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Cubic systems with seven invariant straight lines of configuration $(3,3,1)$

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Abstract. We classify all cubic differential systems with exactly seven invariant straight lines (taking into account their parallel multiplicity) which form a configuration of type $(3,3,1)$. We prove that there are six different topological classes of such systems. For every class we carried out the qualitative investigation on the Poincaré disc. Some properties of cubic systems with invariant straight lines are given.

Mathematics subject classification: 34C05.
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1 Introduction and statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{2}$$

associated to system (1). Here the condition $\gcd(P,Q) = 1$ means that the right-hand sides of the system (1) have no non-constant common factor.

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ (respectively $n = 3$), then system (1) is called quadratic (respectively cubic).

A function $f : D \subset \mathbb{R}^2 \to \mathbb{C}$, $f \neq const$, is said to be an *elementary invariant* (or a *Darboux invariant*) for (2) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ with $deg(K_f) \leq n-1$ such that the identity

$$\mathbb{X}(f) \equiv f(x,y)K_f(x,y), \ (x,y) \in D \tag{3}$$

holds. The polynomial K_f is called the *cofactor of* f. Denote by $I_{\mathbb{X}}$ the set of all elementary invariants of (2); $I_a = \{f \in \mathbb{C}[x,y] | f \in I_{\mathbb{X}}\}$, $I_e = \{exp(\frac{g}{h}) | g,h \in \mathbb{C}[x,y],\ gcd(g,h) = 1,\ exp(\frac{g}{h}) \in I_{\mathbb{X}}\}$. The elements from I_a (respectively I_e) are called *algebraic invariants* (respectively *exponential invariants*) of (2). In [6] it is shown that if $f = exp(\frac{g}{h}) \in I_e$, $h \neq const$, then $h \in I_a$ and $\mathbb{X}(g) = gK_h + hK_f$.

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We say that the exponential invariant $f = exp(\frac{g}{h})$ is *irreducible*, if there do not exist exponential invariants $f_1 = exp(\frac{g_1}{h_1})$ and $f_2 = exp(\frac{g_2}{h_2})$ such that $\frac{g}{h} = \frac{g_1}{h_1} + \frac{g_2}{h_2}$, $deg(h_1) \le deg(h_2)$ and h_1 divides h_2.

Let $f \in \mathbb{C}[x, y]$ and $f = f_1^{n_1} \cdots f_s^{n_s}$ be its factorization in irreducible factors over $\mathbb{C}[x, y]$. Then $f \in I_a$ if and only if $f_j \in I_a, j = \overline{1, s}$. Moreover, $K_f = n_1 K_{f_1} + \cdots + n_s K_{f_s}$. If $f_j \in I_a \bigcup I_e$, $\lambda_j \in \mathbb{C}$, $j = \overline{1, s}$, then $f_1^{\lambda_1} \cdots f_s^{\lambda_s} \in I_X$.

We say that an algebraic invariant $f \in I_a$ has the *parallel multiplicity* equal to m if m is the greatest positive integer such that f^m divides $X(f)$. If $f \in I_a$ has the parallel multiplicity equal to $m \ge 2$, then $exp(1/f), ..., exp(1/f^{m-1}) \in I_e$.

It is easy to see that in general case there is no correlation between the parallel multiplicity of the algebraic invariant $f \in I_a$ and the parallel multiplicity of its factors f_j, $j = \overline{1, s}$. For example, for a vector field $\mathbb{X} = x^3 \frac{\partial}{\partial x} + y(2x^2 + y^2)\frac{\partial}{\partial y}$, we have that the invariant $f = x^2 + y^2$ has the parallel multiplicity equal to two, while for each of its factors $f_{1,2} = x \pm iy$, $i^2 = -1$, the parallel multiplicity is equal to one. For the vector field [12]: $\mathbb{X} = 2x^3 \frac{\partial}{\partial x} + y(3x^2 + y^2)\frac{\partial}{\partial y}$, each of the invariants $f_{1,2} = x \pm iy$ has the parallel multiplicity equal to two, but their product $f = x^2 + y^2$ has the parallel multiplicity equal to one.

We say that the system (1) is *Darboux integrable* if there exists a non-constant function of the form

$$f = f_1^{\lambda_1} \cdots f_s^{\lambda_s}, \tag{4}$$

where $f_j \in I_a \bigcup I_e$ and $\lambda_j \in \mathbb{C}$, $j = \overline{1, s}$, such that either $f = const$ is a first integral (i.e. $K_f \equiv 0$) or f is an integrating factor (i.e. $K_f \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}$) for (1). One can show that (4) is a first integral (an integrating factor) for (1) if and only if

$$\lambda_1 K_{f_1}(x, y) + \cdots + \lambda_s K_{f_s}(x, y) \equiv 0$$

$$\Big(\lambda_1 K_{f_1}(x, y) + \cdots + \lambda_s K_{f_s}(x, y) \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\Big), \ (x, y) \in \mathbb{R}^2.$$

If $f \in I_a$ (respectively $f \in I_e$), then $f(x, y) = 0$ (f) is called *invariant algebraic curve* (respectively *invariant exponential function*) for polynomial system (1).

Later on, we will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0$, $(\alpha, \beta) \ne (0, 0)$. If some of the coefficients α, β, γ of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real*.

By present a great number of works have been dedicated to the investigation of the polynomial differential systems with invariant straight lines. Here we indicate some problems and the corresponding works concerning the polynomial differential system with invariant straight lines. The problem of estimation for the number of invariant straight lines which a polynomial differential system can have was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles ([17, $n = 2$], [11, $n = 3$], [10]); the problem of coexistence of the invariant straight lines and the singular points of the center type for the cubic system ([8, 20]). An interesting relation between the number of invariant straight lines and the possible number of directions for them was established in [2].

The qualitative study of quadratic systems with degenerate infinity was carried out in [18]. For cubic differential systems with degenerate infinity in [4,13] and [5] the integrability problems, the center and the isochronocity problems were studied.

The classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities is given in [12].

The cubic system with exactly eight and exactly seven invariant straight lines has been studied in [12,14] and the one with six real invariant straight lines along two (respectively three) directions has been study in [15] (respectively [16]).

In this paper a qualitative investigation of cubic systems with exactly seven invariant straight lines (real or complex) of configuration $(3,3,1)$, i.e. six of which form two triplets of parallel straight lines, is given. Our main result is the following one:

Theorem 1. *Assume that a cubic system possesses invariant straight lines of total parallel multiplicity seven and six of them form two triplets of parallel straight lines. Then via an affine transformation and time rescaling this system can be brought to one of the six systems* $1.1)-1.6)$*. Moreover its phase portrait on the Poincaré disc corresponds up to topological equivalence to one of the portraits given in Fig. 1.1 - Fig. 1.6. In the table below for each one of the systems* $1.1)-1.6)$ *the first arrow shows the straight lines and either the first integral (*$\mathcal{F}$*) or integrating factor (*μ*) that corresponds to each system.*

$$1.1)\ \begin{cases} \dot{x}=x(x+1)(x-a), \\ \dot{y}=y(y+1)(y-a), \\ \qquad a\in\mathbb{R}_{+}^{*},\ a\neq 1; \end{cases} \qquad \xrightarrow{7r;\ \mathcal{F}} \quad (16) \quad \longrightarrow \quad \text{Fig. 1.1;}$$

$$1.2)\ \begin{cases} \dot{x}=x^2(x+1), \\ \dot{y}=y^2(y+1); \end{cases} \qquad \xrightarrow{7r;\ \mathcal{F}} \quad (18) \quad \longrightarrow \quad \text{Fig. 1.2;}$$

$$1.3)\ \begin{cases} \dot{x}=x\left((x-a)^2+1\right), \\ \dot{y}=y\left((y-a)^2+1\right),\ a\neq 0; \end{cases} \qquad \xrightarrow{3r+4c_0;\ \mu} \quad (22) \quad \longrightarrow \quad \text{Fig. 1.3;}$$

