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On asymptotic representation of singular solutions of the model elliptic equation near boundary and formulation of singular boundary conditions

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Abstract. In the work the asymptotic representation of singular solution of the elliptic model Sobolev problem near components of arbitrary dimensions of boundary is specified. Using this asymptotical representation of solutions, the singular boundary conditions are formulated. The solvability of boundary problem with singular boundary conditions is proved.

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1 Introduction

This work is the continuation of [1], devoted to integral and asymptotic representation of singular solutions of elliptic equations near components of small dimensions of boundary. The problem of representation of solutions near boundary is interesting not only in itself, but also in connection with reduction of the boundary value problem to integral, integro-differential or differential equations on the boundary. In [2, 3] S. L. Sobolev for the first time formulated and studied the boundary value problem for polyharmonic equation in a domain with boundary, consisting of a submanifold of diverse dimensions (and afterwards this problem was named the Sobolev boundary problem).

Later the work [4] was published, where the Sobolev boundary value problem is studied for a general elliptic equation of order $2m$. In this work it is proved that the number of boundary conditions on the submanifold of boundary depends on the order of regularity of solutions $u(x)$ from Sobolev space $H^s(\Omega)$ near submanifold.

Moreover, it was proved that the solution of the elliptic equation admits asymptotic representation with respect to the power $p^{-\nu}$ and $\ln r$ (where $r = dist(x, \mathbb{R}^q)$), any explicit formulae to compute the coefficients have been done.

Using the integral representation of solution of the boundary value problem with Green function, in [1] the asymptotic representation of the components of the singular solutions generated by distributions with support on the submanifold $\mathbb{R}^q$ was obtained.

2 On elliptic model problem. Asymptotic representation of component of singular solution near boundary

Let $\mathbb{R}^n$ be a Euclidean n-dimensional space, $\mathbb{R}^q \subset \mathbb{R}^n$ a subspace of $\mathbb{R}^n$, $x = (x', x'') = (x_1, ..., x_q, x_{q+1}, ..., x_n)$ a point of $\mathbb{R}^n$, $D_x = (D_{x'}, D_{x''})$, $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$. By $C^\alpha(\Omega)$, $C_0^\alpha(\Omega), C^\alpha(\mathbb{R}^q)$, $C_0^\alpha(\mathbb{R}^q)$ we denote the usual Hölder spaces, spaces of functions with finite support in Ω and $\mathbb{R}^q$, respectively, $H^s(\Omega)$, $H^s(\mathbb{R}^q)$, $s \in \mathbb{R}^1$, are Hilbertian Sobolev spaces in Ω and $\mathbb{R}^q$, respectively [5,6].

Let $\mathcal{L}(D_x)$ be a homogeneous elliptic operator of order $2m$ with constant coefficients. In domain Ω we consider the elliptic equation

$$\mathcal{L}(D_x)\, u(x) = f(x), \tag{1}$$

where

$$u(x) \in H^s(\Omega), \; f(x) \in H^{s-2m}(\Omega), \; s \in \mathbb{R}^1. \tag{2}$$

First of all we consider the problem of asymptotic behavior of singular solutions $u(x) \in H^s(\Omega)$ near submanifold $\mathbb{R}^q$, and obtain the formulae of asymptotic representation of solutions, generated by distributions with support on the $\mathbb{R}^q$. For this we observe that it is known [5,6] that the non-zero element $f(x) \in H^{s-2m}(\mathbb{R}^n)$ is concentrated in $\mathbb{R}^q$ if and only if $s < 2m - \theta/2$ ($\theta = co\dim \mathbb{R}^q = n - q$) and there exist elements $f_\sigma(x') \in H^{s-2m+|\sigma|+\theta/2}(\mathbb{R}^q)$, $|\sigma| \le \tau = [2m - s - \theta/2]$ such that

$$f(x) = \sum_{|\sigma| \le \tau} D_\nu^\sigma \left(f_\sigma(x') \times \delta(x'') \right), \; \nu = x'', \tag{3}$$

where $[\alpha]$ is the integer part of number α, $D_\nu^\sigma = D_{x''}^\sigma = \dfrac{\partial^{\sigma_{q+1}}}{\partial x_{q+1}^{\sigma_{q+1}}} \cdots \dfrac{\partial^{\sigma_n}}{\partial x_n^{\sigma_n}}$, and $f_\sigma(x') \times \delta(x'')$ is the direct product of distributions, $|\sigma| = \sum_{i=q+1}^n \sigma_i$.

