y^5$.

24 anticonformal automorphisms are the following. There are 2 conjugate 'reflections' $n_k$, $k = 1, 2$, in triples of geodesics ($n_k$, $n'_k$ and $n''_k$ in Fig. 1, *b*). The species of the symmetry $n_k$ is $+3$. There are 3 conjugate 'reflections' $o_k$, $k = 1, 2, 3$, in geodesics ($o_k$ in Fig. 1, *b*). The species of the symmetry $o_k$ is $+1$. There are 6 conjugate 'reflections' $p_k$, $k = 1, 2, \ldots, 6$, in geodesics ($p_k$ in Fig. 1, *b*). The species of the symmetry $p_k$ is $-1$. There are 3 conjugate involutions $q_k$, $k = 1, 2, 3$, without invariant points. The species of the symmetry $q_k$ is 0. There is an isometry $g$ of order 12 without invariant points. Its powers $g^5$, $g^7$, and $g^{11}$ are also of order 12 and without invariant points, the powers $g^3$ and $g^9$ are of order 4 and have no invariant points, $g^2 = y$. There are 2 isometries $h_k$, $k = 1, 2$, of order 6 without invariant points. Their powers $h_k^5$ are also of order 6 and have no invariant points, $h_k^2$ and $h_k^4$ are the above 'rotations' of order 3.

Thus we made sure that the symmetry type of the second Riemann surface is $\{-1, 0, +1, +3\}$.

The third Riemann surface is obtained from a regular 10-gon with angle of $2\pi/5$ by the indentification of opposite sides with translations (Fig. 1, *c*). For more detailed description of this Riemann surface, its conformal automorphisms and subgroups of the full group of conformal automorphisms refer to [1].

10 conformal automorphisms are powers of a 'rotation' $l$ of order 10 having one invariant point. Its powers $l^3$, $l^7$, and $l^9$ are also of order 10. The powers $l^2$, $l^4$, $l^6$, and $l^8$ are 'rotations' of order 5 with 3 invariant points. The 'inversion' $l^5$ of order 2 has 6 invariant points and is a hyperelliptic involution.

10 anticonformal automorphisms are the following. There are 5 conjugate 'reflections' $m_k$, $k = 1, 2, \ldots, 5$, in geodesics ($m_k$ in Fig. 1, *c*). The species of the symmetry $m_k$ is $-1$. There are 5 conjugate 'reflections' $w_k$, $k = 1, 2, \ldots, 5$, in geodesics ($w_k$ in Fig. 1, *c*). The species of the symmetry $w_k$ is $-1$.

Thus we made sure that the symmetry type of the third Riemann surface is $\{-1, -1\}$.

## 3 Automorphism groups

For each of the 3 full automorphism groups of Riemann surfaces, a Cayley table of compositions of isometries has been compiled. Examining the Caley table has permitted to determine all geometrically different subgroups of these groups. Due to the so-called principle of symmetry elements it is equivalent to that all subgroups has been divided into classes of groups conjugate in the full automorphism group of the respective Riemann surface. In order to avoid additional symbols, for a subgroup $G'$ we use a designation of the form $\Gamma'/K$, where $\Gamma'$ is the isometry group on the universal covering plane which corresponds to $G'$ (here we use orbifold symbols [12] for $\Gamma'$). A symbol may correspond to a group on the Riemann surface or to a class of conjugate groups (in the full automorphism group). Some symbols have additional marks. Below we list groups of conformal and anticonformal automorphisms for the 3 Riemann surfaces, limiting ourselves with groups of order not less than 5. The groups of conformal automorphisms were listed in [1].