$$1.4)\begin{cases} \dot{x}=x\left(-a+2(a+1)y+x^2-3y^2\right), \\ \dot{y}=-ay-(a+1)x^2+(a+1)y^2 \\ \qquad +3x^2y-y^3,\ \ a\in(0;1),\ a\neq 1/2; \end{cases} \qquad \xrightarrow{1r+6c_1;\ \mu} \quad (24) \longrightarrow \quad \text{Fig. 1.4;}$$

$$1.5)\begin{cases} \dot{x}=x\left(1+2ay-x^2+3y^2\right),\ a>0, \\ \dot{y}=a+y-ax^2+ay^2-3x^2y+y^3; \end{cases} \qquad \xrightarrow{1r+6c_1;\ \mu} \quad (26) \longrightarrow \quad \text{Fig. 1.5;}$$

$$1.6)\ \begin{cases} \dot{x}=x(x^2+2y-3y^2), \\ \dot{y}=-x^2+y^2+3x^2y-y^3; \end{cases} \qquad \xrightarrow{1r+6c_1;\ \mu} \quad (27) \longrightarrow \quad \text{Fig.1.6.}$$

2 Some properties of the cubic systems with invariant straight lines

We consider the real cubic differential system

$$\begin{cases} \frac{dx}{dt}=\sum\limits_{r=0}^{3}P_r\left(x,y\right)\equiv P\left(x,y\right), \\ \frac{dy}{dt}=\sum\limits_{r=0}^{3}Q_r\left(x,y\right)\equiv Q\left(x,y\right), \end{cases} \qquad \gcd\left(P,Q\right)=1, \tag{5}$$

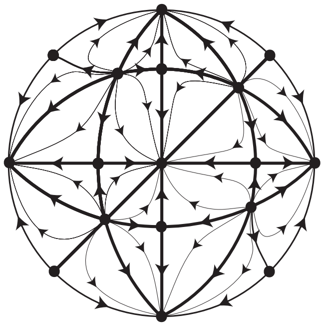


Fig. 1.1

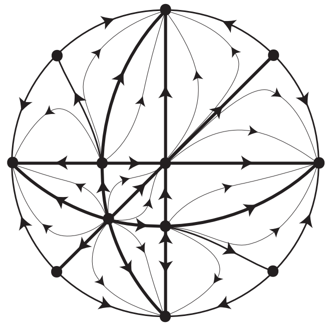


Fig. 1.2

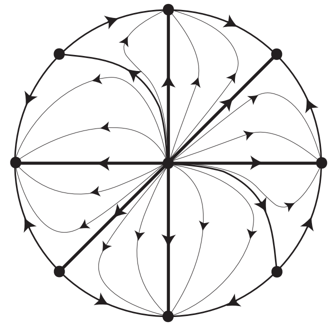


Fig. 1.3

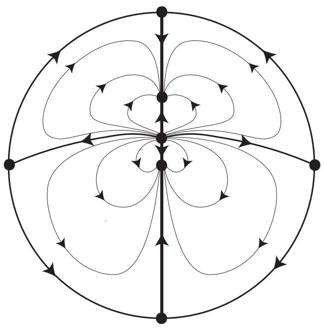


Fig. 1.4

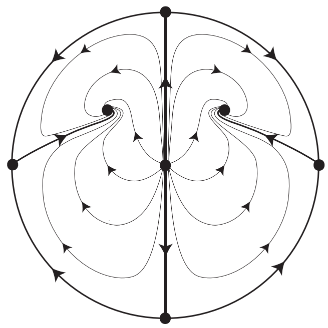


Fig. 1.5

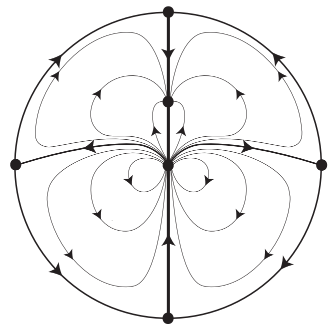


Fig. 1.6

where $P_r(x,y) = \sum\limits_{j+l=r} a_{jl}x^jy^l,\ Q_r(x,y) = \sum\limits_{j+l=r} b_{jl}x^jy^l,\ |P_3(x,y)| + |Q_3(x,y)| \not\equiv 0.$

In general case, by a *straight lines configuration* we understand the $\mathbb{R}^2$ plane with a given number of straight lines. Therefore, to each cubic system with invariant straight lines one can associate a configuration consisting of given straight lines. It is easy to show that the converse statement is not valid, i.e. not for any configuration of straight lines a cubic system can be build for which the straight lines should be invariant or, in other words, this configuration cannot be realized in the class of cubic systems.

The goal of this section is to determine such properties for invariant straight lines which will allow to construct configurations of straight lines realizable for (5).

2.1 Points and straight lines

2.1) *In the finite part of the phase plane the system* (5) *has at most nine singular points.*

2.2) *In the finite part of the phase plane on any straight line there are at most three singular points of the system* (5)*.*

2.3) *In the finite part of the phase plane the system* (5) *has no more than eight invariant straight lines* [1, 14].

2.4) *At infinity the system* (5) *has at most four distinct singular points (in the Poincaré compactification [18]) if* $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$. *In the case* $yP_3(x,y) - xQ_3(x,y) \equiv 0$ *the infinity is degenerate, i.e. consists only of singular points.*

2.5) *If* $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$, *then the infinity represents for* (5) *a non singular invariant straight line.*

2.6) *Through one point more than four distinct invariant straight lines of system* (5) *cannot pass.*

We say that straight lines $l_j \equiv \alpha_j x + \beta_j y + \gamma_j \in \mathbb{C}[x,y], j = 1,2$, are *parallel* if $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. Otherwise those straight lines are called *concurrent*. If an invariant straight line l has the parallel multiplicity equal to m, then we will consider that we have m parallel invariant straight lines identical with l.

2.7) *The intersection point* (x_0, y_0) *of two concurrent invariant straight lines* l_1 *and* l_2 *of system* (5) *is a singular point for this system. If* $l_1, l_2 \in \mathbb{R}[x,y]$ *or* $l_2 \equiv \bar{l_1}$, *i.e. the straight lines* l_1 *and* l_2 *are complex conjugate, then* $x_0, y_0 \in \mathbb{R}$.

2.8) *A complex straight line* $l \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$ *can pass through at most one point with real coordinates* M_0. *If the complex straight line passes through such a point, then it is described by an equation of the form:* $y = \alpha x + \beta$, $Im\,\alpha \neq 0$, *and* M_0 *is the intersection point of the straights* l *and* $\bar{l}$.

Definition 1. A complex straight line whose equation is verified by a single point with real coordinates will be called *relatively complex straight line.*

Unlike the complex straight lines, a real straight line $ax + by + c = 0$, $a, b, c \in \mathbb{R}$, $a^2 + b^2 \neq 0$, passes through an infinite number of real points and through an infinite number of points with at least one complex coordinate. Indeed, if $x_0, y_0 \in \mathbb{R}$ and $ax_0 + by_0 + c = 0$, then this straight line passes through complex points $(x_0 + \alpha b, y_0 - \alpha a)$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

2.9) *A straight line that passes through two distinct real points or through two complex conjugate points is real.*

Proof. The case when the points through which the straight line passes are real is trivial. If the straight line passes through complex points (x_0, y_0) and $(\overline{x_0}, \overline{y_0})$, $|Im(x_0)| + |Im(y_0)| \neq 0$, then it is described by the equation $Im(y_0)x - Im(x_0)y + Im(x_0)y_0 - Im(y_0)x_0 = 0$. □

The next two properties are a consequence of the relation $\mathbb{X}(\overline{l}\,) = \overline{\mathbb{X}(l)}$.