In [1], using the Green function of boundary value problem, the integral representation of solution of equation (1) near $\mathbb{R}^q$ is obtained, from which the asymptotic representation of singular part of solution $u(x)$ in Ω is obtained.

Really, let $G(x,y) = E(x-y) + g(x,y)$ be the Green function of homogeneous Dirichlet problem in the ball B_R of radius R (sufficiently large), where $E(x)$ is a fundamental solution of equation (1) in $\mathbb{R}^n$, and $g(x,y)$ is the solution of equation (1) in Ω, satisfying the condition $g(x,y)|_{B_R} = E(x-y)|_{\partial B_R}$.

Write the formulae of integral representation of solution of Dirichlet problem

$$u(x) = \int_{\mathbb{R}^n} G(x,y) f(y) dy$$

for $f(x) \in C_0^\infty$. After that approximate $f(x)$ with functions $f_\varepsilon(x) \in C_0^\infty(\mathbb{R}^n)$, $f_\varepsilon(x) \underset{\varepsilon \to 0}{\to} f(x)$ in $H^{s-2m}(\mathbb{R}^n)$, then integrating by parts with respect to variable x'', and passing to the limit as $\varepsilon \to 0$, we obtain the integral representation of solution $u(x)$:

$$u(x) = \sum_{|\sigma|\le\tau} \int_{\mathbb{R}^q} \bar{D}^{\sigma}_{x''} E\left(x'-y', x''\right) f_\sigma\left(y'\right) dy' + \tilde{u}\left(x\right) =$$
$$= \sum_{|\sigma|\le\tau} \int_{\mathbb{R}^q} \bar{D}^{\sigma}_{x''} E\left(z', x''\right) f_\sigma\left(x'-z'\right) dz' + \tilde{u}\left(x\right) \equiv \sum_{|\sigma|\le\tau} v_\sigma\left(x\right) + \tilde{u}\left(x\right), \quad (4)$$

where $\tilde{u}(x)$ is a regular, bounded function, $\bar{D}_{x''} = -D_{x''}$.

It is known [7, 8] that

$$\left|D^{\sigma}_{x''} E\left(x'-y', x''\right)\right| \le c\left|x-y\right|^{2m-n-|\sigma|}\left|\ln\left|x-y\right|\right|,$$

where $\ln|x-y|$ is dropped for $2m-n-|\sigma|<0$. Moreover, if $2m-n-|\sigma|<0$, then $E^{(\sigma)}\left(z', x''\right)$ are homogeneous functions of degrees $2m-n-|\sigma|$ and if $n-2m+|\sigma|\ge q$, i.e. $n-q-2m+|\sigma| = \theta-2m+|\sigma| \overset{def}{\equiv} \alpha_\sigma \ge 0$, then the integrals $v_\sigma(x)$ are singular or hypersingular integrals with homogeneous kernels [7,8]. Now consider the singular and hypersingular integrals $v_\sigma(x)$. In [1], using the known procedure of regularization of divergent integrals (separation of the finite part in the Hadamard sense), by separating the singular and regular parts, the asymptotic representations of the divergent integrals $v_\sigma\left(x\right)$ near $\mathbb{R}^q$ are obtained. For convenience, here we shortly expose this known procedure [1].