For the first Riemann surface the full automorphism group (of order 96) is denoted by $*832/K_1$, where the group $K_1$ is generated by translations that map opposite sides of the regular 8-gon with angle of $\pi/4$. There are 2 groups of conformal and anticonformal automorphisms of order 48. The group $3*4/K_1$ contains even powers of 'rotations' $r_k$, $k = 1, 2, 3$, all powers of 'rotations' $v_k$, $k = 1, 2, 3, 4$, isometries $t_k$, $t_k^5$, $k = 1, 2, 3, 4$, all 'reflections' $a_k$, $k = 1, 2, \dots, 6$, isometries $c = a_1a_2a_5$, $c^3$ (of order 4), $f_k$, $f_k^5$, $f_k^7$, $f_k^{11}$, $k = 1, 2, 3, 4$. The group $*433/K_1$ contains even powers of 'rotations' $r_k$, $k = 1, 2, 3$, all powers of 'rotations' $v_k$, $k = 1, 2, 3, 4$, isometries $t_k$, $t_k^5$, $k = 1, 2, 3, 4$, all 'reflections' $b_k$, $k = 1, 2, \dots, 12$, isometries $d_k$, $d_k^3$, $d_k^5$, $d_k^7$, $k = 1, 2, 3$. There are 3 conjugate groups of order 32, denoted by $*842/K_1$, one of them contains all powers of 'rotation' $r_1$, even powers of 'rotations' $r_2$, $r_3$, 'rotations' $s_1$, $s_2$, $s_3$, $s_4$, all 'reflections' $a_k$, $k = 1, 2, \dots, 6$, 'reflections' $b_1$, $b_2$, $b_3$, $b_4$, isometries $c$, $c^3$, $d_1$, $d_1^3$, $d_1^5$, $d_1^7$.

Enumerate groups of order 16. There are 3 conjugate groups $*882/K_1$, one of them contains all powers of 'rotation' $r_1$ and 'reflections' $a_1$, $a_2$, $a_3$, $a_4$, $b_1$, $b_2$, $b_3$, $b_4$. There are 3 conjugate groups $8*2/K_1$, one of them contains all powers of 'rotation' $r_1$, 'reflections' $a_5$, $a_6$, isometries $c$, $c^3$, $d_1$, $d_1^3$, $d_1^5$, $d_1^7$. The group $*444/K_1$ contains even powers of 'rotations' $r_k$, $k = 1, 2, 3$, all 'reflections' $a_k$, $k = 1, 2, \dots, 6$, isometries $c$, $c^3$. There are 3 conjugate groups $4*4/K_1$, one of them contains even powers of 'rotations' $r_k$, $k = 1, 2, 3$, 'reflections' $b_1$, $b_2$, $b_3$, $b_4$, isometries $d_1$, $d_1^3$, $d_1^5$, $d_1^7$. There are 3 conjugate groups $2*42/K_1$, one of them contains even powers of 'rotation' $r_1$, 'rotations' $s_1$, $s_2$, $s_3$, $s_4$, 'reflections' $a_1$, $a_2$, $a_3$, $a_4$, isometries $d_1$, $d_1^3$, $d_1^5$, $d_1^7$. There are 3 conjugate groups $*4222/K_1$, one of them contains even powers of 'rotation' $r_1$, 'rotations' $s_1$, $s_2$, $s_3$, $s_4$, 'reflections' $a_5$, $a_6$ and $b_1$, $b_2$, $b_3$, $b_4$, isometries $c$, $c^3$.

There are 2 conjugate classes, each of them consisting of 4 groups of order 12. One of groups $*3232/K_1$ contains all powers of 'rotation' $v_1$, 'inversion' $i$, isometries $t_1$, $t_1^5$, 'reflections' $b_1$, $b_3$, $b_5$, $b_7$, $b_{11}$, $b_{12}$. One of groups $32\times/K_1$ contains all powers of 'rotation' $v_1$, 'inversion' $i$, isometries $t_1$, $t_1^5$, $c$, $c^3$ and $f_1$, $f_1^5$, $f_1^7$, $f_1^{11}$.

Enumerate groups of order 8. There are 3 conjugate groups $*4242/K_1$, one of them contains even powers of 'rotation' $r_1$ and 'reflections' $a_1$, $a_2$, $a_3$, $a_4$. There are 3 conjugate groups $4*22/K_1$, one of them contains even powers of 'rotation' $r_1$, 'reflections' $a_5$, $a_6$, isometries $c$, $c^3$. There are 3 conjugate groups $2*44/K_1$, one of them contains even powers of 'rotation' $r_1$ and 'reflections' $b_1$, $b_2$, $b_3$, $b_4$. There are 3 conjugate groups $42\times/K_1$, one of them contains even powers of 'rotation' $r_1$ and isometries $d_1$, $d_1^3$, $d_1^5$, $d_1^7$. There are 6 conjugate groups $*22222/K_1$, one of them contains 'rotations' $s_1$, $s_3$, 'inversion' $i$, 'reflections' $a_5$, $a_6$ and $b_1$, $b_3$. There are 6 conjugate groups $22*2/K_1$, one of them contains 'rotations' $s_1$, $s_3$, 'inversion' $i$, 'reflections' $b_2$, $b_4$, isometries $c$, $c^3$.

