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Moment analysis of the telegraph random process

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Abstract. We consider the Goldstein-Kac telegraph process $X(t),\ t > 0$, on the real line $\mathbb{R}^1$ performed by the random motion at finite speed c and controlled by a homogeneous Poisson process of rate $\lambda > 0$. Using a formula for the moment function $\mu_{2k}(t)$ of $X(t)$ we study its asymptotic behaviour, as c, λ and t vary in different ways. Explicit asymptotic formulas for $\mu_{2k}(t)$, as $k \to \infty$, are derived and numerical comparison of their effectiveness is given. We also prove that the moments $\mu_{2k}(t)$ for arbitrary fixed $t > 0$ satisfy the Carleman condition and, therefore, the distribution of the telegraph process is completely determined by its moments. Thus, the moment problem is completely solved for the telegraph process $X(t)$. We obtain an explicit formula for the Laplace transform of $\mu_{2k}(t)$ and give a derivation of the the moment generating function based on direct calculations. A formula for the semi-invariants of $X(t)$ is also presented.

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1 Preliminaries

Consider the one-dimensional stochastic process performed by a particle that starts at the time instant $t = 0$ from the origin $x = 0$ of the real line $\mathbb{R}^1$ and moves with some finite constant speed c. The initial direction of the motion (positive or negative) is taken on with equal probabilities 1/2. The motion is driven by a homogeneous Poisson process of rate $\lambda > 0$ as follows. As a Poisson event occurs, the particle instantaneously takes on the opposite direction and keeps moving with the same speed c until the next Poisson event occurrence, then it takes on the opposite direction again independently of its previous motion, and so on. This random motion has first been studied by Goldstein [12] and Kac [16] and was called the *telegraph process* afterwards (the latter article [16] is a reprinting of an earlier 1956 work).

Let $X(t)$ denote the particle's position on $\mathbb{R}^1$ at an arbitrary time instant $t > 0$. Since the speed c is finite, then, at the time instant $t > 0$, the distribution $\Pr\{X(t) \in dx\}$ is concentrated in the finite interval $[-ct, ct]$ which is the support of the distribution of $X(t)$. The density $f(x,t),\ x \in \mathbb{R}^1,\ t \geq 0$, of the distribution $\Pr\{X(t) \in dx\}$ has the structure

$$f(x,t) = f_s(x,t) + f_{ac}(x,t),$$

where $f_s(x,t)$ and $f_{ac}(x,t)$ are the densities of the singular (with respect to the Lebesgue measure on the line) and of the absolutely continuous components of the distribution of $X(t)$, respectively.

The singular component of the distribution is, obviously, concentrated at two terminal points $\pm ct$ of the interval $[-ct, ct]$ and corresponds to the case when no one Poisson event occurs until the moment t and, therefore, the particle does not change its initial direction (the probability of this event is $e^{-\lambda t}$).

The density $f_{ac}(x,t)$ of the absolutely continuous components of the distribution corresponds to the case when at least one Poisson event occurs by moment t and, therefore, the particle changes its initial direction (the probability of this event is $1 - e^{-\lambda t}$). The support of this part of the distribution is the open interval $(-ct, ct)$.

The principal result by Goldstein [12] and Kac [16] states that the density $f = f(x,t)$, $x \in [-ct, ct]$, $t \geq 0$, satisfies the following hyperbolic partial differential equation

$$\frac{\partial^2 f}{\partial t^2} + 2\lambda \frac{\partial f}{\partial t} - c^2 \frac{\partial^2 f}{\partial x^2} = 0, \tag{1}$$

which is referred to as the *telegraph* or *damped wave* equation and can be found by solving (1) with the initial conditions

$$f(x,t)|_{t=0} = \delta(x), \qquad \left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0,$$

where $\delta(x)$ is the Dirac delta-function. This means that the transition density $f(x,t)$ of the process $X(t)$ is the fundamental solution (i.e. the Green's function) of the telegraph equation (1).

The explicit form of the density $f(x,t)$ is given by the formula (see, for instance, [29, Section 0.4] or [27, Theorem 1]):

$$\begin{aligned} f(x,t) = {} & \frac{e^{-\lambda t}}{2} \left[\delta(ct - x) + \delta(ct + x)\right] + \\ & + \frac{e^{-\lambda t}}{2c} \left[\lambda I_0 \left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0 \left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \Theta(ct - |x|), \end{aligned} \tag{2}$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases}$$

and $I_0(z)$ is the modified Bessel function of order zero (that is, the Bessel function with imaginary argument) given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{z}{2}\right)^{2k}.$$

The first term of (2)

$$f_s(x,t) = \frac{e^{-\lambda t}}{2} \left[\delta(ct - x) + \delta(ct + x)\right] \tag{3}$$

represents the density of the singular part of the distribution of $X(t)$ concentrated at two terminal points $\pm ct$ of the interval $[-ct, ct]$, while the second term of (2)

$$f_{ac}(x,t) = \frac{e^{-\lambda t}}{2c} \left[\lambda I_0 \left(\frac{\lambda}{c} \sqrt{c^2t^2 - x^2} \right) + \frac{\partial}{\partial t} I_0 \left(\frac{\lambda}{c} \sqrt{c^2t^2 - x^2} \right) \right] \Theta(ct - |x|), \quad (4)$$

is the density of the absolutely continuous part of the distribution of $X(t)$ concentrated in the open interval $(-ct, ct)$.