Let $n\ge 3$, $r = \left|x''\right|$, $\rho = \left|x'\right|$. Denote by

$$\mathcal{P}_\alpha\left(x', z'\right) f\left(x'\right) = \sum_{\lambda=0}^{\alpha} \sum_{|k'|=\lambda} \frac{f^{(k')}\left(x'\right)}{k'!}\left(-z'\right)^{k'} \equiv \sum_{\lambda=0}^{\alpha} P_\lambda\left(x', z'\right) f\left(x'\right)$$

the segment of the Taylor expansion of the function $f\left(x'-z'\right)$ near the point $z'=0$, where $k' = \left(k_1, \ldots, k_q\right)$,

$$v_{\sigma 0}\left(x\right) = \int_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right)\left(f_\sigma\left(x'-z'\right) - \mathcal{P}_{\alpha_\sigma-1}\left(x', z'\right) f_\sigma - \theta\left(z'\right) P_{\alpha_\sigma}\left(x', z'\right) f_\sigma\right) dz' \quad (5)$$

is the regularization (finite part) of the divergent integral $v_\sigma(x)$ at the point $z'=0$, $\theta\left(z'\right)=1$ for $\left|z'\right|\le 1$ and $\theta\left(z'\right)=0$ for $\left|z'\right|>1$. In [1] it is proved that the integrals $v_\sigma(x)$ could be presented in the form

$$v_\sigma\left(x\right) = v_{\sigma 0}\left(x\right) - \int_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right) P_{\alpha_\sigma-1}\left(z', D'_x\right) f_\sigma\left(x'\right) dz' -$$
$$- \int_{|z'|<1} P_{\alpha_\sigma}(z', D'_x)\, f_\sigma dz' \equiv v_{\sigma 0}\left(x\right) + \sum_{\lambda=0}^{\alpha_\sigma-1} \int_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right) P_\lambda(z', D'_x)\, f_\sigma\left(z'\right) dz' + \quad (6)$$
$$+ \int_{|z'|<1} \bar{E}^{(\sigma)}\left(z', x''\right) P_{\alpha_\sigma}\left(z', D'_x\right) f_\sigma dz' \equiv v_{\sigma 0}\left(x\right) + \sum_{\lambda=0}^{\alpha_\sigma-1} \mathcal{I}_\lambda\left[f_\sigma\right] + \mathcal{I}_{\alpha_\sigma}\left[f_\sigma\right],$$

where

$$\mathcal{I}_\lambda[f_\sigma]=(-1)^\lambda\sum_{|k'|=\lambda}A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}r^{-\alpha_\sigma+\lambda}\equiv Q_{\sigma\lambda}\left(\omega'',D'_x\right)f_\sigma\left(x'\right)r^{-\alpha_\sigma+\lambda},\quad(7)$$

$$A_{\sigma k'}\left(\omega''\right)=\int\limits_{\mathbb{R}^q}E^{(\sigma)}\left(\xi',\omega''\right)\xi'^{k'}d\xi',\ \ \omega''=x''/\left|x''\right|,\quad(8)$$

and

$$\mathcal{I}_{\alpha_\sigma}[f_\sigma]=-A_\sigma\left(D'_x\right)f_\sigma\left(x'\right)\ln r+B_\sigma\left(D'_x\right)f_\sigma\left(x'\right)+o\left(r\right),$$

with $o(r)\to 0$ as $r\to 0$,

$$A_\sigma\left(D'_x\right)f_\sigma\left(x'\right)=(-1)^{\alpha_\sigma}\sum_{|k'|=\alpha_\sigma}a_{\sigma k'}\frac{f_\sigma^{(k')}\left(x'\right)}{k'!},\ \ a_{\sigma k'}=\int\limits_{|\omega'|=1}E^{(\sigma)}\left(\omega',0\right)\left(\omega'\right)^{k'}d\omega',\quad(9)$$

$$B_\sigma\left(D'_x\right)f_\sigma\left(x'\right)=(-1)^{\alpha_\sigma}\sum_{|k'|=\alpha_\sigma}b_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!},\quad(10)$$

and $b_{\sigma k'}(\omega'')$ is the integral

$$b_{\sigma k'}(\omega'')=\int\limits_{|\omega'|=1}\omega'^{k'}d\omega'\left(\int\limits_0^1E^{(\sigma)}(\rho\omega',\omega'')\rho^{|k'|+q-1}d\rho+\int\limits_1^\infty\left(E^{(\sigma)}(\rho\omega',\omega'')-E^{(\sigma)}(\omega',0)\right)\frac{1}{\rho}d\rho\right).$$