There are 8 conjugate groups $3*33/K_1$ of order 6, one of them contains all powers of 'rotation' $v_1$ and 'reflections' $b_1$, $b_5$, $b_7$.

For the second Riemann surface the full automorphism group (of order 48) is denoted by $*642/K_2$, where the group $K_2$ is generated by translations that map pairs of sides of the semiregular equilateral 12-gon with angles of $\pi/3$ and $2\pi/3$ as described above. There are 6 groups of conformal and anticonformal automorphisms of order 24. The group $*662/K_2$ contains all powers of 'rotation' $u_1$, odd powers of 'rotation' $u_2$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $o_k$, $k = 1, 2, 3$, all 'reflections' $p_k$, $k = 1, 2, \ldots, 6$, all involutions $q_k$, $k = 1, 2, 3$. The group $*443/K_2$ contains even powers of 'rotation' $u_1$, odd powers of 'rotations' $x_k$, $k = 1, 2, \ldots, 6$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $n_k$, $k = 1, 2$, all 'reflections' $p_k$, $k = 1, 2, \ldots, 6$, all isometries $h_k$, $h_k^5$, $k = 1, 2$. The group $*3222/K_2$ contains even powers of 'rotation' $u_1$, all 'rotations' $z_k$, $k = 1, 2, \ldots, 6$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $n_k$, $k = 1, 2$, all 'reflections' $o_k$, $k = 1, 2, 3$, all involutions $q_k$, $k = 1, 2, 3$, all isometries $h_k$, $h_k^5$, $k = 1, 2$. The group $6*2/K_2$ contains all powers of 'rotation' $u_1$, odd powers of 'rotation' $u_2$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $n_k$, $k = 1, 2$, odd powers of isometry $g$ and all isometries $h_k$, $h_k^5$, $k = 1, 2$. The group $4*3/K_2$ contains even powers of 'rotation' $u_1$, odd powers of 'rotations' $x_k$, $k = 1, 2, 3$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $o_k$, $k = 1, 2, 3$, all involutions $q_k$, $k = 1, 2, 3$, odd powers of isometry $g$. The group $2*32/K_2$ contains even powers of 'rotation' $u_1$, all 'rotations' $z_k$, $k = 1, 2, \ldots, 6$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $p_k$, $k = 1, 2, \ldots, 6$, odd powers of isometry $g$.

Enumerate groups of order 12. There are 2 conjugate groups $*663/K_2$, one of them contains all powers of 'rotation' $u_1$, all 'reflections' $o_k$, $k = 1, 2, 3$, and 'reflections' $p_1$, $p_2$, $p_3$. There are 2 conjugate groups $6*3/K_2$, one of them contains all powers of 'rotation' $u_1$, 'reflections' $p_4$, $p_5$, $p_6$ and all involutions $q_k$, $k = 1, 2, 3$. The group $3*22/K_2$ contains even powers of 'rotation' $u_1$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $n_k$, $k = 1, 2$, and all isometries $h_k$, $h_k^5$, $k = 1, 2$. The group $2*33/K_2$ contains even powers of 'rotation' $u_1$, 'inversion' $j$, isometries $y$, $y^5$, all 'reflections' $o_k$, $k = 1, 2, 3$, and all involutions $q_k$, $k = 1, 2, 3$. The group $*3232/K_2$ contains even powers of 'rotation' $u_1$, 'inversion' $j$, isometries $y$, $y^5$, and all 'reflections' $p_k$, $k = 1, 2, \ldots, 6$. The group $32\times/K_2$ contains even powers of 'rotation' $u_1$, 'inversion' $j$, isometries $y$, $y^5$, and odd powers of isometry $g$. There are 2 conjugate groups $*3322/K_2$, one of them contains even powers of 'rotation' $U_1$, 'rotations' $z_1$, $z_3$, $z_5$, 'reflection' $n_1$, all 'reflections' $o_k$, $k = 1, 2, 3$, and isometries $h_2$, $h_2^5$. There are 2 conjugate groups $32*/K_2$, one of them contains even powers of 'rotation' $u_1$, 'rotations' $z_1$, $z_3$, $z_5$, 'reflection' $n_2$, all involutions $q_k$, $k = 1, 2, 3$, and isometries $h_1$, $h_1^5$.

