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Matrix algorithm for Polling models with PH distribution

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Abstract. Polling systems provide performance evaluation criteria for a variety of demand-based, multiple-access schemes in computer and communication systems [1]. For studying this systems it is necessary to find their important characteristics. One of the important characteristics of these systems is the k-busy period [2]. In [3] it is showed that analytical results for k-busy period can be viewed as the generalization of classical Kendall functional equation [4]. A matrix algorithm for solving the generalization of classical Kendall functional equation is proposed. Some examples and numerical results are presented.

Mathematics subject classification: 34C05, 58F14.
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1 Introduction

In this paper we study one of the important characteristics for queueing system of Polling type, the k-busy period. A Polling model is a system of multiple queues accessed by a single server in cyclic order. We consider a queueing system of Polling type with semi-Markov switching. Handling mechanism for this system is given by Polling table $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, r\}$, where the function f shows that at the stage $j, j = \overline{1, n}$, the user number $k, k = \overline{1, r}$, is served. The items (messages) of the user k, arrive according to Poisson distribution with parameter $\tilde{\lambda}_k$. The service time for the items of class k is a random variable B_k with the distribution function $B_k(x) = P\{B_k < x\}$. Duration of the orientation from one user to another one is a random variable C_k with the distribution function $C_k(x) = P\{C_k < x\}$. In this paper, the matrix algorithm of determining the k-busy period for Polling systems is obtained, and some numerical examples are presented.

2 The k-busy period

Definition 2.1 *The k-busy period* is a measure of the time that expires from when a server begins to process, after an empty queue, to when the k-queue becomes empty again for the first time [3].

Denote by Π_k^δ the length of the k-busy period, and by

$$\Pi_k^\delta(x) = P\{\Pi_k^\delta < x\},$$

its distribution function. Consider that

$$\pi_k^\delta(s) = \int_0^\infty e^{-sx} d\Pi_k^\delta(x)$$

is the Laplace-Stieltjes transform of distribution function of k-busy period.
The following result is known [3]:
Theorem 2.1 *The function $\pi_k^\delta(s)$ is determined from the equation*

$$\pi_k^\delta(s) = c_k(s + \tilde{\lambda}_k - \tilde{\lambda}_k \pi_k(s))\pi_k(s), \tag{2.1}$$

where

$$\pi_k(s) = \beta_k(s + \tilde{\lambda}_k - \tilde{\lambda}_k \pi_k(s)), \tag{2.2}$$

and $c_k(s)$ and $\beta_k(s)$ denote the Laplace-Stieltjes transforms of distribution functions $C_k(x)$ and $B_k(x)$,

$$c_k(s) = \int_0^\infty e^{-sx} dC_k(x),$$

$$\beta_k(s) = \int_0^\infty e^{-sx} dB_k(x).$$

A matrix algorithm for solving the generalization of classical Kendall functional equation (2.1) is proposed. For this, the matrix algorithm for solving Kendall functional equation in Polling models [5] was used. It has no analytical solution, but it can be solved numerically with the accuracy required. Both distributions $B_k(x)$ and $C_k(x)$ were considered distributions of Phase Type (PH). All results were obtained in terms of the Laplace-Stieltjes transform.

3 Laplace-Stieltjes Transform of Phase Type distribution

Phase type distributions are getting to be very commonly used these days after Neuts [6] made them very popular and easily accessible. They are very often referred to as the PH distribution. The PH distribution has became very popular in stochastic modeling because it allows numerical tractability of some difficult problems and in addition, several distributions encountered in queueing seem to resemble the PH distribution.
Phase type distributions are distributions of the time until absorption in an absorbing CTMC (Continuous Time Markov Chain). Consider an $(n+1)$ absorbing CTMC

with the state space $\{0, 1, \ldots, n\}$ and let the state 0 be the absorbing state. The transition matrix Q of this absorbing Markov chain is given as

$$Q = \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

where the $n \times n$ matrix T satisfies $T_{ii} < 0$, for $1 \leq i \leq n$, and $T_{ij} \geq 0$, for $i \neq j$. $Te + T^0 = 0$, $\alpha^t e = 1$, and $(\alpha^t, 0)$ is the initial probability vector of Q. We suppose that all states $1, \ldots, n$ are transient.