During last decades the Goldstein-Kac telegraph process $X(t)$ and its numerous generalizations have become the subject of intense researches provided both by great theoretical importance and fruitful applications in statistical physics, financial modeling, transport phenomena in physical and biological systems, hydrology and some other fields. Some properties of the solution space of the Goldstein-Kac telegraph equation (1) were studied by Bartlett [2]. The process of one-dimensional random motion at finite speed governed by a Poisson process with a time-depending parameter was considered by Kaplan [17]. The relationships between the Goldstein-Kac model and physical processes, including some emerging effects of the relativity theory, were thoroughly examined by Bartlett [1], Cane [5,6]. Formulas for the distributions of the first-exit time from a given interval and of the maximum displacement of the telegraph process were obtained by Pinsky [29, Section 0.5], Foong [10], Masoliver and Weiss [25, 26]. The behaviour of the telegraph process with absorbing and reflecting barriers was studied by Foong and Kanno [11], Orsingher [28]. A one-dimensional stochastic motion with an arbitrary number of velocities and governing Poisson processes was examined by Kolesnik [21]. The telegraph-type processes with random velocities were studied by Stadje and Zacks [32]. Probabilistic methods of solving the Cauchy problems for the telegraph equation (1) were developed by Kac [16], Kisynski [18], Kabanov [15], Turbin and Samoilenko [33]. A generalization of the Goldstein-Kac model for the case of a damped telegraph process with logistic stationary distributions was given by Di Crescenzo and Martinucci [8]. A random motion with velocities alternating at Erlang-distributed random times was studied by Di Crescenzo [7]. Formulas for the occupation time distributions of the telegraph process were recently obtained by Bogachev and Ratanov [4]. A generalization of the Goldstein-Kac telegraph process to the $\mathbb{R}^d$, $d \geq 1$, space with an arbitrary *finite number* of cyclically changing directions was thoroughly examined by Lachal [24]. A similar motion in the plane $\mathbb{R}^2$ with an arbitrary *finite number of directions* and uniform mechanism of their change was studied by Kolesnik and Turbin [23].

Moments of any stochastic process are one of the most interesting and useful objects both from theoretical and practical points of view. This especially concerns the telegraph process $X(t)$ which is the basis for many important models in financial mathematics, biology, physics and other fields. For example, the knowledge of moments enables to construct various moment-type estimators in statistics (see, for instance,[14]). However, despite the great variety of existing works on the subject and of the results obtained, the moment problem for the Goldstein-Kac telegraph process was not properly solved so far. In particular, it was not clear whether the distribution of $X(t)$ was completely determined by its moments.

The most enigmatic fact is that the transition density (2) of the one-dimensional telegraph process $X(t)$ has much more complicated form than the transition densities of its two- and four-dimensional counterparts with a *continuum number of directions* (for the transition density of the 2D and 4D-motions see [22, Theorem 2] and [20, Theorem 2], respectively). While the transition density (2) contains special functions, the densities of the 2D- and 4D-motions have very simple exponential form that enables to explicitly compute the moments (see [19, Theorems 1 and 3, respectively]). Note also that the moments of a special multidimensional random motion with a cyclic mechanism of choosing new directions were computed by Samoilenko [31].

In this article we give a detailed moment analysis of the Goldstein-Kac telegraph process $X(t)$. In Section 2 we study the asymptotic behaviour of the moment function as c, λ and t vary in different ways. In Section 3 we obtain an explicit formula for the Laplace transform of the moment function of $X(t)$. In Section 4 we give the complete solution of the moment problem for the telegraph process $X(t)$. We show that, for arbitrary $t > 0$, the moments of $X(t)$ satisfy the Carleman condition and, therefore, the distribution of $X(t)$ is completely determined by its moments. In Section 5 we derive the moment generating function by direct computations and give a formula for the semi-invariants of the telegraph process $X(t)$.

2 Asymptotic Behaviour of Moments

Consider the moment function of the Goldstein-Kac telegraph process $X(t)$ defined by the formula

$$\mu_n(t) = E[X(t)]^n, \quad n \geq 1,$$

where E means the expectation.

It is known (see, for instance,[14, Theorem 2.1]) that, for arbitrary $t > 0$, the moments of $X(t)$ are given by the formula

$$\begin{aligned} &\mu_{2k}(t) = e^{-\lambda t}\, c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, t^{k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right], \\ &\mu_{2k+1}(t) = 0, \qquad\qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{5}$$

where $I_\nu(z)$ is the modified Bessel function of order ν

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\, \Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu},$$

and $\Gamma(x)$ is the Euler gamma-function. Note that formula (5) slightly differs from that of [14, Theorem 2.1]), however one can easily check that both these representations of the moment function $\mu_{2k}(t)$ are equivalent. For our purposes it is more convenient to use just the representation (5).

From (5) we can easily obtain the first and the second moments of the telegraph process $X(t)$:

$$\mu_1(t) = 0, \qquad \mu_2(t) = \frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2}\,(1 - e^{-2\lambda t}), \tag{6}$$

and this coincides with [27, Formula (28)].

In this section we thoroughly study the asymptotic behaviour of the moment function given by (5). Clearly, we need to examine the behaviour of the even-order moments $\mu_{2k}(t), \; k = 1, 2, \dots,$ only.

2.1. *Asymptotic behaviour with respect to* $c \to \infty, \; \lambda \to \infty$*, (*$t$ *and* k *are fixed).* In this subsection we consider the case when, under fixed t and k, the speed of the motion c and the intensity of switching Poisson process λ both go to infinity in such a way that the following Kac condition holds:

$$c \to \infty, \qquad \lambda \to \infty, \qquad \frac{c^2}{\lambda} \to \rho, \quad \rho > 0. \tag{7}$$