There are 3 conjugate classes, each of them consisting of 3 groups of order 8. One of groups $*4422/K_2$ contains all powers of 'rotation' $x_1$, all 'reflections' $n_k$, $k = 1, 2$, and 'reflections' $p_1$, $p_4$. One of groups $*22222/K_2$ contains 'rotations' $z_1$, $z_4$, 'inversion' $j$, all 'reflections' $n_k$, $k = 1, 2$, 'reflection' $o_1$, and involution $q_1$. One of groups $2*222/K_2$ contains 'inversion' $j$, isometries $u_1^3$, $u_2^3$, 'reflections' $o_1$, $p_2$, $p_5$, and involution $q_1$.

Enumerate groups of order 6. The group $*3333/K_2$ contains even powers of 'rotation' and all 'reflections' $o_k$, $k = 1, 2, 3$. There are 2 conjugate groups $3{*}33/K_2$, one of them contains even powers of 'rotation' $u_1$ and 'reflections' $p_1$, $p_2$, $p_3$. The group $33 \times /K_2$ contains even powers of 'rotation' $u_1$ and all involutions $q_k$, $k = 1$, 2, 3. There are 2 conjugate groups $33 * /K_2$, one of them contains even powers of 'rotation' $u_1$, 'reflection' $n_2$, and isometries $h_1$, $h_1^5$.

For the third Riemann surface the full automorphism group (of order 20) is denoted by $*(10)52/K_3$, where the group $K_3$ is generated by translations that map opposite sides of the regular 10-gon with angle of $2\pi/5$. There are 2 groups of conformal and anticonformal automorphisms of order 10. The group $5 * 5/K_3$ contains even powers of 'rotation' $l$ and all 'reflections' $m_k$, $k = 1, 2, \ldots, 5$. The group $*555/K_3$ contains even powers of 'rotation' $l$ and all 'reflections' $w_k$, $k = 1, 2, \ldots, 5$.

Now we have a complete list of subgroups of the full automorphism groups of the 3 Riemann surfaces together with subgroups of conformal automorphisms listed in [1].

Thus, for the 3 Riemann surfaces of genus 2 we have described all conformal and anticonformal automorphisms and all automorphism groups of order not less than 5. Adding groups of conformal automorphisms from [1] we have listed altogether $19 + 7 = 26$ non-conjugate groups for the first Riemann surface, $22 + 7 = 29$ non-conjugate groups for the second Riemann surface, $3 + 2 = 5$ non-conjugate groups for the third Riemann surface.

## 4 Methods of finding isohedral tilings

Definition 4. Consider all possible pairs $(W, G)$ where the tiling $W$ of the Riemann surface is isohedral with respect to the group $G$. Two pairs $(W, G)$ and $(W', G')$ are said to belong to one Delone class if there exists a homeomorphism $\phi$ of the surface that maps the tiling $W$ onto the tiling $W'$ so that the relation $G = \phi^{-1}G'\phi$ holds.

The tilings of one Delone class are also called homeomeric [13] or equivariantly equivalent [6]. We distinguish between fundamental and non-fundamental Delone classes depending on whether the group $G$ acts one time (or simply) transitively on the set of tiles or not.

Our task is to classify fundamental isohedral tilings on the 3 Riemann surfaces of genus 2. In the author's work [1] fundamental isohedral tilings were obtained for groups of conformal automorphisms of the Riemann surfaces. Now we are going to obtain fundamental isohedral tilings for groups found in the previous section. Following the paper [1] we need to find fundamental isohedral tilings on the covering hyperbolic plane for appropriate isometry group.

Discuss methods for finding fundamental isohedral tilings of the hyperbolic plane. In [1] we used the method of adjacency symbols, which is analogous to the method by Delone [14,15] and was explained in detail in [10]. Because isometry groups of the hyperbolic plane found in the previous section are not complicated, in order to avoid doing superfluous work we can apply the method of obtaining fundamental domains (and tilings) developed in [4,5]. Given any discrete group with compact fundamental domain, one should construct certain graphs on the orbifold corresponding to the group and then cut the orbifold along these graphs. One more method of obtaining tilings with transitivity properties [6, 7] is based on Delaney–Dress symbols. Some algorithms were developed which produce Delaney–Dress symbols corresponding to tilings on the hyperbolic plane (as well as the Euclidean plane and the sphere).