2.10) *The complex conjugate straight lines* l *and* $\bar{l}$ *can be invariant lines for system* (1) *only together.*

2.11) *The complex conjugate invariant straight lines* l *and* $\bar{l}$ *have the same parallel multiplicity.*

If a straight line l is real, then $(x_0, y_0) \in l$ implies $(\overline{x_0}, \overline{y_0}) \in l$. If (x_0, y_0) is a singular point of system (5), then $(\overline{x_0}, \overline{y_0})$ is also a singular point for this system. From this, and from **2.2), 2.7)** and **2.10)** we obtain the following two properties:

2.12) *The number of complex singular points on a real invariant straight line of system* (5) *is even and is at most two.*

2.13) *A real invariant straight line either intersects none of the complex invariant straight lines of the system* (5) *in complex points, or it intersects exactly two complex conjugate invariant straight lines in complex points.*

Let us consider the polynomial

$$\psi(x,y) = P \cdot (P \cdot \partial Q/\partial x + Q \cdot \partial Q/\partial y) - Q \cdot (P \cdot \partial P/\partial x + Q \cdot \partial P/\partial y)\,, \qquad (6)$$

i.e. $P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$. The condition $\gcd(P,Q) = 1$ does not allow $\psi(x,y)$ to be identically zero with respect to variables x and y. If $\alpha x + \beta y + \gamma = 0$ is an invariant straight line for (5) with parallel multiplicity equal to m, then $(\alpha x + \beta y + \gamma)^m$ divides $\psi(x,y)$ (see [7]). From here and from the fact that if the identity $yP_3(x,y) - xQ_3(x,y) \equiv 0$ holds the degree of the polynomial $\psi(x,y)$ is at most six follows

2.14) *The cubic system with at least seven invariant straight lines has non-degenerate infinity and, therefore, there exist at most four directions (slopes) for these lines.*

2.2 The parallel invariant straight lines

Definition 2. A straight line which does not pass through any real point will be called *absolutely complex straight line.*

From this definition and from the fact that the point of intersection of two real or two complex conjugate straight lines is real, the next two properties result:

2.15) *A complex invariant straight line* ($l \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$) *of the system* (5) *is absolutely complex if and only if it is parallel with its conjugate line.*

2.16) *Through one point (respectively complex point) of any absolutely complex (respectively relatively complex) straight line at most one real straight line can pass.*

If the straight line l is absolutely complex, then it is described by an equation of the form $\alpha x + \beta y - \gamma = 0$ with $\alpha, \beta \in \mathbb{R}$, $(\alpha,\, \beta) \neq (0,0)$, $\gamma \in \mathbb{C} \setminus \mathbb{R}$. If $\alpha \neq 0$ (can pass $\alpha = 0$), then the non-degenerate linear transformation $X = \alpha x + \beta y, Y = y$ (can pass $X = \beta y, Y = x$) makes the straight line l to be parallel with the ordinate axis. Taking into account this, we obtain:

2.17) *Via a non-degenerate linear transformation of the phase plane any absolutely complex straight line can be made parallel to one of the axes of the coordinate system, i.e. it is described by one of the equations* $x = \gamma$ *or* $y = \gamma$, $\gamma \in \mathbb{C} \setminus \mathbb{R}$. *Moreover, if we have such two straight lines* l_1 *and* l_2, $l_1 \nparallel l_2$, $l_1 \parallel \overline{l_1}$, $l_2 \parallel \overline{l_2}$, *then by a suitable transformation we can at the same time make the straight line* l_1 *to be parallel with the coordinate axis* Ox, *and the straight line* l_2 *to be parallel to* Oy *axis.*

Taking into account the form of the equation of a relatively complex straight line, brought in the property **2.8)**, and the form of the equations of the real straight lines and also of absolutely complex ones, we obtain the following property:

2.18) *A real straight line as well as an absolutely complex line cannot be parallel with a relatively complex straight line.*

Concerning two parallel invariant straight lines we have the property

2.19) *If l_1 and l_2 are two parallel invariant straight lines of the system (1), then either*

a) $l_1, l_2 \in \mathbb{R}[x, y]$, or

b) l_1 is real, and l_2 is absolutely complex, or

c) l_1 and l_2 are absolutely complex and $l_2 = \overline{l_1}$, or

d) l_1 and l_2 are relatively complex straight lines and $l_2 \neq \overline{l_1}$.

Let l_1, l_2, l_3 be a triplet of parallel straight lines. Then each of these straight lines is described by an equation of the form

$$\alpha x + \beta y - \gamma_j = 0, \qquad (\alpha, \beta) \neq (0, 0).$$

If these straight lines are invariant for cubic system (5), then the identity (3) is written as:

$$\alpha P(x, y) + \beta Q(x, y) \equiv c_0 \cdot \prod_{j=1}^{3} l_j\,,\ c_0 = const \neq 0. \tag{7}$$

From (7) and from properties **2.3)** and **2.18)** the next four statements concerning triplets of parallel invariant straight lines follows:

2.20) *The system* (5) *can not have more than three invariant straight lines parallel among themselves.*

2.21) *If the cubic system* (5) *has a triplet of parallel invariant straight lines, then all its singular points lie on these straight lines.*

2.22) *The cubic system* (5) *cannot have more than two triplets of parallel invariant straight lines.*

2.23) *If l_1, l_2, l_3 form a triplet of parallel invariant straight lines of cubic system* (5)*, then either*

a) $l_1, l_2, l_3 \in \mathbb{R}[x, y]$, or

b) $l_1, l_2, l_3 \in \mathbb{C}[x, y]$ and $\bar{l_j} \notin \{l_1, l_2, l_3\}$, $j = 1, 2, 3$, or

c) $l_1 \in \mathbb{R}[x, y]$, $l_{2,3} \in \mathbb{C}[x, y] \setminus \mathbb{R}[x, y]$ and $l_3 = \overline{l_2}$.

We mention that in the case b) of the property **2.23)** all straight lines l_1, l_2, l_3 are relatively complex.

2.3 Multiple invariant straight lines

Remark 1. All the straight lines mentioned in properties **2.3)**, **2.20)** − **2.23)** are considered with their parallel multiplicity. For example, in the case a) of the property **2.23)** the cases $l_1 \equiv l_2 \not\equiv l_3$ and $l_1 \equiv l_2 \equiv l_3$ are admissible. In the first case l_1 has parallel multiplicity equal to two, but in the second case the multiplicity of invariant straight lines is equal to three.

The next two statements are a consequence of the relation (7).

2.24) *The parallel multiplicity of an invariant straight line of the cubic system* (5) *is at most three.*

2.25) *The parallel multiplicity of any absolutely complex invariant straight line of the cubic system* (5) *is equal to one.*

2.26) *If the cubic system* (5) *has two concurrent invariant straight lines* l_1, l_2 *and* l_1 *has the parallel multiplicity equal to* m, $1 \leq m \leq 3$, *then this system cannot have more than* $3 - m$ *singular points on* $l_2 \setminus l_1$.