Taking into account the well-known asymptotic formula for the modified Bessel function (see, for instance,[13, Formula 8.451(5)]):

$$I_\nu(z) \; \sim \; \frac{e^z}{\sqrt{2\pi z}} \, , \qquad z \to +\infty, \tag{8}$$

as well as the formula (see [13, Formula 8.339(2)])

$$\Gamma \left(k + \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2^k} \, (2k-1)!!, \qquad k \geq 0, \quad (-1)!! = 1, \tag{9}$$

we obtain

$$\begin{aligned}
\lim_{\substack{c, \, \lambda \to \infty \\ (c^2/\lambda) \to \rho}} \mu_{2k}(t) &= 2^{k-1/2} t^{k+1/2} \Gamma \left(k + \frac{1}{2} \right) \times \\
&\times \lim_{\substack{c, \, \lambda \to \infty \\ (c^2/\lambda) \to \rho}} \left[e^{-\lambda t} c^{2k} \lambda^{-k+1/2} \left(I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t) \right) \right] \sim \\
&\sim 2^{k-1/2} t^{k+1/2} \Gamma \left(k + \frac{1}{2} \right) \lim_{\substack{c, \, \lambda \to \infty \\ (c^2/\lambda) \to \rho}} \left[e^{-\lambda t} c^{2k} \lambda^{-k+1/2} \, \frac{2 e^{\lambda t}}{\sqrt{2\pi \lambda t}} \right] = \\
&= 2^k \, t^k \, \frac{1}{\sqrt{\pi}} \, \Gamma \left(k + \frac{1}{2} \right) \lim_{\substack{c, \, \lambda \to \infty \\ (c^2/\lambda) \to \rho}} \left(\frac{c^{2k}}{\lambda^k} \right) = \\
&= 2^k \, t^k \, \frac{1}{\sqrt{\pi}} \, \frac{\sqrt{\pi}}{2^k} \, (2k-1)!! \, \rho^k = \\
&= \rho^k \, t^k \, (2k-1)!!
\end{aligned}$$

and this coincides with the moment function of the one-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = \rho$.

2.2. *Asymptotic behaviour with respect to* $t \to \infty, \; \lambda \to \infty$*, (*$c$ *and* k *are fixed).* Similarly to the asymptotic analysis of Subsection 2.1 and by using (8) and (9) we can easy show that for $t \to \infty$ or $\lambda \to \infty$ (or both t and λ tend to infinity), under fixed c and k, the following asymptotic formula holds:

$$\mu_{2k}(t) \sim \left(\frac{c^2 t}{\lambda}\right)^k (2k-1)!! \tag{10}$$

From (10) we see that the moments $\mu_{2k}(t)$ increase like t^k as $t \to \infty$ (for fixed c, λ and k). Conversely, the moments $\mu_{2k}(t)$ decrease like λ^{-k} as $\lambda \to \infty$ (for fixed c, t and k).

2.3. *Asymptotic behaviour with respect to $k \to \infty$, (c, t and λ are fixed).* Asymptotic analysis with respect to $k \to \infty$ is much more complicated due to the absence of general asymptotic formulas with respect to the index ν of the modified Bessel function $I_\nu(z)$ (except the very particular case when the argument z has a special form depending on index ν). Nevertheless, we are able to obtain asymptotic formulas for the moment function $\mu_{2k}(t)$, as $k \to \infty$, due to the special form of the indices of the modified Bessel functions in (5). This result is presented by the following theorem.

Theorem 1. *For any fixed c, λ and t the following asymptotic formula holds:*

$$\mu_{2k}(t) \sim e^{-\lambda t} (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1}\right), \qquad k \to \infty. \tag{11}$$

The refined asymptotic formula has the form:

$$\mu_{2k}(t) \sim e^{-\lambda t} (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1} + \frac{(\lambda t)^2}{4k+2} + \frac{(\lambda t)^3}{(4k+2)(2k+3)}\right), \qquad k \to \infty. \tag{12}$$

Proof. First we need to establish the following asymptotic formulas for the modified Bessel functions:

$$I_{k+1/2}(z) \sim \sqrt{\frac{2}{\pi}} \frac{z^{k+1/2}}{(2k+1)!!}, \qquad k \to \infty, \tag{13}$$

$$I_{k-1/2}(z) \sim \sqrt{\frac{2}{\pi}} \frac{z^{k-1/2}}{(2k-1)!!}, \qquad k \to \infty. \tag{14}$$

Let us prove (13). Using the series representation of the modified Bessel function (see, for instance,[13, Formula 8.445]) we have

$$\begin{aligned}
I_{k+1/2}(z) &= z^{k+1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,\Gamma\left((l+k+1/2)+1\right)} \left(\frac{z}{2}\right)^{2l} = \\
&= z^{k+1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,(l+k+1/2)\,\Gamma\left(l+k+1/2\right)} \left(\frac{z}{2}\right)^{2l} = \\
&\quad \text{(see formula(9))} \\
&= \frac{z^{k+1/2}}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{z^{2l}\; 2^{l+k}}{l!\,(l+k+1/2)\,(2l+2k-1)!!\;2^{2l+k+1/2}} = \\
&= \sqrt{\frac{2}{\pi}}\; z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{l!\,(2l+2k+1)\,(2l+2k-1)!!\;2^{l}} = \\
&= \sqrt{\frac{2}{\pi}}\; z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!} \sim \\
&\sim \sqrt{\frac{2}{\pi}}\; \frac{z^{k+1/2}}{(2k+1)!!}\,, \qquad k\to\infty,
\end{aligned}$$

proving (13). Similarly, we have

$$\begin{aligned}
I_{k-1/2}(z) &= z^{k-1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,\Gamma(l+k+1/2)} \left(\frac{z}{2}\right)^{2l} = \\
&= \frac{z^{k-1/2}}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{z^{2l}}{l!\,(2l+2k-1)!!\;2^{l-1/2}} = \\
&= \sqrt{\frac{2}{\pi}}\; z^{k-1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!} \sim \\
&\sim \sqrt{\frac{2}{\pi}}\; \frac{z^{k-1/2}}{(2k-1)!!}\,, \qquad k\to\infty,
\end{aligned}$$

and (14) is also proved.