The probability distribution $F(x)$ of the time until absorbtion in the state 0, corresponding to the initial probability vector $(\alpha^t, 0)$, is given by:

$$F(x) = 1 - \alpha^t e^{Tx} e, \text{ for } x \geq 0. \tag{3.2}$$

The phase type distribution with parameter α^t and T is usually written as PH distribution with representation (α^t, T). Let find the Laplace-Stieltjes transform of phase type distribution with representation (α^t, T):

$$\frac{dF(x)}{dx} = -\frac{d}{dx}\alpha^t e^{Tx} e = -\alpha^t [\frac{d}{dx} e^{Tx}] e,$$

where $e^{Tx} = \sum_{k=1}^{\infty} \frac{(Tx)^i}{i!}$.

$$\frac{de^{Tx}}{dx} = e^{Tx} \cdot \frac{d(Tx)}{dx} = e^{Tx} \cdot T,$$

$$\frac{dF(x)}{dx} = -\alpha^t e^{Tx} Te = \alpha^t e^{Tx}(-Te) = \alpha^t e^{Tx} T^0.$$

$$f(s) = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx} \alpha^t e^{Tx} T^0 dx = \alpha^t \int_0^\infty e^{-sx} I e^{Tx} dx T^0$$
$$= \alpha^t \int_0^\infty e^{-(sI-T)x} dx T^0 = \alpha^t (sI - T)^{-1} T^0.$$

The Laplace-Stieltjes transform $f(s)$ of the PH distribution with representation (α^t, T), is:

$$f(s) = \alpha^t (sI - T)^{-1} T^0. \tag{3.3}$$

4 Matrix form for Kendall equation

We know that

$$\pi_k(s) = \beta_k(s + \tilde{\lambda}_k(1 - \pi_k(s))). \tag{4.1}$$

Suppose that $B_k(x)$ is a PH distribution with representation (α_k^t, T), where

$$T_k = \begin{pmatrix} -\lambda_k & \lambda_k & \dots & 0 & 0 \\ 0 & -\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda_k & \lambda_k \\ 0 & 0 & \dots & 0 & -\lambda_k \end{pmatrix},$$

and

$$T_k{}^0 = \begin{pmatrix} 0 & 0 & \dots & \lambda_k \end{pmatrix}^t.$$

The Laplace-Stieltjes transform $\beta_k(s)$ of probability distribution $B_k(x)$ of the time until absorbtion in the state 0 is

$$\beta_k(s) = \int_0^\infty e^{-sx} dB_k(x) = \alpha^t (sI - T_k)^{-1} T_k{}^0.$$

Then, from equation (4.1) we obtain:

$$\pi_k(s) = \alpha_k^t([s + \tilde{\lambda}_k(1 - \pi_k(s))]I + A_k)^{-1} A_k e,$$

where $A_k = -T_k$.

Denote $g_k(s) = s + \tilde{\lambda}_k(1 - \pi_k(s))$, then

$$a_k(s) = 1 - \pi_k(s) = 1 - \alpha_k^t (g_k(s)I + A_k)^{-1} A_k e = \alpha_k^t e - \alpha_k^t (g(s)_k I + A)^{-1} A_k e =$$
$$= \alpha_k^t [I - (g_k(s)I + A)^{-1} A_k] e = \alpha_k{}^t (g_k(s)I + A_k)^{-1} [g_k(s)I + A_k - A_k] e =$$
$$= \alpha_k^t g_k(s)(g_k(s)I + A_k)^{-1} e.$$