Proof. If $m = 1$ ($m = 3$), then the property **2.26)** is a consequence of the properties **2.2)** and **2.7)**. Assume $m = 2$ and $l_j = \alpha_j x + \beta_j y + \gamma_j$, $j = 1, 2$, $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. From the equalities

$$\begin{aligned} &\alpha_1 P(x,y) + \beta_1 Q(x,y) = l_1^2 K_1(x,y), \\ &\alpha_2 P(x,y) + \beta_2 Q(x,y) = l_2 K_2(x,y), \ \forall (x,y) \in \mathbb{R}^2, \end{aligned} \tag{8}$$

we have

$$P = \frac{\beta_2 l_1^2 K_1 - \beta_1 l_2 K_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \ Q = \frac{\alpha_1 l_2 K_2 - \alpha_2 l_1^2 K_1}{\alpha_1\beta_2 - \alpha_2\beta_1}. \tag{9}$$

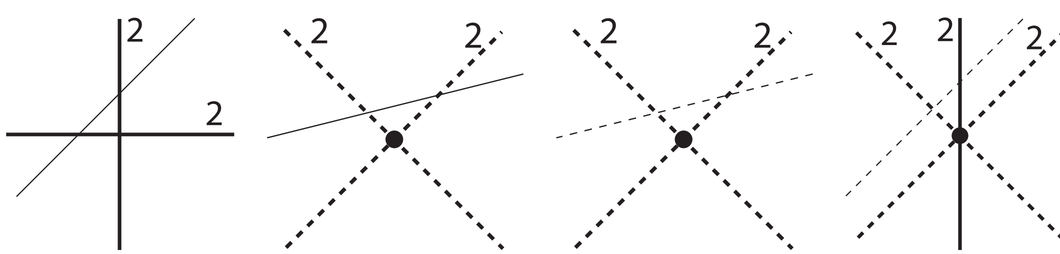
Assume that on $l_2 \setminus l_1$ at least two distinct singular points of the system (5) lie. Then, from (8) it follows $K_1 = c_0 l_2$, $c_0 = const$, and from (9) it follows that l_2 is a common factor of the polynomials P and Q, in contradiction to the condition $\gcd(P, Q) = 1$. □

We say that three straight lines are in generic position if no one pair of the lines is parallel and no more that two lines pass through the same point.

2.27) *For the cubic system* (5) *the total parallel multiplicity of three invariant straight lines in generic position is at most four.*

Proof. Let $l_j \equiv \alpha_j x + \beta_j y + \gamma_j = 0$, $j = 1, 2, 3$, be three invariant straight lines of the system (5). The properties **2.7)** and **2.26)** do not allow one of the lines l_1, l_2, l_3 to have degree of invariance equal to three.

Let us suppose that each of the invariant straight lines l_1 and l_2 has the degree of invariance equal to two. It is enough to examine the cases when the straight lines l_1, l_2, l_3 form one of the configurations from Fig. 2.1.



a) b) c) d)

Fig. 2.1

In the case of configuration a) of Fig. 2.1 we can consider $l_1 = x$ and $l_2 = y$. Then the system (5) can be written in the form

$$\dot{x} = x^2(a_{20} + a_{30}x + a_{21}) \equiv P(x,y), \ \ \dot{y} = y^2(b_{02} + b_{12}x + b_{03}y) \equiv Q(x,y). \tag{10}$$

Requiring from (10) to have the invariant straight line $l_3 = \alpha x - y + \beta$, $\alpha\beta \neq 0$, i.e. the following identity

$$\alpha P(x, \alpha x + \beta) - Q(x, \alpha x + \beta) \equiv 0 \tag{11}$$

to be satisfied with respect to x, we obtain that $a_{20} = -\beta a_{21}$, $a_{30} = -\alpha a_{21}$, $b_{20} = -\beta b_{30}$, $b_{12} = -\alpha b_{30}$. In this case, we have $P(x,y) = a_{21}x^2 l_3$, $Q(x,y) = -b_{30}l_3$. Therefore, $\deg(\gcd(P,Q)) > 0$, is not allowed.

Now let us consider the configurations b) and c) from Fig. 2.1. By an affine transformation, we can get $l_1 = y - ix$ and $l_2 = y + ix$, $i^2 = -1$. Then the system (5) obtains the form

$$\begin{cases} \dot{x} = a_{20}x^2 - 2b_{20}xy - a_{20}y^2 + a_{30}x^3 - (a_{03} + 2b_{30})x^2y \\ \qquad +(3a_{30} - 2b_{21})xy^2 + a_{03}y^3 \equiv P(x,y), \\ \dot{y} = b_{20}x^2 + 2a_{20}xy - b_{20}y^2 + b_{30}x^3 + b_{21}x^2y - (2a_{03} + b_{30})xy^2 \\ \qquad +(2a_{30} - b_{21})y^3 \equiv Q(x,y). \end{cases} \tag{12}$$

The identity (11), written for (12), leads us to the degenerate system

$$\begin{cases} \dot{x} = l_3\big[a_{03}x^2 + (2b_{21} - 4\alpha a_{03})xy - a_{03}y^2\big], \\ \dot{y} = -l_3\big[(b_{21} - 2\alpha a_{03})x^2 - 2a_{03}xy + (2\alpha a_{03} - b_{21})y^2\big], \end{cases}$$

where $l_3 = \alpha x - y + \beta$, $\beta(\alpha^2 + 1) \neq 0$.

In the case of configuration d) of Fig. 2.1 we can consider $l_1 = y - ix$ and $l_2 = x$. The family of cubic systems for which the straight line l_1 is invariant with parallel multiplicity equal to two is described by (12). In order the straight line $l_2 = x$ to be invariant for (12) with parallel multiplicity equal to two, it is necessary that $a_{20} = a_{03} = b_{20} = 0$, $b_{21} = 3a_{30}/2$. In this conditions, (12) looks as

$$\dot{x} = x^2(a_{30}x - 2b_{30}y), \ \dot{y} = (2b_{30}x^3 + 3a_{30}x^2y - 2b_{30}xy^2 + a_{30}y^3)/2. \tag{13}$$

From (13) it is seen that on the straight line l_2 only a singular point $(0,0) lies$, which means that the configuration d) from Fig. 2.1 is not realized in the class of cubic systems. □

3 The proof of Theorem 1

3.1 Configurations of straight lines and classification of cubic systems

The goal of this section is to classify the cubic systems with exactly seven invariant straight lines, six of which form two triplets of parallel lines. It is obvious that these straight lines can be only of three directions. Taking into account this fact and the properties **2.19)–2.21), 2.23)** and **2.25)**, the following configurations are possible:

3.1) $(3r, 3r, 1r)$; 3.2) $(\mathbf{3(2)r}, \mathbf{3r}, \mathbf{1r})$; 3.3) $(\mathbf{3(3)r}, \mathbf{3r}, \mathbf{1r})$;
3.4) $(3(2)r, 3(2)r, 1r)$; 3.5) $(\mathbf{3(3)r}, \mathbf{3(2)r}, \mathbf{1r})$; 3.6) *(3(3)r, 3(3)r, 1r)*;
3.7) $(\mathbf{3r}, \mathbf{1r+2c_0}, \mathbf{1r})$; 3.8) $(\mathbf{3(2)r}, \mathbf{1r+2c_0}, \mathbf{1r})$; 3.9) $(\mathbf{3(3)r}, \mathbf{1r+2c_0}, \mathbf{1r})$;
3.10) $(1r+2c_0, 1r+2c_0, 1r)$; 3.11) $(3c_1, 3c_1, 1r)$; 3.12) $(3(2)c_1, 3(2)c_1, 1r)$;
3.13) *(3(3)c₁, 3(3)c₁, 1r)*.

By $(3r, 3r, 1r)$ we denoted the configuration which consists of seven distinct real straight lines $l_1, ..., l_7 \in \mathbb{R}[x, y]$, of which l_1, l_2, l_3 and l_4, l_5, l_6, form two triplets of parallel straight lines, i.e. $l_1 \parallel l_2 \parallel l_3$, $l_4 \parallel l_5 \parallel l_6$ and $l_j \nparallel l_k$, $(j, k) = (1, 4), (1, 7), (4, 7)$. In the case of configuration $(3r, 1r + 2c_0, 1r)$ we have seven straight lines $l_1, ..., l_7$, of which l_1, l_2, l_3, l_4 and l_7 are real, l_5, l_6 are absolutely complex, l_1, l_2, l_3 and l_4, l_5, l_6 form two triplets of distinct parallel straight lines and $l_j \nparallel l_k$, $(j, k) = (1, 4), (1, 7), (4, 7)$. The configuration $(3(2)c_1, 3(2)c_1, 1r)$ consists of a real straight line l_7 and relatively complex lines $l_1, ..., l_6$, $l_1 \equiv l_2 \parallel l_3$, $l_4 \equiv l_5 \parallel l_6$, $l_4 = \overline{l_1}$, $l_6 = \overline{l_3}$, $l_j \nparallel l_k$, $(j, k) = (1, 4), (1, 7), (4, 7)$. The straight lines l_1 and l_2 (or l_4 and l_5) are considered relatively complex straight lines which have the parallel multiplicity equal to two and so on.