Therefore, by applying formulas (13) and (14) just now proved, we obtain:

$$\begin{aligned}
\mu_{2k}(t) &= e^{-\lambda t}\; c^{2k}\; 2^{k-1/2}\; \lambda^{-k+1/2}\; t^{k+1/2}\; \Gamma\left(k+\frac{1}{2}\right)\; \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] \sim \\
&\sim e^{-\lambda t} c^{2k} 2^{k-1/2} \lambda^{-k+1/2} t^{k+1/2} \frac{\sqrt{\pi}}{2^k} (2k-1)!! \sqrt{\frac{2}{\pi}} \left[\frac{(\lambda t)^{k+1/2}}{(2k+1)!!} + \frac{(\lambda t)^{k-1/2}}{(2k-1)!!}\right] = \\
&= e^{-\lambda t}\; c^{2k}\; \lambda^{-k+1/2}\; t^{k+1/2}\; (2k-1)!!\; \frac{(\lambda t)^{k-1/2}}{(2k-1)!!} \left[1 + \frac{\lambda t}{2k+1}\right] = \\
&= e^{-\lambda t}\; (ct)^{2k}\; \left(1 + \frac{\lambda t}{2k+1}\right), \qquad k\to\infty,
\end{aligned}$$

yielding (11).

Formula (12) can be proved in the same manner by applying, instead of (13) and (14), the refined asymptotic formulas for the modified Bessel function (see also Remark 1 below):

$$I_{k+1/2}(z) \sim \frac{z^{k+5/2} + (4k+6)z^{k+1/2}}{\sqrt{2\pi}\,(2k+3)!!}, \qquad k \to \infty, \tag{15}$$

$$I_{k-1/2}(z) \sim \frac{z^{k+3/2} + (4k+2)z^{k-1/2}}{\sqrt{2\pi}\,(2k+1)!!}, \qquad k \to \infty. \tag{16}$$

The theorem is thus completely proved. □

Remark 1. One can write down more accurate asymptotic formulas by taking arbitrary finite number of terms in the series expansions of the functions $I_{k+1/2}(z)$ and $I_{k-1/2}(z)$:

$$\begin{aligned} I_{k+1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!}, \\ I_{k-1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k-1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!}. \end{aligned} \tag{17}$$

Since the index k is presented in the denominators of (17) and, therefore, each term of these series tends to zero as $k \to \infty$, then for arbitrary integer $n \geq 0$ the following formulas hold:

$$\begin{aligned} I_{k+1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k+1/2} \sum_{l=0}^{n} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!} + R^{+}_{k,n}(z), \\ I_{k-1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k-1/2} \sum_{l=0}^{n} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!} + R^{-}_{k,n}(z), \end{aligned} \tag{18}$$

where the remainders $R^{\pm}_{k,n}(z) \to 0$, as $k \to \infty$, for any fixed z and $n \geq 0$. Note that formulas (13) and (14) follow, as $k \to \infty$, from (18) for $n = 0$, while (15) and (16) follow, as $k \to \infty$, from (18) for $n = 1$, respectively. One can also obtain the upper bounds for the remainders $R^{\pm}_{k,n}(z)$ and, therefore, to evaluate the rate of their convergence to zero, as $k \to \infty$, however this is not our concern here.

Remark 2. Asymptotic formulas (11) and (12) show that the behaviour of the moment function $\mu_{2k}(t)$ with respect to $k \to \infty$ depends on the factor ct as follows:

If $ct < 1$, then $\mu_{2k}(t) \to 0$, as $k \to \infty$;
If $ct = 1$, then $\mu_{2k}(t) \to e^{-\lambda t}$, as $k \to \infty$;
If $ct > 1$, then $\mu_{2k}(t) \to \infty$, as $k \to \infty$.

This enables us to make some interesting and somewhat unexpected conclusions concerning the asymptotic behaviour of the moment function $\mu_{2k}(t)$, as $k \to \infty$.

Since ct is the total length of an arbitrary sample path of the Goldstein-Kac telegraph process $X(t)$ at the time instant $t > 0$ whose distribution is concentrated in the interval $[-ct, ct]$, then $[-1, 1]$ is the critical interval in the following sense. If $[-ct, ct] \subset [-1, 1]$, then the moments $\mu_{2k}(t)$ are finite and tend to zero, as $k \to \infty$. If $[-ct, ct] = [-1, 1]$, then the moments $\mu_{2k}(t)$ are finite and tend to $e^{-\lambda t}$, as $k \to \infty$. Finally, if $[-ct, ct] \supset [-1, 1]$, then the moments $\mu_{2k}(t)$ tend to ∞, as $k \to \infty$. In terms of the time t this means that for $t < \frac{1}{c}$, the moments are finite and tend to zero, as $k \to \infty$; at the time instant $t = \frac{1}{c}$, the moments are finite and tend to $e^{-\lambda/c}$, as $k \to \infty$; for $t > \frac{1}{c}$, the moments tend to ∞, as $k \to \infty$.

Numerical computations of moments according to formula (5) and their approximations (for increasing k) by means of the asymptotic functions

$$g_0(t) = e^{-\lambda t} \, (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1} \right),$$
$$g_1(t) = e^{-\lambda t} \, (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1} + \frac{(\lambda t)^2}{4k+2} + \frac{(\lambda t)^3}{(4k+2)(2k+3)} \right),$$

obtained in Theorem 1 are given in the following table below (for the particular values of the parameters $c = 0.6, \ t = 1.5, \ \lambda = 2.5$):

k	$\mu_{2k}(1.5)$	$g_0(1.5)$	$g_1(1.5)$
100	$0.175030 \cdot 10^{-10}$	$0.169015 \cdot 10^{-10}$	$0.174926 \cdot 10^{-10}$
500	$0.415508 \cdot 10^{-47}$	$0.412600 \cdot 10^{-47}$	$0.415498 \cdot 10^{-47}$
1000	$0.722360 \cdot 10^{-93}$	$0.719826 \cdot 10^{-93}$	$0.722356 \cdot 10^{-93}$
5000	$0.626552 \cdot 10^{-459}$	$0.626113 \cdot 10^{-459}$	$0.626553 \cdot 10^{-459}$
10000	$0.166654 \cdot 10^{-916}$	$0.166597 \cdot 10^{-916}$	$0.166655 \cdot 10^{-916}$

We see that the second asymptotic function $g_1(t)$ yields a better approximation (for increasing k) of the moment function $\mu_{2k}(t)$ than the first asymptotic function $g_0(t)$. In particular, we see that the function $g_1(t)$ provides stabilization in the second digit already for $k = 100$, while the function $g_0(t)$ does so only for $k = 500$. Note also that in this example $ct = 0.6 \cdot 1.5 = 0.9 < 1$ and the moments $\mu_{2k}(1.5)$ tend to zero, as $k \to \infty$, very rapidly.