Hence, for divergent integrals $v_\sigma(x')$ (singular and hypersingular) we obtain the representations

$$\begin{gathered}v_\sigma(x)=v_{\sigma_0}(x)+\sum_{\lambda=0}^{\alpha_\sigma-1}Q_{\sigma\lambda}\left(D'_x\right)f_\sigma(x')r^{-\alpha_\sigma+\lambda}-\\-A_\sigma\left(D'_x\right)f_\sigma(x')\ln r+B_\sigma\left(D'_x\right)f_\sigma(x')+o\left(r\right),\end{gathered}\quad(11)$$

where the functions $v_{\sigma_0}(x)$ and operators $Q_{\sigma\lambda}(D'_x)$, $A_\sigma(D'_x)$, $B_\sigma(D'_x)$ are defined by (5), (7), (9) and (10), $o(r)$ tends to zero as r tends to zero.

3 Asymptotic representation of singular part of integer solution near $\mathbb{R}^q$

Here, using the asymptotical representation of components $v_\sigma(x)$ of singular solution $u(x)$ near $\mathbb{R}^q$, generated by distribution $f(x)$, concentrated on the manifold $\mathbb{R}^q$, the asymptotic representation of integer solution $v(x)$ near $\mathbb{R}^q$ is obtained.

Really, summing the equality (11) ovwr σ for all σ such that $\alpha_\sigma \geq 0$, the asymptotic representation for the singular part $v(x)$ of the solution $u(x)$ near $\mathbb{R}^q$ is obtained:

$$v(x)=\sum_{\sigma:\,\alpha_\sigma\geq 0} v_{\sigma 0}(x)+\sum_{\sigma:\,\alpha_\sigma>0}\sum_{\lambda=0}^{\alpha_\sigma-1} Q_{\sigma\lambda}\left(D'_x\right) f_\sigma\left(x'\right) r^{-\alpha_\sigma+\lambda}+\sum_{\sigma:\,\alpha_\sigma\geq 0} A_\sigma\left(D'_x\right)\ln r+ \\ +\sum_{\sigma:\,\alpha_\sigma\geq 0} B_\sigma\left(D'_x\right) f_\sigma\left(x'\right)+o\left(r\right)\equiv v_0(x)+w(x)+w_0\left(x\right)+B\left(D'_x\right) f\left(x'\right)+o\left(r\right). \tag{12}$$

Here by $v_0\left(x\right)$, $w\left(x\right)$ and $w_0\left(x\right)$ we denoted the first three sums of right hand side of equality (12), $o\left(r\right)\to 0$ when $r\to 0$. The equality (12) is the asymptotic representation of singular part of solution $u(x)$ near submanifold $\mathbb{R}^q$ with respect to the power $r^{-\nu}$ and $\ln r$. But in order to obtain an asymptotic ordered representation with respect to the ascending order of power $r^{-\nu}$ and $\ln r$ it is necessary to transform the equality (12). For this, at first, we consider the function $w(x)$ and transform it into an ordered sum with respect to the ascending order of power $r^{-\nu}$. Since $-\alpha_\mu+\lambda=-\left(\mu-\lambda+\theta-2m\right)$, the expression $\mu-\lambda+\theta-2m$ is constant on the any straight line $\mu-\lambda+\theta-2m=\nu$. Therefore, it is natural to denote $\mu-\lambda+\theta-2m=\nu$, and to obtain an ordered sum with respect to the ascending order of power $r^{-\nu}$ it remains to change the order of summing over λ, μ and ν. From the inequality $\nu=\mu-\lambda+\theta-2m\geq 1$ it follows that $\mu\geq 2m+\lambda-\theta+1\geq 2m-\theta+1$ and, since $\mu=|\sigma|\geq 0$, we have $\mu\geq\mu_1=\max\left(0,2m-\theta+1\right)$. Therefore, $\mu_1\leq|\sigma|=\mu\leq\tau$ and $\nu_1\leq\nu\leq\nu_2$, where $\nu_1=\mu_1+\theta-2m,\ \nu_2=\tau+\theta-2m$.