Next, we will examine the configurations $3.1) - 3.13)$ and their realization in the class of cubic systems.

3.1.1 Unrealizable configurations

The configurations 3.2), 3.3), 3.5), 3.7), 3.8) and 3.9) (denoted bold) are not realizable in the class of cubic systems of differential equations, that is there are no cubic systems with real coefficients invariant straight lines of which would form one of the configuration mentioned above. So, the properties **2.2), 2.7), 2.21)** and **2.26)** do not allow the realization of configurations 3.2), 3.3), 3.5) and 3.9); the properties **2.2), 2.7), 2.16)** and **2.21)** do not allow the realization of configurations 3.7) and 3.8).

3.1.2 Subconfigurations of configurations with eight lines

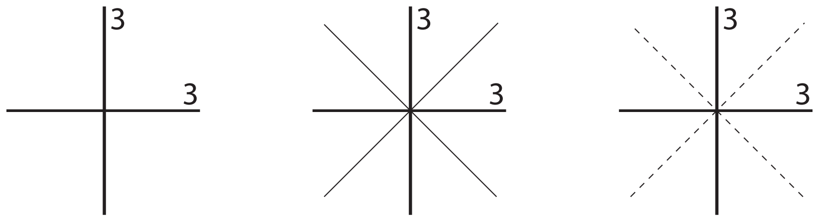
We will show that there are no cubic systems with exactly seven invariant straight lines in the finite phase plane for which al least one of the configurations 3.6) and 3.13) is realized.

In the case of configuration 3.6), i.e. *(3(3)r, 3(3)r, 1r)*, we can consider $l_1 = l_2 = l_3 = x$ and $l_4 = l_5 = l_6 = y$. Then, the system (5) is written as:

$$\dot{x} = x^3 \equiv P(x, y), \;\; \dot{y} = \omega y^3 \equiv Q(x, y), \;\; \omega \neq 0. \tag{14}$$

The property **2.27)** imposes the straight line l_7 to pass through the origin of coordinates, that is to be described by an equation of the form $y = \alpha x$, $\alpha \neq 0$. The identity (11) which ensures the invariance of the straight line l_7, written for (14) (when $\beta = 0$), gives us $\alpha x^3(\alpha^2\omega - 1) = 0 \; \forall x \in \mathbb{R}$, which implies $\alpha = \pm\sqrt{1/\omega}$.

Thus, in the finite phase plane, the system (14) has either exactly six or eight invariant straight lines (see Fig. 3.1).



a) b) c)

Fig. 3.1

In the case of configuration 3.13) (*3*(*3*)c_1, *3*(*3*)c_1, *1r*), we consider the straight lines $l_1 = l_2 = l_3 = y - ix$ and $l_4 = l_5 = l_6 = y + ix$, $i^2 = -1$. In order the straight lines be invariant for (5) it is necessary that the identity $Q(x,y) - iP(x,y) \equiv (y - ix)^3$ be satisfied (see (7)). This identity leads us to the system

$$\dot{x} = x(x^2 - 3y^2), \;\; \dot{y} = y(3x^2 - y^2),$$

which, besides the invariant straight lines $y - ix = 0$ and $y + ix = 0$ with parallel multiplicities equal to three, has also the real invariant straight lines $x = 0$ and $y = 0$.

3.1.3 Realizable configurations

In this subsection we will show that the configurations 3.1), 3.4), 3.10) and 3.12) are respectively realized by systems 1.1), 1.2), 1.3) and 1.6) and the configuration 3.11) – by the systems 1.4) and 1.5) from Theorem 1.

Configuration 3.1) (3*r*,3*r*,1*r*). We may consider $l_1 = x$; $l_2 = x + 1$; $l_3 = x - a$, $a > 0$; $l_4 = y$; $l_5 = y + 1$; $l_6 = y - b$, $b > 0$. The cubic system (5), for which the given straight lines are invariant, looks as:

$$\dot{x} = x(x+1)(x-a), \;\; \dot{y} = \omega y(y+1)(y-b), \;\; a > 0, \, b > 0, \;\; \omega \neq 0. \qquad (15)$$

The properties **2.1)** and **2.7)** impose the straight line l_7 to be described by: α) the equation $y = x$ and so, $b = a$, (see Fig. 3.2) or β) the equation $y = -x/a$ and then $b = 1/a$. The case β) is reduced to the case α) by the transformation $x \to -ax$, $y \to y, a \to 1/a$.

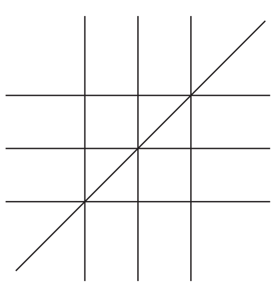


Fig. 3.2

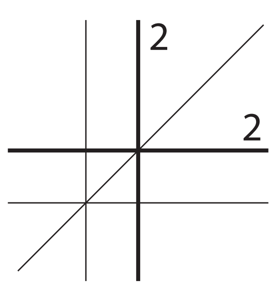


Fig. 3.3

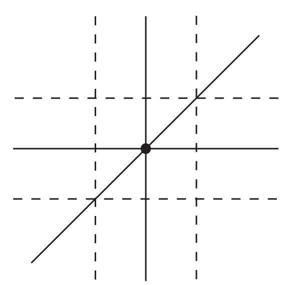


Fig. 3.4

Substituting into the system (15) $b = a$ and requiring the straight line $y = x$ to be invariant for (15), we obtain $\omega = 1$. Thus, we come to the system 1.1) from Theorem 1, which has the following invariant straight lines and a first integral:

$$\begin{gathered} l_1 \equiv x = 0,\ l_2 \equiv x + 1 = 0,\ l_3 \equiv x - a = 0,\ l_4 \equiv y = 0, \\ l_5 \equiv y + 1 = 0,\ l_6 \equiv y - b = 0,\ l_7 \equiv y - x = 0; \\ \mathcal{F} \equiv \left(\frac{x}{y}\right)^{a+1} \left(\frac{y+1}{x+1}\right)^{a} \frac{y-a}{x-a} = C. \end{gathered} \tag{16}$$

The inequality $a \neq 1$ was imposed to the system 1.1) to exclude the existence of the invariant straight line $l_8 \equiv y + x = 0$.

Configuration 3.4) (3(2)$\boldsymbol{r}$,3(2)$\boldsymbol{r}$,1$\boldsymbol{r}$). Let the real straight lines $l_1 \equiv l_2 \parallel l_3$, $l_4 = l_5 \parallel l_6$, $l_1 \nparallel l_4$ be given. By an affine transformation of coordinates we can make them to be $l_1 = x$, $l_3 = x + 1$, $l_4 = y$, $l_6 = y + 1$. The cubic system (5) possessing the invariant straight lines l_1, l_3 and l_6, where l_1 as well as l_4 have the parallel multiplicity equal to two, looks as:

$$\dot{x} = x^2(x + 1),\ \dot{y} = \omega y^2(y + 1),\ \omega \neq 0. \tag{17}$$

The properties **2.7), 2.21), 2.26)** and **2.27)**, allow the coexistence of straight line l_7 together with straight lines $l_1, ..., l_6$ only in the configuration given in Fig. 3.3.