3 Laplace Transform of Moment Function

In this section we derive an explicit formula for the Laplace transform of the moment function $\mu_{2k}(t), \ k \geq 1$, given by (5). We show that, despite the fairly complicated form of the moment function (5), its Laplace transform has a very simple form. This result is presented by the following theorem.

Theorem 2. *The Laplace transform of moment function* (5) *is given by the formula:*

$$\mathcal{L}_t \left[\mu_{2k}(t) \right] (s) = \frac{c^{2k} \, (2k)!}{s^{k+1} \, (s + 2\lambda)^k} \, , \qquad \text{Re} \, s > 0. \tag{19}$$

Proof. Applying the Laplace transformation to (5) we have:

$$\begin{aligned}\mathcal{L}_t\left[\mu_{2k}(t)\right](s) = c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \times \\ \times \mathcal{L}_t\left[e^{-\lambda t}\, t^{k+1/2}\, \left(I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right)\right](s) = \\ = c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \times \\ \times \mathcal{L}_t\left[t^{k+1/2}\, \left(I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right)\right](s+\lambda).\end{aligned} \tag{20}$$

According to [3, Table 4.16, Formulas 6 and 7]

$$\mathcal{L}_t\left[t^{k+1/2}\, I_{k+1/2}(\lambda t)\right](s) = \frac{1}{\sqrt{\pi}}\, 2^{k+1/2}\, \lambda^{k+1/2}\, k!\, \frac{1}{(s^2-\lambda^2)^{k+1}}\,,$$
$$\mathcal{L}_t\left[t^{k+1/2}\, I_{k-1/2}(\lambda t)\right](s) = \frac{1}{\sqrt{\pi}}\, 2^{k+1/2}\, \lambda^{k-1/2}\, k!\, \frac{s}{(s^2-\lambda^2)^{k+1}}\,.$$

Substituting these expressions into (20) we obtain

$$\begin{aligned}\mathcal{L}_t\left[\mu_{2k}(t)\right](s) &= c^{2k}\, 2^{2k}\, \Gamma\left(k+\frac{1}{2}\right)\, \frac{k!}{\sqrt{\pi}}\, \frac{s+2\lambda}{((s+\lambda)^2-\lambda^2)^{k+1}} = \\ &\quad \text{(see Formula (9))} \\ &= c^{2k}\, 2^{2k}\, \frac{\sqrt{\pi}}{2^k}(2k-1)!!\, \frac{k!}{\sqrt{\pi}}\, \frac{s+2\lambda}{((s+\lambda)^2-\lambda^2)^{k+1}} = \\ &= c^{2k}\, k!\, 2^k\, (2k-1)!!\, \frac{s+2\lambda}{(s(s+2\lambda))^{k+1}} = \\ &= c^{2k}\, (2k)!!\, (2k-1)!!\, \frac{s+2\lambda}{(s(s+2\lambda))^{k+1}} = \\ &= \frac{c^{2k}\, (2k)!}{s^{k+1}\, (s+2\lambda)^k}.\end{aligned}$$

The theorem is proved. □

In particular, for $k=1$, we obtain from (19) the formula for the Laplace transform of the second moment

$$\mathcal{L}_t\left[\mu_2(t)\right](s) = \frac{2c^2}{s^2\, (s+2\lambda)}. \tag{21}$$

On the other hand, applying Laplace transformation to (6) we have:

$$\begin{aligned}
\mathcal{L}_t\left[\mu_2(t)\right](s) &= \mathcal{L}_t\left[\frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right)\right](s) = \\
&= \frac{c^2}{\lambda}\,\mathcal{L}_t[t](s) - \frac{c^2}{2\lambda^2}\,\Big(\mathcal{L}_t[1](s) - \mathcal{L}_t[e^{-2\lambda t}](s)\Big) = \\
&= \frac{c^2}{\lambda}\,\frac{1}{s^2} - \frac{c^2}{2\lambda^2}\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right) = \\
&= \frac{c^2}{\lambda}\,\frac{1}{s^2} - \frac{c^2}{2\lambda^2}\,\frac{2\lambda}{s(s+2\lambda)} = \\
&= \frac{c^2}{\lambda}\left(\frac{1}{s^2} - \frac{1}{s(s+2\lambda)}\right) = \\
&= \frac{2c^2}{s^2\,(s+2\lambda)}
\end{aligned}$$

and this coincides with (21).