Denote $(g_k(s)I + A_k)^{-1} e = y_k(s)$, then the matrix form for Kendall equation is

$$a_k(s) = \alpha_k{}^t g_k(s) y_k(s), \tag{4.2}$$

where $y_k(s)$ can be found by solving these simultaneous linear equations

$$(g_k(s)I + A_k) y_k(s) = e. \tag{4.3}$$

5 Matrix Algorithm for Solving Kendall Equation

We have to calculate

$$a_k(s) = \alpha_k^t g_k(s) y_k(s), \tag{5.1}$$

where $g_k(s) = s + \lambda_k a_k(s)$ and $y_k(s) = (g_k(s)I + A_k)^{-1} e$.
For calculating $y_k(s)$ it is necessary to solve these simultaneous linear equations

$$(g_k(s)I + A_k) y_k(s) = e, \tag{5.2}$$

where $e = (11\dots1)^t$, $A_k = -T_k$ and

$$T_k = \begin{pmatrix} -\lambda_k & \lambda_k & \dots & 0 & 0 \\ 0 & -\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda_k & \lambda_k \\ 0 & 0 & \dots & 0 & -\lambda_k \end{pmatrix},$$

and

$$T_k^0 = \begin{pmatrix} 0 & 0 & \dots & \lambda_k \end{pmatrix}^t.$$

The simultaneous linear equations (5.2) have the analytical solution. Let write these simultaneous linear equations in explicit form:

$$\begin{pmatrix} g_k(s)+\lambda_k & -\lambda_k & \dots & 0 & 0 \\ 0 & g_k(s)+\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & g_k(s)+\lambda_k & -\lambda_k \\ 0 & 0 & \dots & 0 & g_k(s)+\lambda_k \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\begin{gathered} y_{n-1} = 1/(g_k(s)+\lambda_k) = \omega_k, \\ y_i = (1+\lambda_k y_{i+1})\omega_k = \omega_k + \omega_k\lambda_k y_{i+1}, i = \overline{1, n-2}, \\ y_0 = \omega_k + \omega_k\lambda_k y_1 = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)}. \end{gathered} \tag{5.3}$$

First prove relation (5.3).

$$\begin{gathered} y_0 = \omega_k + \omega_k\lambda_k y_1 = \omega_k + \omega_k\lambda_k(\omega_k + \omega_k\lambda_k y_2) = \\ = \omega_k + \omega_k^2\lambda_k + (\omega_k\lambda_k)^2 y_2 = \omega_k + \omega_k^2\lambda_k + (\omega_k\lambda_k)^2(\omega_k + \omega_k\lambda_k y_3) = \\ = \omega_k + \omega_k^2\lambda_k + \omega_k^3\lambda_k^2 + (\omega_k\lambda_k)^3 y_3 = \omega_k(1 + \omega_k\lambda_k + (\omega_k\lambda_k)^2 + (\omega_k\lambda_k)^3\frac{y_3}{\omega_k}) = \\ = \dots = \omega_k(1 + \omega_k\lambda_k + (\omega_k\lambda_k)^2 + \dots + (\omega_k\lambda_k)^{n-1}\frac{y_{n-1}}{\omega_k}) = \omega_k\sum_{j=0}^{n-1}(\lambda_k\omega_k)^j = \\ = \frac{\omega_k(1-(\lambda_k\omega_k)^n)}{1-\lambda_k\omega_k} = \frac{\omega_k(1-(\lambda_k\omega_k)^n)}{\omega_k(\frac{1}{\omega_k}-\lambda_k)} = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)+\lambda_k-\lambda_k} = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)}, \end{gathered}$$

because

$$|\omega_k\lambda_k| = |\frac{\lambda_k}{g_k(s)+\lambda_k}| < 1.$$

So, the solution of the simultaneous linear equations (5.2) is:

$$y_0 = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)},$$

$$y_{i+1} = \frac{y_i - \omega_k}{\lambda_k\omega_k}, i = \overline{1, n-2},$$

$$y_{n-1} = \omega_k.$$

Then $a_k(s) = \alpha_k^t g_k(s) y_k(s)$ will be

$$a_k(s) = \begin{pmatrix}1 & 0 & \dots & 0 & 0\end{pmatrix} g_k(s) \begin{pmatrix} \frac{1-(\lambda_k\omega_k)^n}{g_k(s)} \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix} = 1-(\lambda_k\omega_k)^n.$$