Therefore, $l_7 = y - x$ and this straight line is invariant for system (17) if $\omega = 1$, that is if (17) coincides with the system 1.2) from Theorem 1. This system has the following invariant straight lines $l_1, ..., l_7$ and a first integral $\mathcal{F}$:

$$\begin{gathered} l_{1,2} \equiv x = 0,\ l_3 \equiv x + 1 = 0,\ l_{4,5} \equiv y = 0,\ l_6 \equiv y + 1 = 0,\ l_7 = y - x; \\ \mathcal{F} \equiv x^{-1} e^{-1/x}(x + 1) y e^{1/y}(y + 1)^{-1} = C. \end{gathered} \tag{18}$$

Configuration 3.10) (1$\boldsymbol{r}$+2$\boldsymbol{c}_0$,1$\boldsymbol{r}$+2$\boldsymbol{c}_0$,1$\boldsymbol{r}$) (Fig. 3.4). Setting $l_1 = x$, $l_{2,3} = x - (a \pm bi)$, $l_4 = y$, $l_{5,6} = y - (c \pm di)$, $a, b, c, d \in \mathbb{R}$, $bd \neq 0$, $i^2 = -1$, we arrive at the system

$$\dot{x} = x((x - a)^2 + b^2), \quad \dot{y} = \omega y((y - c)^2 + d^2), \tag{19}$$

where $\omega = \pm 1$. We denote by $O_{j,k}$ the points of intersection of straight lines l_j and l_k.

The properties **2.1), 2.2) 2.7)** and **2.13)** impose the straight line l_7 to pass through points $O_{1,4}(0, 0)$, $O_{2,5}(a + bi, c + di)$, $O_{3,6}(a - bi, c - di)$ (or through points $O_{1,4}(0, 0)$, $O_{2,6}(a + bi, c - di)$, $O_{3,5}(a - bi, c + di)$). The substitution $d \to -d$ reduces the case $O_{1,4},\ O_{2,6},\ O_{3,5} \in l_7$ to the case $O_{1,4},\ O_{2,5},\ O_{3,6} \in l_7$. From $O_{1,4}(0, 0) \in l_7$ it results that l_7 is described by an equation of the form $y = \alpha x$, and from $O_{2,5},\ O_{3,6} \in l_7$ ($O_{2,6},\ O_{3,5} \in l_7$) we obtain that $c = \alpha a$, $d = \alpha b$. Taking into account these relations, the invariance condition (11), (the case $\beta = 0$) of the straight line $l_7 = y - \alpha x$ for the system (19) looks as:

$$\alpha x(\alpha^2 \omega - 1)((x - a)^2 + b^2) \equiv 0,$$

from which we obtain $\omega = 1$, $\alpha = 1$ (or $\alpha = -1$). Thus, the system (19) for which the straight line $l_7 = y - x$ is invariant is written as follows:

$$\dot{x} = x((x-a)^2 + b^2), \quad \dot{y} = y((y-a)^2 + b^2), \tag{20}$$

$$\left(\dot{x} = x((x-a)^2 + b^2), \quad \dot{y} = y((y+a)^2 + b^2)\right). \tag{21}$$

The substitutions $x \to x$, $y \to -y$ reduce the system (21) to the system (20), and the substitutions $x \to bx$, $y \to by$, $a \to ab$, $t \to b^2\tau$ reduce the system (20) to the system 1.3) from Theorem 1, which has the following invariant straight lines and integrating factor:

$$\begin{gathered} l_1 = x,\ l_{2,3} = x - a \mp i,\ l_4 = y,\ l_{5,6} = y - a \mp i,\ l_7 = y - x; \\ \mu = 1/(l_1 l_2 l_3 l_4 l_5 l_6). \end{gathered} \tag{22}$$

If we set $a = 0$ in the cubic system 1.3), then besides the straight lines $l_1, ..., l_7$, given in (22), the cubic system will have the invariant straight line $l_8 = y + x$.

Remark 2. Let l_1, l_2, l_3 be three relatively complex distinct lines and $l_1 \parallel l_2 \parallel l_3$. If their real points are collinear, then, by an affine transformation, their equations can be brought to the form $y - ix = 0$, $y - ix - 1 = 0$, $y - ix - a = 0$, $i^2 = -1$, where $a \in (0, 1)$.

Configuration 3.11) ($3c_1$,$3c_1$,$1r$). Via affine transformations we can make the relatively complex straight lines $l_1, ..., l_6$, $l_1 \parallel l_2 \parallel l_3$, $l_4 \parallel l_5 \parallel l_6$, $l_1 \nparallel l_4$ to be described by equations: $l_{1,4} \equiv y \mp ix = 0$, $l_{2,5} \equiv y \mp ix - 1 = 0$, $l_{3,6} \equiv y \mp ix - a \mp bi$, $(a, b) \neq (0, 0), (1, 0)$, $i^2 = -1$. From the identity $Q(x, y) - iP(x, y) \equiv l_1 l_3 l_5$ we find that the cubic systems for which the straight lines $l_1, ..., l_6$ are invariant look as:

$$\begin{cases} \dot{x} = ax - by - bx^2 - 2(a+1)xy + by^2 - x^3 + 3xy^2 \equiv P(x, y), \\ \dot{y} = bx + ay + (a+1)x^2 - 2bxy - (a+1)y^2 - 3x^2y + y^3 \equiv Q(x, y). \end{cases} \tag{23}$$

Next, we will require from (23) to have one more invariant straight line l_7. First, we will consider that $l_7 = x - c$, $c \in \mathbb{R}$. This straight line is invariant for (23) if and only if the identity $P(c, y) \equiv 0$ holds with respect to y. This identity gives the equalities $c = b = 0$, which, in turn, reduce (23) to the system 1.4) from Theorem 1. This system has the following invariant straight lines and integrating factor:

$$\begin{gathered} l_1 = y - ix,\ l_2 = y - ix - 1,\ l_3 = y - ix - a,\ l_4 = y + ix, \\ l_5 = y + ix - 1 = 0,\ l_6 = y + ix - a,\ l_7 = x; \quad \mu = 1/(l_1 l_2 l_3 l_4 l_5 l_6). \end{gathered} \tag{24}$$

We mention that if $a = 1/2$, then the system 1.4), besides the straight lines $l_1, ..., l_7$, has one more invariant straight line.

Now let us require for system (23) to have an invariant straight line l_7 of the form $y = \alpha x - \beta$, $\alpha, \beta \in \mathbb{R}$. From the identity $\alpha P(x, \alpha x - \beta) - Q(x, \alpha x - \beta) \equiv 0$, i.e. $2\alpha(1+\alpha^2)x^3 - (1+\alpha^2)(1+a-\alpha b-3\beta)x^2 + b(1+\alpha^2)(2\beta-1)x + \beta(\beta-1)(a+\alpha b-\beta) \equiv 0$, we find that $\alpha = 0$, $a = \beta = 1/2$, and (23) takes the form

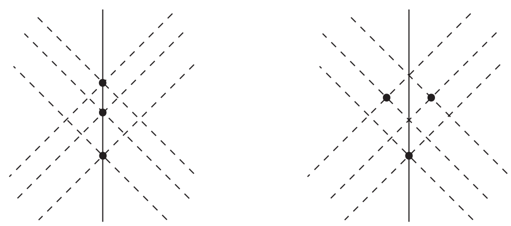
$$\dot{x} = x - 2bx^2 - 2x^3 - 2by - 6xy + 2by^2 + 6xy^2,\ \dot{y} = (2y-1)(2bx + 3x^2 + y - y^2). \tag{25}$$

The substitutions $x \to y/2$, $y \to (x+1)/2$, $t \to -4t$, $b \to a/2$, reduce (25) to the system 1.5) from Theorem 1. This system has the following invariant straight lines and integrating factor:

$$\begin{gathered} l_1 = y - ix + i,\ l_2 = y + ix - i,\ l_3 = y - ix - i, l_4 = y + ix + i = 0, \\ l_5 = y - ix + a,\ l_6 = y + ix + a,\ l_7 = x; \quad \mu = 1/(l_1 l_2 l_3 l_4 l_5 l_6). \end{gathered} \tag{26}$$

If $a = 0$, then the system 1.5) has the straight line $l_8 = y$.