For $w(x)$ we obtain the representation

$$w(x)=\sum_{\mu=\mu_1}^{\tau}\sum_{\lambda=0}^{\alpha_\mu-1}\left(\sum_{|\sigma|=\mu}\sum_{|k'|=\lambda} A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}\right) r^{-\alpha_\mu+\lambda}\equiv \\ \equiv\sum_{\mu=\mu_1}^{\tau}\sum_{\lambda=0}^{\alpha_\mu-1} P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right) r^{\lambda-\alpha_\mu}= \\ =\sum_{\nu=\nu_1}^{\tau+\theta-2m}\sum_{\substack{\mu,\lambda:\\ \mu-\lambda=\nu+2m-\theta}} P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right) r^{-\nu}=\sum_{\nu=\nu_1}^{\tau+\theta-2m} M_\nu\left(\omega'',D'_x\right) f\left(x'\right) r^{-\nu}, \tag{13}$$

where by $P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right)$ we denoted the double sum over σ and k' from right hand side of equality (13),

$$P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right)=\sum_{|\sigma|=\mu}\sum_{|k'|=\lambda} A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}, \tag{14}$$

where $A_{\sigma k'}\left(\omega''\right)=\int_{\mathbb{R}^q}\bar{E}^{(\sigma)}\left(\xi',\omega''\right)\xi'^{k'}d\xi'$. It remains to transform the expression

$w_0(x)$. Detailing the structure of sum, which defines the function $w_0(x)$, after ordered summation over σ, we obtain

$$w_0(x) = M_0\left(D'_x\right) f\left(x'\right) \ln r = \sum_{\mu,\lambda:\, \mu-\lambda=\nu+2m-\theta} P_{\mu\lambda}\left(D'_x\right) f\left(x'\right) \ln r, \tag{15}$$

where $P_{\mu\lambda}\left(D'_x\right) f\left(x'\right) = \sum\limits_{|\sigma|=\mu} \sum\limits_{|k'|=\lambda} a_{\sigma k'} \frac{f_\sigma^{(k')}(x')}{k'!}$, $a_{\sigma k'} = \int\limits_{|\omega'|=1} E^{(\sigma)}\left(\omega', 0\right) \omega'^{k'} d\omega'$.

Thus, substituting in (13) these functions $w(x)$ and $w_0(x)$ with their transformed expressions, we obtain the following

Theorem 1. *Let functions $f_\sigma\left(x'\right) \in C_0^{\alpha_\sigma+1}\left(\mathbb{R}^q\right)$. Then the singular part $v(x)$ of solution $u(x)$ near $\mathbb{R}^q$ is represented by*

$$\begin{gathered} v(x) = v_0(x) + \sum_{\nu=1}^{\tau+\theta-2m} M_\nu\left(\omega'', D'_x, f\right) r^{-\nu} + \\ + M_0\left(D'_x\right) f\left(x'\right) \ln r + \sum_{\sigma:\, \alpha_\sigma \geq 0} B_\sigma\left(\omega'', D'_x\right) f\left(x'\right) + o\left(r\right), \end{gathered} \tag{16}$$

where $M_\nu\left(\omega'', D'_x, f\right)$, $M_0\left(D'_x\right) f\left(x'\right)$ is defined by formulae (13), (15), respectively, $o\left(r\right) \to 0$ when $r \to 0$.