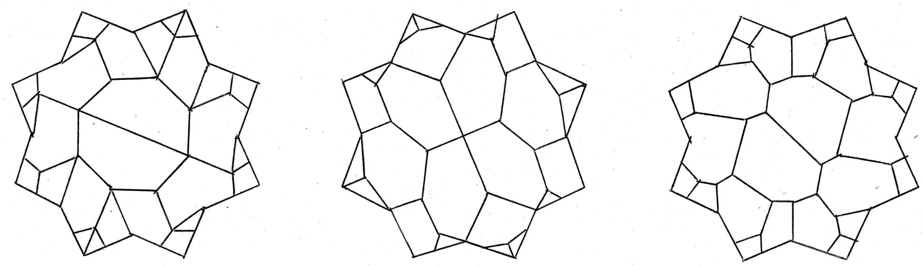
For the description of fundamental isohedral tilings on the hyperbolic plane we use adjacency symbols as in [14, 15].

After the tilings on the hyperbolic plane are found, we check if they are compatible wifh groups on Riemann surfaces.

## 5 Enumeration of isohedral tilings

Summarize the obtained classification of fundamental tilings on the 3 Riemann surfaces. For each fundamental Delone class of isohedral tilings on a Riemann surface we give an adjacency symbol associated with a fundamental Delone class of isohedral tilings on the universal covering hyperbolic plane. Because covering mappings for all 3 Riemann surfaces are known, such an adjacency symbol fully determines a Delone class on the Riemann surface.

On the first Riemann surface the classification of fundamental isohedral tilings is as follows. For the Coxeter group $*832/K_1$ there is one Delone class given by the adjacency symbol $(a\bar{a}_4b\bar{b}_6c\bar{c}_{16})$ (Fig. 1, *a*). For the group $3*4/K_1$ there are 2 Delone classes given by the adjacency symbols $(ab_3ba_{16}c\bar{c}_{16})$ and $(ab_3ba_4c\bar{c}_8d\bar{d}_4)$. For the Coxeter group $*433/K_1$ there is one Delone class given by the adjacency symbol $(a\bar{a}_6b\bar{b}_6c\bar{c}_8)$. For the Coxeter group $*842/K_1$ there is one Delone class given by the adjacency symbol $(a\bar{a}_4b\bar{b}_8c\bar{c}_{16})$. For the Coxeter group $*882/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_{16}c\bar{c}_{16})$. For the group $8*2/K_1$ there are 2 Delone classes with the adjacency symbols $(ab_8ba_8c\bar{c}_8)$ and $(ad_4b\bar{b}_4c\bar{c}_4da_8)$. For the Coxeter group $*444/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_8b\bar{b}_8c\bar{c}_8)$. For the group $4*4/K_1$ there are 2 Delone classes with the adjacency symbols $(ab_4ba_{16}c\bar{c}_{16})$ and $(a\bar{a}_4bc_4cb_4d\bar{d}_8)$. For the group $2*42/K_1$ there are 3 Delone classes with the adjacency symbols $(a\bar{a}_8b\bar{b}_8cc_8)$, $(a\bar{a}_4b\bar{b}_{16}cc_{16})$ and $(a\bar{a}_4bb_4c\bar{c}_4d\bar{d}_8)$. For the Coxeter group $*4222/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_4c\bar{c}_4d\bar{d}_8)$. For the Coxeter group $*3232/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_6c\bar{c}_4d\bar{d}_6)$. For the group $32\times/K_1$ there are 3 Delone classes given by the adjacency symbols $(ab_3ba_4cc_4d\bar{e}_4e\bar{d}_4)$, $(ad_3b\bar{c}_3c\bar{b}_3da_4ef_3fe_4)$, and $(ab_3ba_3cc_3dg_3e\bar{f}_3f\bar{e}_3gd_3)$ (Fig. 2). For the Coxeter group $*4242/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_8c\bar{c}_4d\bar{d}_8)$. For the group $4*22/K_1$ there are 2 Delone classes with the adjacency symbols $(ab_4ba_8c\bar{c}_4d\bar{d}_8)$ and $(ab_4ba_4c\bar{c}_4d\bar{d}_4e\bar{e}_4)$. For the group $2*44/K_1$ there are 2 Delone classes with the adjacency symbols $(a\bar{a}_8b\bar{b}_{16}cc_{16})$ and $(a\bar{a}_4b\bar{b}_4c\bar{c}_8dd_8)$. For the group $42\times/K_1$ there are 6 Delone classes with the adjacency symbols $(aa_{12}b\bar{c}_{12}c\bar{b}_{12})$, $(a\bar{c}_4b\bar{d}_8c\bar{a}_4d\bar{b}_8)$, $(aa_4b\bar{e}_4cd_4dc_4e\bar{b}_4)$, $(a\bar{d}_3bb_3c\bar{e}_8d\bar{a}_3e\bar{c}_8)$, $(af_3b\bar{d}_4c\bar{e}_3d\bar{b}_4e\bar{c}_3fa_4)$, and