Remark 3. One can check that, under the Kac condition (7), function (19) turns into the Laplace transform of the moment function of Brownian motion. Really, for function (19) we have

$$\begin{aligned}
\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left\{\mathcal{L}_t\left[\mu_{2k}(t)\right](s)\right\} &= \frac{(2k)!}{s^{k+1}} \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{(s+2\lambda)^k}\right\} = \\
&= \frac{(2k)!!\,(2k-1)!!}{s^{k+1}} \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{(2\lambda)^k}\,\frac{1}{\left(\frac{s}{2\lambda}+1\right)^k}\right\} = \\
&= \frac{2^k\,k!\,(2k-1)!!}{s^{k+1}}\,\frac{1}{2^k} \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{\lambda^k}\right\} = \\
&= \frac{\rho^k\,k!\,(2k-1)!!}{s^{k+1}}.
\end{aligned}$$

On the other hand, for the Laplace transform of the moment function of the one-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = \rho$ derived in Subsection 2.1 above, we obtain the formula

$$\begin{aligned}
\mathcal{L}_t\Big[\rho^k\,t^k\,(2k-1)!!\Big](s) &= \rho^k\,(2k-1)!!\,\mathcal{L}_t[t^k](s) = \\
&= \rho^k\,(2k-1)!!\,\frac{\Gamma(k+1)}{s^{k+1}} = \\
&= \frac{\rho^k\,k!\,(2k-1)!!}{s^{k+1}}
\end{aligned}$$

exactly coinciding with the previous one.

4 Moment Problem

In this section we give the complete solution of the moment problem for the Goldstein-Kac telegraph process $X(t)$. We show that, for any fixed $t > 0$, the moments of $X(t)$ satisfy the Carleman condition and, therefore, the distribution of $X(t)$ is completely determined by its moments. This result is given by the following theorem.

Theorem 3. *For any fixed $t > 0$ the moments $\mu_{2k}(t)$ of the telegraph process $X(t)$, given by* (5), *satisfy the Carleman condition:*

$$\sum_{k=1}^{\infty} [\mu_{2k}(t)]^{-1/(2k)} = \infty. \tag{22}$$

Proof. To prove the theorem it suffices to show that the general term of the series on the left-hand side of (22) does not tend to zero, as $k \to \infty$. First, we prove that, for arbitrary $k \geq 1$, the following inequality holds:

$$\mu_{2k}(t) < (ct)^{2k} \; (1 + \lambda t) \; e^{\lambda^2 t^2/2}, \qquad k \geq 1. \tag{23}$$

By using formulas (9) and (17) we have:

$$\begin{aligned}
\mu_{2k}(t) &= e^{-\lambda t} \; c^{2k} \; 2^{k-1/2} \; \lambda^{-k+1/2} \; t^{k+1/2} \; \Gamma\left(k + \frac{1}{2}\right) \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t} \; c^{2k} \; 2^{k-1/2} \; \lambda^{-k+1/2} \; t^{k+1/2} \; \frac{\sqrt{\pi}}{2^k} \; (2k-1)!! \; \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t} \; c^{2k} \; \lambda^{-k+1/2} \; t^{k+1/2} \; \sqrt{\frac{\pi}{2}} \; (2k-1)!! \; \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] < \\
&< c^{2k} \; \lambda^{-k+1/2} \; t^{k+1/2} \; (2k-1)!! \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!! \; (2l+2k+1)!!} + \right. \\
&\qquad \left. + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!! \; (2l+2k-1)!!} \right] = \\
&= c^{2k} \; \lambda^{-k+1/2} \; t^{k+1/2} \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(2k-1)!!}{(2l+2k+1)!!} \; \frac{(\lambda t)^{2l}}{(2l)!!} + \right. \\
&\qquad \left. + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(2k-1)!!}{(2l+2k-1)!!} \; \frac{(\lambda t)^{2l}}{(2l)!!} \right] < \\
&< c^{2k} \; \lambda^{-k+1/2} \; t^{k+1/2} \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!} + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!} \right],
\end{aligned}$$

where in the last step we have used the fact that, for any $k \geq 1$, the following inequalities hold

$$\frac{(2k-1)!!}{(2l+2k+1)!!} < 1, \qquad \frac{(2k-1)!!}{(2l+2k-1)!!} \leq 1, \qquad \text{for any } \; l \geq 0.$$

Now taking into account that

$$\sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!} = \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{2^l \; l!} = \sum_{l=0}^{\infty} \frac{1}{l!} \; \left(\frac{\lambda^2 t^2}{2} \right)^l = e^{\lambda^2 t^2/2}$$

we obtain

$$\begin{aligned} \mu_{2k}(t) &< c^{2k} \; \lambda^{-k+1/2} \; t^{k+1/2} \; (\lambda t)^{k-1/2} \; (1+\lambda t) \; e^{\lambda^2 t^2/2} = \\ &= (ct)^{2k} \; (1+\lambda t) \; e^{\lambda^2 t^2/2}, \end{aligned}$$

proving (23). From (23) we have the inequality:

$$[\mu_{2k}(t)]^{-1/(2k)} > (ct)^{-1} \; (1+\lambda t)^{-1/(2k)} \; e^{-\lambda^2 t^2/(4k)}, \qquad k \geq 1.$$

Then, by passing to the limit, as $k \to \infty$, in this last inequality, we obtain:

$$\lim_{k\to\infty} [\mu_{2k}(t)]^{-1/(2k)} \geq (ct)^{-1} > 0$$

for any c and $t > 0$. Hence, the sequence $[\mu_{2k}(t)]^{-1/(2k)}$ does not tend to zero as $k \to \infty$ and, therefore, the series (4.1) is divergent. The theorem is thus completely proved. □

5 Moment generating function

In this section we obtain a formula for the generating function of the moments $\mu_{2k}(t)$, $k \geq 1$, in an explicit form. Taking into account the well-know connection between the moments and the characteristic function of a stochastic process, this can be done by applying the known formula for the characteristic function of the Goldstein-Kac telegraph process (see, for instance,[9, Proposition 2.1] or [28, Theorem 2.3]). Instead, we give an alternative way of deriving the moment generating function based on direct computations and use of some properties of the modified Bessel functions.

For arbitrary complex number z such that

$$|z| < \frac{\lambda^2}{c^2},$$

introduce the function

$$\psi(z,t) = \sum_{k=0}^{\infty} z^k \; \frac{\mu_{2k}(t)}{(2k)!}. \tag{24}$$

The explicit form of function (24) is given by the following theorem.