In the case of the system 1.4) we have Fig. 3.5α), and in the case 1.5) we have Fig. 3.5β).



α) β)

Fig. 3.5

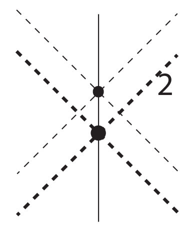


Fig. 3.6

Configuration 3.12) (3(2)c_1,3(2)c_1,1r). Assume $l_1 = l_2 \parallel l_3$, $l_4 = l_5 \parallel l_6$, $l_1 \nparallel l_4$. We can make $l_1 = y - ix$, $l_3 = y - ix - 1$ and so, $l_4 = y + ix$, $l_6 = y + ix - 1$, $i^2 = -1$. The cubic system for which the straight lines $l_1, ..., l_6$ are invariant coincides with the system 1.6) from Theorem 1. It is obvious that for system 1.6) the straight line $x = 0$ is invariant (see configuration from Fig 3.6). At the same time for this system, the polynomial (6) looks as: $\psi(x, y) = 2x(3y-1)(x^2+y^2)^2(1+x^2-2y+y^2)$, from which we deduce that $l_7 = x$ has the parallel multiplicity equal to one, and the system 1.6) has not other invariant straight lines besides the straight lines

$$\begin{gathered} l_1 = l_2 = y - ix,\ l_3 = y - ix - 1,\ l_4 = l_5 = y + ix, \\ l_6 = y + ix - 1,\ l_7 = x;\ \mu = 1/(l_1^2 l_3 l_4^2 l_6). \end{gathered} \tag{27}$$

By μ in (27) is denoted an integrating factor for system 1.6).

4 Qualitative study of systems 1.1) – 1.6)

In this section, the qualitative study of the systems 1.1) – 1.6) from Theorem 1 will be done. For this purpose, the finite as well as infinite singular points will be examined in order to determine the topological structures of their neighborhoods. Using this information, as well as the information provided by the existence of invariant straight lines, we will construct the phase portraits of systems 1.1) – 1.6) on Poincarè disk.

We denote by SP singular points; λ_1 and λ_2 the eigenvalues of SP; TSP – type of singular points; S – saddle ($\lambda_1 \lambda_2 < 0$); TS – topological saddle; N^s – stable node

$(\lambda_1, \lambda_2 < 0)$; N^i – unstable node $(\lambda_1, \lambda_2 > 0)$; $DN^{s(i)}$ – "decritic" stable (unstable) node $(\lambda_1 = \lambda_2 \neq 0)$; $NT^{s(i)}$ – topological stable (unstable) node; $S-N^{s(i)}$ – saddle-node with stable (unstable) parabolic sector; $F^{s(i)}$ – stable (unstable) focus; $P^{s(i)}$ – stable (unstable) parabolic sector; H – hyperbolic sector; E – elliptic sector.

4.1 The systems 1.1), 1.3), 1.4), 1.5)

In the first and the fourth columns of Tab. 4.1 we indicated the singular points of the systems 1.1), 1.3) – 1.5); in the second and the fifth columns the eigenvalues corresponding to the respective singular point are given and in the third and sixth columns the types of the singularities are established. All these points are simple and together with the invariant straight lines, complectly determine the phase portrait of each of systems 1.1), 1.3) – 1.5).

Tab. 4.1

System 1.1)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(-1,-1)$	$1+a;$ $1+a$	DN^i	$O_7(a,-1)$	$1+a;$ $a(1+a)$	N^i
$O_2(-1,0)$	$-a;\ 1+a$	S	$O_8(a,0)$	$-a;\ a(1+a)$	S
$O_3(-1,a)$	$1+a;$ $a(1+a)$	N^i	$O_9(a,a)$	$a(1+a);$ $a(1+a)$	DN^i
$O_4(0,-1)$	$-a;\ 1+a$	S	$X_{1_\infty}(1,0,0)$	$-1;\ -1$	DN^s
$O_5(0,0)$	$-a;\ -a$	DN^s	$X_{2,3_\infty}(1,\pm1,0)$	$-1;\ 2$	S
$O_6(0,a)$	$-a;\ a(1+a)$	S	$Y_\infty(0,1,0)$	$-1;\ -1$	DN^s

System 1.3)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$a^2+1;\ a^2+1$	DN^i	$X_{1_\infty}(1,0,0)$	$-1;\ -1$	DN^s
$Y_\infty(0,1,0)$	$-1;\ -1$	DN^s	$X_{2,3_\infty}(1,\pm1,0)$	$-1;\ 2$	S

System 1.4)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$-a;\ -a$	DN^s	$O_2(0,1)$	$a-1;\ a-1$	DN^s
$O_3(0,a)$	$a(1-a);$ $a(1-a)$	DN^i	$X_{1_\infty}(1,0,0)$	$-1;\ 2$	S
$Y_\infty(0,1,0)$	$-2;\ 1$	S			

System 1.5)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,-a)$	$a^2+1;\ a^2+1$	DN^i	$O_2(-1,0)$	$-2\pm2ai$	F^s
$O_3(1,0)$	$-2\pm2ai$	F^s	$X_{1_\infty}(1,0,0)$	$-2;\ 1$	S
$Y_\infty(0,1,0)$	$2;\ -1$	S			

4.2 System 1.2)

For 1.2) we have Tab 4.2.

Tab. 4.2

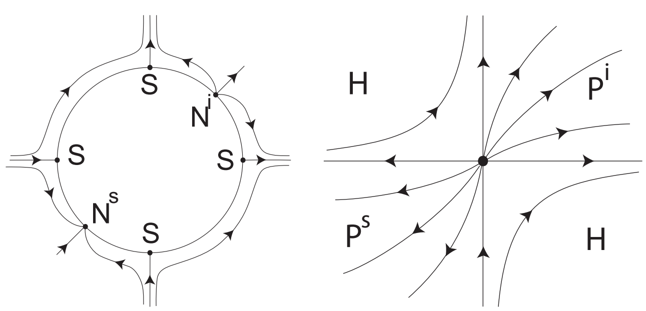
PS	$\lambda_1;\lambda_2$	TPS	PS	$\lambda_1;\lambda_2$	TPS
$O_1(0,0)$	0; 0	P^iHP^sH	$X_{1\infty}(1,0,0)$	$-1;-1$	ND^s
$O_2(-1,0)$	1; 0	$S{-}N^i$	$X_{2\infty}(1,-1,0)$	$-1;2$	S
$O_3(-1,-1)$	1; 1	ND^i	$X_{3\infty}(1,1,0)$	$-1;2$	S
$O_4(0,-1)$	0; 1	$S{-}N^i$	$Y_\infty(0,1,0)$	$-1;-1$	ND^s

In Tab. 4.2 all the singular points are simple except the point $O_1(0,0)$, which is nilpotent, and the points $O_2(-1,0)$ and $O_4(0,-1)$, which are semi-hyperbolic.