$a$ $b$ $c$

Figure 2. Fundamental isohedral tilings for group $32\times/K_1$

$(ag_3b\bar{e}_3cc_3d\bar{f}_3e\bar{b}_3f\bar{d}_3ga_4)$. For the Coxeter group $*22222/K_1$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_4c\bar{c}_4d\bar{d}_4e\bar{e}_4)$. For the group $22*2/K_1$ there are 8 Delone classes with the adjacency symbols $(aa_{12}bb_{12}c\bar{c}_{12})$, $(a\bar{a}_4b\bar{b}_6cc_6dd_6)$, $(ac_4bb_4ca_8d\bar{d}_8)$, $(a\bar{a}_4bb_4c\bar{c}_8dd_8)$, $(aa_4be_4c\bar{c}_4d|bard_4eb_4)$, $(aa_4b\bar{b}_4c\bar{c}_4dd_4e\bar{e}_4)$, $(a\bar{a}_4be_4cc_4dd_4eb_4)$, and $(ad_3bb_3cc_3da_4e\bar{e}_4f\bar{f}_4)$. For the group $3*33/K_1$ there is one Delone classe with the adjacency symbol $(ab_3ba_4c\bar{c}_6d\bar{d}_6e\bar{e}_4)$.

On the second Riemann surface the classification of fundamental isohedral tilings is as follows. For the Coxeter group $*642/K_2$ there is one Delone class given by the adjacency symbol $(a\bar{a}_4b\bar{b}_8c\bar{c}_{12})$. For the Coxeter group $*662/K_2$ there is one Delone class given by the adjacency symbol $(a\bar{a}_4b\bar{b}_{12}c\bar{c}_{12})$ (Fig. 1, $b$). For the Coxeter group $*443/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_6b\bar{b}_8c\bar{c}_8)$. For the Coxeter group $*3222/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_4c\bar{c}_4d\bar{d}_6)$. For the group $6*2/K_2$ there are 2 Delone classes with the adjacency symbols $(ab_6ba_8c\bar{c}_8)$ and $(ad_4b\bar{b}_4c\bar{c}_4da_6)$. For the group $4*3/K_2$ there are 2 Delone classes with the adjacency symbols $(ab_4ba_{12}c\bar{c}_{12})$ and $(a\bar{a}_4bc_4cb_4d\bar{d}_6)$. For the group $2*32/K_2$ there are 3 Delone classes with the adjacency symbols $(a\bar{a}_4b\bar{b}_{12}cc_{12})$, $(a\bar{a}_6b\bar{b}_6cc_6)$, and $(a\bar{a}_4b\bar{b}_4cc_4d\bar{d}_6)$. For the Coxeter group $*663/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_6b\bar{b}_{12}c\bar{c}_{12})$. For the group $6*3/K_2$ there are 2 Delone classes with the adjacency symbols $(ab_6ba_{12}c\bar{c}_{12})$ and $(ad_4b\bar{b}_6c\bar{c}_4da_6)$. For the group $3*22/K_2$ there are 2 Delone classes with the adjacency symbols $(ab_3ba_3c\bar{c}_4d\bar{d}_8)$ and $(ab_3ba_4c\bar{c}_4d\bar{d}_4e\bar{e}_4)$. For the group $2*33/K_2$ there are 2 Delone classes with the adjacency symbols $(a\bar{a}_6b\bar{b}_{12}cc_{12})$ and $(a\bar{a}_4bb_4c\bar{c}_6d\bar{d}_6)$. For the Coxeter group $*3232/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_6c\bar{c}_4d\bar{d}_6)$. For the group $32\times/K_2$ there are 7 Delone classes with the adjacency symbols $(a\bar{c}_9bb_9c\bar{a}_9)$, $(a\bar{c}_4b\bar{d}_6c\bar{a}_4d\bar{b}_6)$, $(ad_3b\bar{c}_3c\bar{b}_3da_6ee_6)$, $(ab_3ba_4c\bar{e}_4dd_4e\bar{c}_4)$, $(a\bar{d}_3bb_3c\bar{e}_6d\bar{a}_3e\bar{c}_6)$, $(a\bar{e}_3bc_3cb_3d\bar{f}_4e\bar{a}_3f\bar{d}_4)$, and $(ab_3ba_3c\bar{f}_3dd_3e\bar{g}_3f\bar{c}_3g\bar{e}_3)$ (Fig. 3). For the Coxeter group $*3322/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_4c\bar{c}_6d\bar{d}_6)$. For the group $32*/K_2$ there are 5 Delone classes with the adjacency symbols $(ac_4b\bar{b}_4ca_6dd_6)$, $(ab_3ba_6cc_6d\bar{d}_6)$, $(ab_3ba_4ce_4d\bar{d}_4ec_4)$, $(ab_3ba_4c\bar{c}_4)$, and $(ae_3bc_3cb_3dd_3ea_4f\bar{f}_4)$. For the Coxeter group $*4422/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_4b\bar{b}_4c\bar{c}_8d\bar{d}_8)$. For the Coxeter group $*22222/K_2$ there