4 Formulation of the boundary value problem with singular boundary conditions

In the general theory of elliptic boundary value problems in domain Ω with smooth boundary $\partial\Omega$, the boundary problem is reduced to a system of pseudodifferential equations on the boundary $\partial\Omega$. This system is a system of regular integral (Fredholm) equations in the case of smooth solutions up to $\partial\Omega$ or a system of differential equations in the case of singular solutions.

Here, using the obtained formulae (16) of asymptotic representation of singular parts of solutions $u(x)$ near boundary $\mathbb{R}^q$, we formulate, firstly, the formal model boundary value problem with singular boundary conditions on the $\mathbb{R}^q$:

In the domain $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$ find the solutions of elliptic equation $L\left(D_x\right) u(x) = 0$ that have near $\mathbb{R}^q$ the given singular asymptotic representation:

$$z(x) = \sum_{\nu=1}^{\tau+\theta-2m} \Phi_\nu\left(\omega'', x'\right) r^{-\nu} + \Phi_0\left(\omega'', x'\right) \ln r + \tilde{z}\left(x\right), \tag{17}$$

where $\tilde{z}\left(x\right)$ is a regular bounded function.

Formally, equating the coefficients by the same power $r^{-\nu}$ and $\ln r$ from equalities (16) and (17), we obtain

$$M_\nu\left(\omega'', D_x'\right) f\left(x'\right) = \sum_{\mu,\lambda:\, \mu-\lambda+\theta-2m=\nu} P_{\mu\lambda}\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_\nu\left(\omega'', x'\right), \quad \nu = \tau+\theta-2m, \dots, 1, \tag{18}$$

$$M_0\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_0\left(\omega'', x'\right). \tag{19}$$

The system of equations (18), (19) is a system of linear partial differential equations with unknown density $f_\sigma\left(x'\right), \ \sigma: \ \mu_0 \le |\sigma| \le \tau$ and the solvability of boundary problem with singular boundary conditions is reduced to the solvability of system of differential equations (18), (19). This system is rather complicated, since the number of unknown densities $f_\sigma\left(x'\right)$, as well as the number of equations, depends on s and on the difference $\theta - 2m$, too.

Now we pass to the study of the structure of equations of system (18)-(19) depending on s, τ and θ. Denote by Π_m the linear space of all homogeneous polynomials of degree m. It is known [9] that the dimension of space Π_m $(\dim \Pi_m)$ is equal to $C_{m+\theta-1}^{\theta-1}$, where C_n^k are the binomial coefficients. Hence, the number of unknown functions $f_\sigma\left(x'\right)$ in the system (18)-(19) is equal to $\Pi = \sum_{m=\mu_0}^{\tau} \dim \Pi_m = \sum_{m=\mu_0}^{\tau} C_{m+\theta-1}^{\theta-1}$ which is greater (for $\theta > 1$) than the number of equations from system (18)-(19). Return to the system of equations (18)-(19). Since $f_\sigma\left(x'\right) \in H^{s-2m+|\sigma|+\theta/2}\left(\mathbb{R}^q\right)$ and $f_\sigma^{(k')}\left(x'\right) \in H^{s-2m+|\sigma|+\theta/2-|k'|}\left(\mathbb{R}^q\right)$, then for any multiindex σ and k' with $|\sigma| - |k'| = \mu - \lambda = \nu - \theta + 2m$ the left hand sides of equations (18), (19) belong to spaces $H^{s+\nu-\theta/2}\left(\mathbb{R}^q\right), \ \nu = \tau+\theta-2m, \dots, 1, 0$. Therefore, the equalities (18), (19) define a bounded operator U from the space $E_1 = \prod_{|\sigma|} H^{s-2m+|\sigma|+\theta/2}\left(\mathbb{R}^q\right), \ |\sigma| \le \tau$, to the space $E_2 = \prod_{\nu} H^{s+\nu-\theta/2}\left(\mathbb{R}^q\right), \nu = 0, 1, \dots, \tau+\theta-2m$.