So, $a_k(s) = 1-(\lambda_k\omega_k)^n$, where $\alpha_k^t e = 1$, and we start with $a_k(s) = 1$ and $\alpha_k^t = (10\dots0)$, the remaining values we give by ourselves ($\lambda_k, \tilde{\lambda}_k$ and s).

6 Matrix Form for Generalization of Classical Kendall Functional Equation

It is known that analytical results for k-busy period can be viewed as a generalization of the classical Kendall functional equation

$$\pi_k^\delta(s) = c_k(s + \tilde{\lambda}_k(1-\pi_k(s)))\pi_k(s). \tag{6.1}$$

Suppose that $C_k(x)$ is a PH distribution with representation (α_k^t, P_k), where

$$P_k = \begin{pmatrix} -\delta_k & \delta_k & \dots & 0 & 0 \\ 0 & -\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\delta_k & \delta_k \\ 0 & 0 & \dots & 0 & -\delta_k \end{pmatrix},$$

and

$$P_k^0 = \begin{pmatrix}0 & 0 & \dots & \delta_k\end{pmatrix}^t.$$

The Laplace-Stieltjes transform is:

$$c_k(s) = \int_0^\infty e^{-sx} dB_k(x) = \alpha^t (sI - P_k)^{-1} {P_k}^0. \tag{6.2}$$

From equation (6.1) we obtain:

$$\pi_k^\delta(s) = \alpha_k^t([(s+\tilde{\lambda}_k(1-\pi_k(s)))\pi_k(s)]I + D_k)^{-1}D_k e,$$

where $D_k = -P_k$. It is known from Section 4 that $\pi_k(s) = 1 - a_k(s)$ and $g_k(s) = s + \tilde{\lambda}_k(1-\pi_k(s))$, then

$$b_k(s) = 1 - \pi_k^\delta(s) = 1 - \alpha_k^t(g_k(s)(1-a_k(s))I + D_k)^{-1}D_k e =$$

$$= \alpha_k^t e - \alpha_k^t(g(s)_k(1-a_k(s))I + D_k)^{-1}D_k e = \alpha_k^t[I - (g_k(s)(1-a_k(s))I + D_k)^{-1}D_k]e =$$

$$= \alpha_k^t(g_k(s)(1-a_k(s))I + D_k)^{-1}[g_k(s)(1-a_k(s))I + D_k - D_k]e =$$

$$= \alpha_k^t g_k(s)(1-a_k(s))(g_k(s)(1-a_k(s))I + D_k)^{-1}e.$$

If we denote $(g_k(s)(1-a_k(s))I + D_k)^{-1}e = \tilde{y}_k(s)$, then

$$b_k(s) = {\alpha_k}^t g_k(s)(1-a_k(s))\tilde{y}_k(s), \tag{6.3}$$

where $\tilde{y}(s)$ can be found by solving these simultaneous linear equations

$$(g_k(s)(1-a_k(s))(s)I + D_k)\tilde{y}_k = e. \tag{6.4}$$

7 Matrix Algorithm for Solving Generalization of Classical Kendall Functional Equation

We have to calculate

$$b_k(s) = \alpha_k^t g_k(s)(1-a_k(s))\tilde{y}_k(s), \tag{7.1}$$

where $g_k(s) = s + \tilde{\lambda}_k a_k(s)$ and $\tilde{y}_k(s) = (g_k(s)(1-a_k(s))I + D_k)^{-1}e$.
To calculate $y_k(s)$ it is necessary to solve these simultaneous linear equations