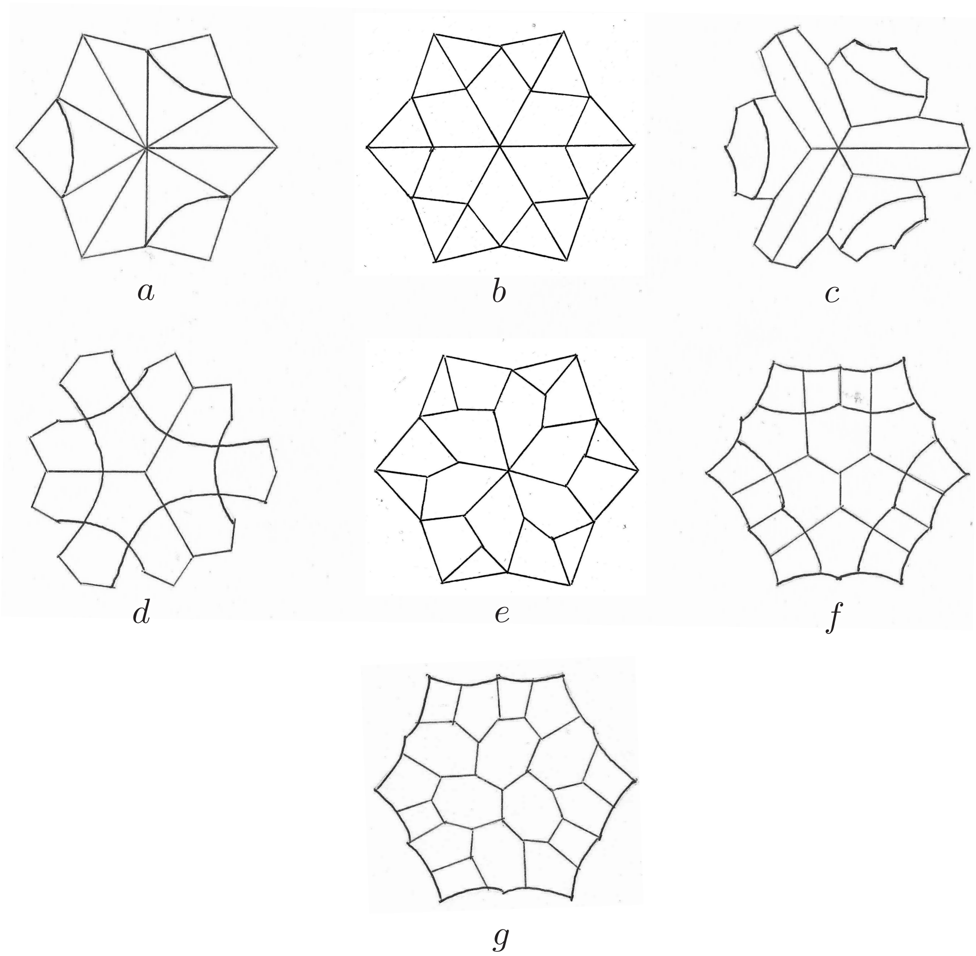



Figure 3. Fundamental isohedral tilings for group $32\times/K_2$

is one Delone class with the adjacency symbol $(a\bar{a}_4 b\bar{b}_4 c\bar{c}_4 d\bar{d}_4 e\bar{e}_4)$. For the group $2{*}222/K_2$ there are 2 Delone classes with the adjacency symbols $(a\bar{a}_4 b\bar{b}_4 c\bar{c}_8 d\bar{d}_8)$ and $(a\bar{a}_4 b\bar{b}_4 c\bar{c}_4 d\bar{d}_4 ee_4)$. For the Coxeter group $*3333/K_2$ there is one Delone class with the adjacency symbol $(a\bar{a}_6 b\bar{b}_6 c\bar{c}_6 d\bar{d}_6)$. For the group $3*33/K_2$ there are 2 Delone classes with the adjacency symbols $(ab_3ba_{12}c\bar{c}_6 d\bar{d}_{12})$ and $(ab_3ba_4c\bar{c}_6d\bar{d}_6e\bar{e}_4)$. For the group $33\times/K_2$ there are 8 Delone classes with the adjacency symbols $(ab_3ba_9c\bar{d}_9d\bar{c}_9)$, $(a\bar{c}_6b\bar{d}_6c\bar{a}_6d\bar{b}_6)$, $(ad_3b\bar{c}_3c\bar{b}_3da_6ef_3fe_6)$, $(ab_3ba_4c\bar{f}_4de_3ed_4f\bar{c}_4)$, $(ab_3ba_4cd_3dc_4e\bar{f}_4f\bar{e}_4)$, $(a\bar{e}_3bc_3cb_3d\bar{f}_6e\bar{a}_3f\bar{d}_6)$, $(ab_3ba_3cd_3dc_3eh_3f\bar{g}_3g\bar{f}_3he_3)$, $(ab_3ba_3c\bar{g}_3de_3ed_3f\bar{h}_3g\bar{c}_3h\bar{f}_3)$. For the group $33*/K_2$ there are 4 Delone classes with the adjacency symbols $(ab_3ba_6ce_4d\bar{d}_4ec_6)$, $(ab_3ba_6cd_3dc_6e\bar{e}_6)$, $(ab_3ba_4c\bar{c}_4de_3ed_4f\bar{f}_4)$, and $(af_3bc_3cb_3de_3ed_3fa_4g\bar{g}_4)$.

Remark that to the Coxeter isometry group of the hyperbolic plane with symbol $*3232$, which has two continuous parameters, there corresponds one Delone class of fundamental tilings, and this Delone class can be realized both on the first Riemann surface for the group $*3232/K_1$ and on the second Riemann surface for the group

$*3232/K_2$. To the isometry group $32\times$ there correspond 10 Delone classes of fundamental tilings. 