Now we begin to investigate the system of equations (18), (19). At first, we will see that the number of equations of system (18)-(19), as well as the condition of solvability of this system, depends on the numbers $\theta - 2m$ and τ. Therefore, we consider two cases: a) $\theta - 2m \le 0$ and b) $\theta - 2m > 0$.

a) Assume that $\theta - 2m \le 0$. In this case the number of equations in the system (18)-(19) is $\tau + \theta - 2m$, which is no more than τ. The system of equations (18)-(19) takes the form

$$\begin{gathered}
P_{\tau 0}\left(\omega'', D_x'\right) f\left(x'\right) = \sum_{|\sigma|=\tau} A_{\sigma 0}\left(\omega''\right) f_\sigma\left(x'\right) = \Phi_{\tau+\theta-2m}\left(\omega'', x'\right), \\
P_{\tau-10}\left(\omega'', D_x'\right) f\left(x'\right) + P_{\tau 1}\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_{\tau+\theta-2m+1}\left(\omega'', x'\right), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
P_{2m+1-\theta 0}\left(\omega'', D_x'\right) f\left(x'\right) + \ldots + P_{\tau\tau-2m-1+\theta}\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_{2m-\theta}\left(\omega'', x'\right), \\
M_0\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_0\left(\omega'', x'\right).
\end{gathered} \tag{20}$$

Now we see that in each of these equations the expressions $P_{\mu 0}(\omega'', f)$ are linear combinations of unknown functions $f_\sigma(x')$ with the coefficients $A_{\sigma 0}$ (the moments of fundamental solution $E(x)$). The system of equations (18)–(19) is of triangular form. Since $\sum_{|\sigma|=\tau} |A_{\sigma 0}(\omega'')| \neq 0$ (otherwise the first condition in (18) is absent), the first equation from (18) is solvable. Assume that functions $f_\sigma(x')$ with $|\sigma| = \tau$ are solutions to the first equation of (18). Substituting this functions $f_\sigma(x')$ with $|\sigma| = \tau$ in the other equations, for functions $f_\sigma(x')$ with $|\sigma| \leq \tau - 1$ we obtain also a triangular system. Continuing this procedure, we express all the functions $f_\sigma(x')$ with $\mu_0 \leq |\sigma| \leq \tau$ only through the functions Φ_τ, $\Phi_{\tau-1}, \ldots, \Phi_{\tau+\theta-2m}$. It means that the system of equations (18)–(19) is solvable.

b) Assume that $\theta > 2m$. In this case the system of equations (18)–(19) contains $\tau + \theta - 2m$ equations, their number is greater than τ. Repeating the above mentioned procedure, we express all the functions $f_\sigma(x')$ with $0 \leq |\sigma| \leq \tau$ by $\Phi_\tau(x'), \Phi_{\tau-1}(x'), \ldots, \Phi_{\tau+\theta-2m}(x')$. Substituting all functions $f_\sigma(x')$ in other equations, we obtain that the first $\theta - 2m$ equations of (18)-(19) become identities, and the functions $\Phi_0(x'), \ldots, \Phi_{\tau+\theta-2m}(x')$ are connected by (18), (19).

From what was mentioned above it follows that the formal model boundary value problem with singular boundary conditions is not solvable for any admissible right hand sides $\Phi_\nu(\omega', x')$. To obtain a solvable singular boundary value problem it is necessary to reformulate this problem in the following way:

In the domain $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$ find the solutions $u(x)$ of the model elliptic equation

$$L(D_x)\, u(x) = 0 \tag{21}$$

that have near $\mathbb{R}^q$ the asymptotic representation (16) with coefficients $M_\nu(\omega'', x')$, satisfying the conditions

$$M_\nu\left(\omega'', D_x'\right) f\left(x'\right) = \Phi_\nu\left(\omega'', x'\right), \;\; \nu = \nu_1, \ldots, \nu_2 = \tau + \theta - 2m. \tag{22}$$

Repeating the similar reasons we obtain

Theorem 2. *For any admissible functions $\Phi_\nu(\omega'', x')$ the model boundary value problem with singular boundary conditions (18), (19) is solvable.*

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