Theorem 4. *For any $t > 0$ the moment generating function (24) has the form:*

$$\psi(z,t) = e^{-\lambda t} \; \left\{ \cosh\left(t\sqrt{\lambda^2 + c^2 z} \right) + \frac{\lambda}{\sqrt{\lambda^2 + c^2 z}} \; \sinh\left(t\sqrt{\lambda^2 + c^2 z} \right) \right\}. \tag{25}$$

Proof. First, we note that, in view of formula (9),

$$\begin{aligned}
\frac{\mu_{2k}(t)}{(2k)!} &= e^{-\lambda t}\, c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, t^{k+1/2}\, \frac{\Gamma\left(k+\frac{1}{2}\right)}{(2k)!} \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t} c^{2k} 2^{k-1/2} \lambda^{-k+1/2} t^{k+1/2} \frac{\sqrt{\pi}}{2^k} \frac{(2k-1)!!}{(2k)!!(2k-1)!!} \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= \sqrt{\frac{\pi\lambda t}{2}}\, e^{-\lambda t} \left(\frac{c^2 t}{\lambda}\right)^k \frac{1}{2^k\, k!} \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= \sqrt{\frac{\pi\lambda t}{2}}\, e^{-\lambda t} \left(\frac{c^2 t}{2\lambda}\right)^k \frac{1}{k!} \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right].
\end{aligned} \tag{26}$$

Substituting this into (24) we have:

$$\begin{aligned}
\psi(z,t) &= e^{-\lambda t} \sqrt{\frac{\pi\lambda t}{2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2 t z}{2\lambda}\right)^k \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t} \sqrt{\frac{\pi\lambda t}{2}} \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2 t z}{2\lambda}\right)^k I_{k+1/2}(\lambda t) + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2 t z}{2\lambda}\right)^k I_{k-1/2}(\lambda t) \right\}.
\end{aligned} \tag{27}$$

Consider separately the series on the right-hand side of (27). Applying the formula (see [30, page 694, Formula 6])

$$\sum_{k=0}^{\infty} \frac{\xi^k}{k!}\, I_{k-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cosh\left(\sqrt{x^2 + 2\xi x}\right), \qquad |2\xi| < |x|,$$

we obtain for the second series in curl brackets of (27):

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2 t z}{2\lambda}\right)^k I_{k-1/2}(\lambda t) &= \sqrt{\frac{2}{\pi\lambda t}}\, \cosh\left(\sqrt{\lambda^2 t^2 + 2\, \frac{c^2 t z}{2\lambda}\, \lambda t}\right) = \\
&= \sqrt{\frac{2}{\pi\lambda t}}\, \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right).
\end{aligned} \tag{28}$$

Similarly, by applying the formula (see [30, page 694, Formula 4 for $\nu = 1/2$])

$$\sum_{k=0}^{\infty} \frac{\xi^k}{k!}\, I_{k+1/2}(x) = \left(\frac{2\xi}{x} + 1\right)^{-1/4} I_{1/2}\left(\sqrt{x^2 + 2\xi x}\right), \qquad |2\xi| < |x|,$$

and taking into account that (see [30, page 730])

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sinh x,$$

we obtain for the first series in curl brackets of (27):

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2tz}{2\lambda}\right)^k I_{k+1/2}(\lambda t) = \left(1+\frac{c^2}{\lambda^2}\, z\right)^{-1/4} I_{1/2}\left(\sqrt{\lambda^2t^2+c^2t^2z}\right) =$$
$$= \frac{\sqrt{\lambda}}{(\lambda^2+c^2z)^{1/4}} \sqrt{\frac{2}{\pi t\sqrt{\lambda^2+c^2z}}} \,\sinh\left(t\sqrt{\lambda^2+c^2z}\right) =$$
$$= \sqrt{\frac{2\lambda}{\pi t}}\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} \,. \tag{29}$$

Substituting (28) and (29) into (27) we finally obtain:

$$\psi(z,t) = e^{-\lambda t} \sqrt{\frac{\pi\lambda t}{2}} \left\{ \sqrt{\frac{2}{\pi\lambda t}} \cosh\left(t\sqrt{\lambda^2+c^2z}\right) + \sqrt{\frac{2\lambda}{\pi t}}\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} \right\} =$$
$$= e^{-\lambda t} \left\{ \cosh\left(t\sqrt{\lambda^2+c^2z}\right) + \frac{\lambda}{\sqrt{\lambda^2+c^2z}} \sinh\left(t\sqrt{\lambda^2+c^2z}\right) \right\},$$

proving (25). The theorem is proved. □

Remark 4. From (24) it follows that the $(2k)$-th moment $\mu_{2k}(t)$, $k \geq 1$, can be obtained by the k-time differentiation of the moment generating function $\psi(z,t)$ with respect to z and by setting then $z=0$ in the expression obtained, that is,

$$\mu_{2k}(t) = (2k)! \left.\frac{\partial^k \psi(z,t)}{\partial z^k}\right|_{z=0}, \qquad k \geq 1.$$

Therefore, according to (25), we have for $k \geq 1$:

$$\mu_{2k}(t) = e^{-\lambda t}\, (2k)! \,\frac{\partial^k}{\partial z^k} \left.\left\{ \cosh\left(t\sqrt{\lambda^2+c^2z}\right) + \lambda\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} \right\}\right|_{z=0} . \tag{30}$$