$$(g_k(s)(1-a_k(s))I + D_k)\tilde{y}_k(s) = e, \tag{7.2}$$

where $e = (11\dots1)^t$, $D_k = -P_k$ and

$$P_k = \begin{pmatrix} -\delta_k & \delta_k & \dots & 0 & 0 \\ 0 & -\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\delta_k & \delta_k \\ 0 & 0 & \dots & 0 & -\delta_k \end{pmatrix},$$

and

$$P_k^0 = \begin{pmatrix} 0 & 0 & \dots & \delta_k \end{pmatrix}^t.$$

The simultaneous linear equations (7.2) have the analytical solution. Denote

$$\mathbf{A} = \begin{pmatrix} g_k(s)(1-a_k(s))+\delta_k & -\delta_k & \dots & 0 & 0 \\ 0 & g_k(s)(1-a_k(s))+\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & g_k(s)(1-a_k(s))+\delta_k \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} y_0 & y_1 & \dots & y_{n-1} \end{pmatrix}^t,$$

$$\mathbf{e} = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^t.$$

These simultaneous linear equations have the matrix form:

$$\mathbf{A} \cdot \mathbf{b} = \mathbf{e}$$

Then

$$\begin{gathered} \tilde{y}_{n-1} = 1/(g_k(s)(1-a_k(s))+\delta_k) = \gamma_k, \\ \tilde{y}_i = (1+\delta_k\tilde{y}_{i+1})\gamma_k = \gamma_k + \gamma_k\delta_k\tilde{y}_{i+1}, i = \overline{1, n-2}, \\ \tilde{y}_0 = \gamma_k + \gamma_k\delta_k\tilde{y}_1 = 1 - \frac{\delta_k}{g_k(s)(1-a_k(s))+\gamma_k}. \end{gathered} \tag{7.3}$$

First prove relation (7.3).

$$\begin{gathered} \tilde{y}_0 = \gamma_k + \gamma_k\delta_k\tilde{y}_1 = \gamma_k + \gamma_k\delta_k(\gamma_k + \gamma_k\delta_k\tilde{y}_2) = \\ = \gamma_k + \gamma_k^2\delta_k + (\gamma_k\delta_k)^2\tilde{y}_2 = \gamma_k + \gamma_k^2\delta_k + (\gamma_k\delta_k)^2(\gamma_k + \gamma_k\delta_k\tilde{y}_3) = \\ = \gamma_k + \gamma_k^2\delta_k + \gamma_k^3\delta_k^2 + (\gamma_k\delta_k)^3\tilde{y}_3 = \gamma_k(1 + \gamma_k\delta_k + (\gamma_k\delta_k)^2 + (\gamma_k\delta_k)^3\frac{y_3}{\gamma_k}) = \\ = \dots = \gamma_k(1 + \gamma_k\delta_k + (\gamma_k\delta_k)^2 + \dots + (\gamma_k\delta_k)^{n-1}\frac{\tilde{y}_{n-1}}{\gamma_k}) = \gamma_k\sum_{j=0}^{n-1}(\delta_k\gamma_k)^j = \\ = \frac{\gamma_k(1-(\delta_k\gamma_k)^n)}{1-\delta_k\gamma_k} = \frac{\gamma_k(1-(\delta_k\gamma_k)^n)}{\gamma_k(\frac{1}{\gamma_k}-\delta_k)} = \\ = \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))+\delta_k-\delta_k} = \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))}, \end{gathered}$$

because

$$|\gamma_k\delta_k| = |\frac{\delta_k}{g_k(s)(1-a_k(s))+\delta_k}| < 1.$$

In this case the solution of the simultaneous linear equations (4.1) is:

$$\tilde{y}_0 = \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))},$$

$$\tilde{y}_{i+1} = \frac{y_i-\gamma_k}{\delta_k\gamma_k}, i=\overline{1,n-2},$$

$$\tilde{y}_{n-1} = \gamma_k.$$

Then $b_k(s) = \alpha_k^t g_k(s)(1-a_k(s))\tilde{y}_k(s)$ will be

$$b_k(s) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix} g_k(s)(1-a_k(s)) \begin{pmatrix} \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))} \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n-2} \\ \tilde{y}_{n-1} \end{pmatrix} = 1-(\delta_k\gamma_k)^n.$$