1) The singular point $O_1(0,0)$ has both eigenvalues equal to zero. To determine the behavior of trajectories in the neighborhood of this point we will use the blow-up method (see, for instance, [9]). According to it, we write the system 1.5) in the polar coordinates $(\rho,\theta): x=\rho cos\theta,\ y=\rho sin\theta$:

$$\begin{cases} \frac{d\rho}{d\tau}=\rho(cos^3\theta+sin^3\theta+\rho(cos^4\theta+sin^4\theta)), \\ \frac{d\theta}{d\tau}=sin\theta cos\theta(sin\theta-cos\theta)(1+\rho(cos\theta+sin\theta)), \end{cases} \tag{28}$$

where $\tau=\rho t$. The singular points of the system (28) with $\rho=0$ and $\theta\in[0,2\pi)$, and their eigenvalues: $\{M_1(0,0),\ M_2(0,\pi),\ M_3(0,\pi/2),\ M_4(0,3\pi/2):\lambda_{1,2}=\pm1\}$; $\{M_5(0,\pi/4):\lambda_{1,2}=1/\sqrt{2}\}$; $\{M_6(0,5\pi/4):\lambda_{1,2}=-1/\sqrt{2}\}$ lead us to Fig. 4.1, a), where we can see that the neighborhood of the point $O_1(0,0)$ is composed from sectors P^iHP^sH (Fig. 4.1, b)).



a) b)

Fig. 4.1

2) The substitution $x\to y,\ y\to x$, sends the singular point $O_4(0,-1)$ to the singular point $O_2(-1,0)$ and vice versa. Therefore, we will study the semi-hyperbolic singular point $O_2(-1,0)$. To determine the type of this point, we make in system 1.2) the transformation $x=Y-1,\ y=X$:

$$\dot{X}=X^2(X+1)=P(X,Y),\ \ \dot{Y}=Y-2Y^2+Y^3=Y+Q(X,Y).$$

From $Y+Q(X,Y)=0$, we have that $Y=\varphi(X)=0$. Substituting $Y=\varphi(X)$ in $P(X,Y)$ we obtain the function $\psi(X)=X^2+X^3$. By [3, Theorem 2, p. 87], the point $O_2(-1,0)$ is an unstable saddle-node, i.e. its neighborhood consists of two hyperbolic sectors and one unstable parabolic sector.

4.3 System 1.6)

This system has two singular points in the plane xOy and two singular points at infinity. The singular point $(0, 0)$ is nilpotent, other singular points are simple (see Tab. 4.3).

Tab. 4.3

SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$0;\ 0$	P^iEP^sE	$O_2(0,1)$	$-1;\ -1$	DN^s
$X_{1_\infty}(1,0,0)$	$-1;\ 2$	S	$Y_\infty(0,1,0)$	$-2;\ 1$	S

We will show that the nilpotent singular point $(0, 0)$ has the sectors: P^i, E, P^s, E. In polar coordinates (ρ, θ) the system 1.6) is written as:

$$\begin{cases} \dot{\rho} = \rho\left(\rho\cos^2\theta + \sin\theta - \rho\sin^2\theta\right), \\ \dot{\theta} = \cos\theta\left(-1 + 2\rho\sin\theta\right). \end{cases}$$

The obtained system, has two singular points M_1 and M_2 with first coordinate zero: $\left\{M_1(0, \frac{\pi}{2});\ \lambda_{1,2} = 1\right\}$, $\left\{M_2(0, \frac{3\pi}{2});\ \lambda_{1,2} = -1\right\}$.

We mark these points on the unit circle (Fig. 4.2, a)) which, being "compressed" in the point $(0, 0)$, give us the behavior of trajectories of differential system 1.6) in the neighborhood of this point (Fig. 4.2, b)).

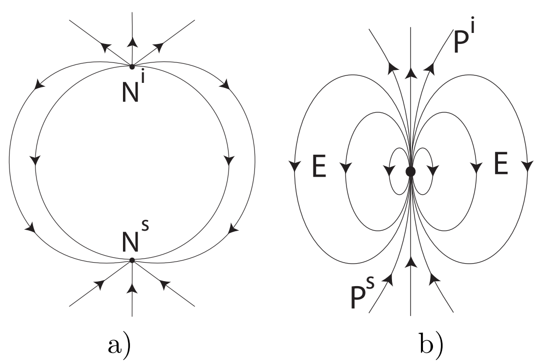


Fig. 4.2

As all the cases are considered, Theorem 1 is proved.

References

[1] Artes J., Grünbaum B. and Llibre J. *On the number of invariant straight lines for polynomial differential systems.* Pacific Journal of Mathematics, 1998, **184**, No. 2, 207–230.

[2] Artes J. and Llibre J. *On the number of slopes of invariant straight lines for polynomial differential systems.* J. of Nanjing University, 1996, **13**, 143–149.

[3] Bautin N. N. and Leontovich E. A. *Methods and rules for the qualitative study of dynamical systems on the plane.* Second edition. Moscow, Nauka, 1990 (in Russian)

[4] Chavarriga J. and Giné J. *Integrability of cubic systems with degenerate infinity.* Differential Equations Dynam. Systems, 1998, **6**, No. 4, 425–438.

[5] Chavarriga J., Giné J., García I. *Isochronous centers of cubic systems with degenerate infinity.* Differential Equations Dynam. Systems, 1999, **7**, No. 2, 221–238.

[6] Christopher C.J. *Invariant algebraic curves and conditions for a center.* Proc. Roy. Soc. Edinburgh, Sect A, 1994, **124**, No. 6, 1209–1229.

[7] Christopher C., Llibre J. and Pereira J.V. *Multiplicity of invariant algebraic curves in polynomial vector fields.* Pacific Journal of Mathematics, 2007, **329**, No. 1, 63–117.

[8] Cozma D. and Şubă A. *The solution of the problem of center for cubic differential systems with four invariant straight lines.* Mathematical analysis and applications (Iaşi, 1997). An. Ştiinţ. Univ. "Al. I. Cuza"(Iaşi), 1998, **44**, suppl., 517–530.

[9] Dumortier, F., *Singularities of vector fields on the plane*, J. Differential Equations, **23** (1977), 53–106.

[10] Suo Guangjian and Sun Jifang. *The n-degree differential system with $(n-1)(n+1)/2$ straight line solutions has no limit cycles.* Proc. of Ordinary Differential Equations and Control Theory, Wuhan, 1987, 216–220 (in Chinese).

[11] Kooij R. *Cubic systems with four line invariants, including complex conjugated lines.* Math. Proc. Camb. Phil. Soc., 1995, **118**, No. 1, 7–19.

[12] Llibre J. and Vulpe N. *Planar cubic polynomial differential systems with the maximum number of invariant straight lines.* Rocky Mountain J. Math., 2006, **36**, No. 4, 1301—1373.

[13] Lloyd N.G., Christopher C.J., Devlin J., Pearson J.M. *Quadratic-like cubic systems.* Differential Equations Dynam. Systems, 1997, **5**, No. 3/4, 329–345.

[14] Lyubimova R.A. *About one differential equation with invariant straight lines.* Differential and integral equations, Gorky Universitet, 1984, **8**, 66–69; 1997, **1**, 19–22 (in Russian).

[15] Puţuntică V. and Şubă A. *The cubic differential system with six real invariant straight lines along two directions.* Studia Universitatis. Seria Ştiinţe Exacte şi Economice, 2008, No. 8(13), 5–16.

[16] Puţuntică V. and Şubă A. *The cubic differential system with six real invariant straight lines along three directions.* Bulletin of ASRM. Mathematics, 2009, no. 2(60), 111–130.

[17] Rychkov G.S. *The limit cycles of the equation* $u(x+1)du = (-x+ax^2+bxu+cu+du^2)dx$. Differentsial'nye Uravneniya, 1972, **8**, No. 12, 2257–2259 (in Russian).

[18] Schlomiuk D. and Vulpe N. *The full study of planar quadratic differential systems possessing a line of singularities at infinity.* J. Dynam. Differential Equations, 2008, **20**, No. 4, 737–775.

[19] Sokulski J. *On the number of invariant lines for polynomial vector fields.* Nonlinearity, 1996, **9**, 479–485.

[20] Şubă A. and Cozma D. *Solution of the problem of the center for cubic differential system with three invariant straight lines in generic position.* Qualitative Theory of Dynamical Systems, Universitat de Lleida. Spaine, 2005, **6**, 45–58.

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