In particular, for $k=1$, formula (30) yields:

$$\mu_2(t) = 2e^{-\lambda t} \,\frac{\partial}{\partial z} \left.\left\{ \cosh\left(t\sqrt{\lambda^2+c^2z}\right) + \lambda\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} \right\}\right|_{z=0} =$$
$$= 2e^{-\lambda t} \left\{ \frac{c^2t}{2}\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} + \frac{\lambda}{\lambda^2+c^2z} \times \right.$$
$$\left.\left. \times \left[\frac{c^2t}{2} \cosh\left(t\sqrt{\lambda^2+c^2z}\right) - \frac{c^2}{2}\, \frac{\sinh\left(t\sqrt{\lambda^2+c^2z}\right)}{\sqrt{\lambda^2+c^2z}} \right] \right\}\right|_{z=0} =$$

$$= 2e^{-\lambda t} \left\{ \frac{c^2 t}{2\lambda} \sinh(\lambda t) + \frac{1}{\lambda} \left[\frac{c^2 t}{2} \cosh(\lambda t) - \frac{c^2}{2\lambda} \sinh(\lambda t) \right] \right\} =$$
$$= 2e^{-\lambda t} \left\{ \frac{c^2 t}{2\lambda} [\sinh(\lambda t) + \cosh(\lambda t)] - \frac{c^2}{2\lambda^2} \sinh(\lambda t) \right\} =$$
$$= 2e^{-\lambda t} \left\{ \frac{c^2 t}{2\lambda} e^{\lambda t} - \frac{c^2}{2\lambda^2} \sinh(\lambda t) \right\} =$$
$$= \frac{c^2 t}{\lambda} - \frac{c^2}{\lambda^2} e^{-\lambda t} \frac{e^{\lambda t} - e^{-\lambda t}}{2} =$$
$$= \frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2} \left(1 - e^{-2\lambda t} \right)$$

and this exactly coincides with (6).

Note that the moment generating function is structurally similar to the characteristic function of the telegraph process $X(t)$ (see, for comparison, [9, Proposition 2.1] or [28, Theorem 2.3]).

Remark 5. We can use some formulas obtained above for deriving an expression for the semi-invariants of the Goldstein-Kac telegraph process $X(t)$. According to the general formula of probability theory, for any fixed $t > 0$, the semi-invariants $\eta_n(t),\ n \geq 1$, of $X(t)$ are expressed in terms of the moments $\mu_n(t),\ n \geq 1$, as follows:

$$\eta_n(t) = n! \sum_{r=0}^{n} \sum_{j,\, l} \frac{(-1)^{j-1}\,(j-1)!}{j_1! \dots j_r!} \left(\frac{\mu_{l_1(t)}}{l_1!} \right)^{j_1} \dots \left(\frac{\mu_{l_r(t)}}{l_r!} \right)^{j_r}, \qquad n \geq 1, \tag{31}$$

where the interior summation is doing with respect to all the non-negative integer numbers j and l such that

$$l_1 j_1 + \dots + l_r j_r = n, \qquad j_1 + \dots + j_r = j.$$

Since, according to (5), all the odd moments are equal to zero, then all the odd semi-invariants are equal to zero too, that is, $\eta_{2k+1}(t) = 0,\ k = 0, 1, 2, \dots$. Therefore, formula (31) takes the form:

$$\eta_{2k}(t) = (2k)! \sum_{r=0}^{2k} \sum_{j,\, l} \frac{(-1)^{j-1}\,(j-1)!}{j_1! \dots j_r!} \left(\frac{\mu_{2l_1(t)}}{(2l_1)!} \right)^{j_1} \dots \left(\frac{\mu_{2l_r(t)}}{(2l_r)!} \right)^{j_r}, \qquad k \geq 1, \tag{32}$$

where

$$l_1 j_1 + \dots + l_r j_r = k, \qquad j_1 + \dots + j_r = j. \tag{33}$$

Each factor of the form $\mu_{2s}(t)/(2s)!$ in (32), according to (26), has the form:

$$\frac{\mu_{2s}(t)}{(2s)!} = \sqrt{\frac{\pi \lambda t}{2}}\, e^{-\lambda t} \left(\frac{c^2 t}{2\lambda} \right)^s \frac{1}{s!} \left[I_{s+1/2}(\lambda t) + I_{s-1/2}(\lambda t) \right].$$

Therefore, the product of such factors in (32), in view of (33), are given by

$$\left(\frac{\mu_{2l_1(t)}}{(2l_1)!}\right)^{j_1} \dots \left(\frac{\mu_{2l_r(t)}}{(2l_r)!}\right)^{j_r} =$$

$$= \prod_{i=1}^{r} \left(\sqrt{\frac{\pi\lambda t}{2}} \, e^{-\lambda t} \, \left(\frac{c^2 t}{2\lambda}\right)^{l_i} \, \frac{1}{l_i!} \, \left[I_{l_i+1/2}(\lambda t) + I_{l_i-1/2}(\lambda t) \right] \right)^{j_i} =$$

$$= \left(\frac{\pi\lambda t}{2}\right)^{j/2} \, e^{-\lambda t j} \, \left(\frac{c^2 t}{2\lambda}\right)^{k} \prod_{i=1}^{r} \left(\frac{1}{l_i!} \, \left[I_{l_i+1/2}(\lambda t) + I_{l_i-1/2}(\lambda t) \right] \right)^{j_i}.$$

By substituting this into (32) we obtain the following formula for the semi-invariants:

$$\eta_{2k}(t) = (2k)! \, \left(\frac{c^2 t}{2\lambda}\right)^{k} \sum_{r=0}^{2k} \sum_{j,\, l} \frac{(-1)^{j-1} \, (j-1)!}{j_1! \dots j_r!} \, \left(\frac{\pi\lambda t}{2}\right)^{j/2} \, e^{-\lambda t j} \times$$
$$\times \prod_{i=1}^{r} \left(\frac{1}{l_i!} \, \left[I_{l_i+1/2}(\lambda t) + I_{l_i-1/2}(\lambda t) \right] \right)^{j_i}. \tag{34}$$

Formula (34) has a fairly complicated form and, apparently, cannot be simplified. Nevertheless, it can be used for computing the semi-invariants for small k.

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