So, $b_k(s) = 1-(\delta_k\gamma_k)^n$, where $\alpha_k^t e = 1$, and we start with $a_k(s) = 1$ and $\alpha_k^t = (10\dots0)$, the remaining values we give by ourselves (δ_k, $\tilde{\lambda}_k$ and s).

8 Conclusion

The main purpose of research of the Polling system is to determine the characteristics of system development. But analytical formulas can not always be used directly, so great attention is paid to numerical algorithms. For finding numerical solutions for the k-busy period, in terms of Laplace-Stieltjes transform, PH distribution was used. A matrix algorithm for solving the generalization of classical Kendall functional equation was obtained. Some numerical examples are presented.

9 Examples

Example 1. The type of distribution function taken for $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1-\alpha_t e^{T_k x} e, \ x>0,$$

$$C_k(x) = 1-\alpha_t e^{P_k x} e, \ x>0,$$

with the following parameters:
$\lambda_k = \{0.5; 0.6; 0.3; 0.4; 0.5; 0.2; 0.6; 0.6; 0.2; 0.1\}$,
$\tilde{\lambda}_k = \{0.2; 0.3; 0.4; 0.2; 0.6; 0.7; 0.8; 0.4; 0.2; 0.3\}$,
$\delta_k = \{0.3; 0.4; 0.1; 0.2; 0.6; 0.8; 0.5; 0.4; 0.4; 0.8\}$,
$s = 0.5$.

The results of the program are presented in Table 1.

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.012692	0.864672	6	0.000060	0.999554
2	0.014688	0.865907	7	0.003161	0.959972
3	0.000978	0.957140	8	0.010383	0.891422
4	0.006395	0.895401	9	0.000542	0.995270
5	0.002997	0.973018	10	0.000017	0.999915

Table 1

Example 2. The type of distribution function taken for $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1 - \alpha_t e^{T_k x} e, x > 0,$$

$$C_k(x) = 1 - \alpha_t e^{P_k x} e, x > 0,$$

with the following parameters:
$\lambda_k = \{0.2; 0.3; 0.1; 0.5; 0.6; 0.7; 0.4; 0.8; 0.4; 0.5; 0.3; 0.7; 0.8; 0.4; 0.6; 0.9; 0.3; 0.4; 0.5; 0.4\}$,
$\tilde{\lambda}_k = \{0.2; 0.3; 0.5; 0.2; 0.3; 0.7; 0.8; 0.4; 0.3; 0.5; 0.1; 0.5; 0.8; 0.4; 0.3; 0.6; 0.4; 0.9; 0.4; 0.2\}$,
$\delta_k = \{0.5; 0.4; 0.8; 0.4; 0.4; 0.7; 0.4; 0.3; 0.8; 0.2; 0.1; 0.5; 0.4; 0.7; 0.5; 0.4; 0.7; 0.9; 0.4; 0.3\}$,
$s = 0.5$.
The results of the program are presented in Table 2.

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.002444	0.986440	11	0.012414	0.750670
2	0.005566	0.956771	12	0.029777	0.796091
3	0.000068	0.999659	13	0.021775	0.763410
4	0.030767	0.812228	14	0.009064	0.954890
5	0.034760	0.766861	15	0.034760	0.807550
6	0.018947	0.881203	16	0.043203	0.644660
7	0.003083	0.960978	17	0.003927	0.980092
8	0.051492	0.569886	18	0.002451	0.984919
9	0.012501	0.951740	19	0.016581	0.864629
10	0.012555	0.785024	20	0.017712	0.851134

Table 2

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