3 of them can be realized on the first Riemann surface for the group $32\times/K_1$, the other 7 ones can be realized on the second Riemann surface for the group $32\times/K_2$. To the isometry group $42\times$ there correspond 10 Delone classes of fundamental tilings, and 6 of them can be realized on the first Riemann surface for the group $42\times/K_1$. To the Coxeter isometry group of the hyperbolic plane with symbol $*22222$, which has 4 continuous parameters, there corresponds one Delone class of fundamental tilings, and this Delone class can be realized both on the first Riemann surface for the group $*22222/K_1$ and on the second Riemann surface for the group $*22222/K_2$. To the isometry group of the hyperbolic plane with symbol $3*33$, which has no parameters, there correspond 2 Delone classes of fundamental tilings with the adjacency symbols $(ab_3ba_{12}c\bar{c}_6d\bar{d}_{12})$ and $(ab_3ba_4c\bar{c}_6d\bar{d}_6e\bar{e}_4)$. On the first Riemann surface the Delone class with the adjacency symbol $(ab_3ba_4c\bar{c}_6d\bar{d}_6e\bar{e}_4)$ can be realized for the group $3*33/K_1$. On the second Riemann surface both Delone classes can be realized for the group $3*33/K_2$.

On the third Riemann surface the classification of fundamental isohedral tilings is as follows. For the Coxeter group $*(10)52/K_3$ there is one Delone class given by the adjacency symbol $(a\bar{a}_4b\bar{b}_{10}c\bar{c}_{20})$ (Fig. 1, *c*). For the group $5*5/K_3$ there are 2 Delone classes with the adjacency symbols $(ab_5ba_{20}c\bar{c}_{20})$ and $(a\bar{a}_4bc_5cb_4d\bar{d}_{10})$. For the Coxeter group $*555/K_3$ there is one Delone class with the adjacency symbol $(a\bar{a}_{10}b\bar{b}_{10}c\bar{c}_{10})$.

Thus uniting the above results with the results of [1] we have a complete classification of fundamental isohedral tilings on the 3 Riemann surfaces for groups of order not less than 5.

Applying the method of obtaining non-fundamental Delone classes of isohedral tilings on the hyperbolic plane from fundamental ones, which is analogous to the method developed in [15] for the Euclidean plane, we can obtain a classification of non-fundamental isohedral tilings on the 3 Riemann surfaces.

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