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Post-optimal analysis of investment problem with Wald's ordered maximin criteria

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Abstract. We consider Markowitz's multicriteria portfolio optimization problem with Wald's ordered maximin criteria. We obtained lower and upper attainable bounds of the stability radius of lexicographically optimal portfolio.

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In the papers [1, 2] we derived the bounds of the stability radius of a Pareto-optimal solution of Markowitz's investment problem with Savage's minimax criteria. In this paper we obtain lower and upper attainable bounds of the stability radius of lexicographical optimum for the Markowitz's multicriteria problem with Wald's maximin criteria.

1 Problem formulation and definitions

Let us consider the multicriterion variant of the investment managing problem based on Markowitz's classical portfolio theory [3]. As a portfolio efficiency criterion we use Wald's maximin criterion. We introduce the following notations: let $N_n = \{1, 2, \ldots, n\}$ be the set of investment projects (assets); N_m be the set of possible financial market states (situation); $x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n \setminus \{\mathbf{0}\}$ be the investment portfolio, where $\mathbf{E} = \{0, 1\}$, $x_j = 1$ if project $j \in N_n$ is implemented, $x_j = 0$ otherwise. As usual $\mathbf{0}$ is the zero vector of the corresponding dimension.

There exist several approaches to the assessment of efficiency (utility) of investment projects (NPV, NFV, IRR et al.) which take into account the uncertainty and risk in different ways (see for example [4,5]). Let N_s be the set of indicators of investment projects efficiency. An investment portfolio x is evaluated by $\sum\limits_{j \in N_n} a_{ijk} x_j$, where a_{ijk} is the efficiency indicator $k \in N_s$ of investment project $j \in N_n$ in the case when the market be in state $i \in N_m$. Therefore we may assume that the input data of the problem are determined by the three-dimensional matrix of investment project efficiency A of size $m \times n \times s$ with elements a_{ijk} from $\mathbf{R}$. Let us introduce the vector objective function

$$f(x, A) = (f_1(x, A_1), f_2(x, A_2), \ldots, f_s(x, A_s)),$$

whose partial objectives are well-known Wald's maximin criteria [6]

$$f_k(x, A_k) = \min_{i \in N_m} A_{ik}x = \min_{i \in N_m} \sum_{j \in N_n} a_{ijk}x_j \to \max_{x \in X}, \qquad k \in N_s,$$

where $A_k \in \mathbf{R}^{m \times n}$ is k-th cut of the matrix $A = [a_{ijk}] \in \mathbf{R}^{m \times n \times s}$, $A_{ik} = (a_{i1k}, a_{i2k}, ..., a_{ink})$ is i-th row of that cut. Thus, following Wald's criterion, the investor shows extreme caution when he/she optimizes the efficiency of the portfolio in assuming that the financial market is in the most unprofitable state, i. e. considering the uncertainty of the market state, the investor chooses the maximin strategy.

The problem of finding the set of lexicographically optimal portfolio $L^s(A)$ will be viewed as the multicriterion (s-criteriion) investment problem $Z^s(A)$ with Wald's ordered criteriion, $s \in \mathbf{N}$, where the set $L^s(A)$ is defined in the following traditional way [7–10]

$$L^s(A) = \{x \in X : \ \ \nexists x' \in X \ (x \underset{A}{\prec} x')\},$$

where

$$x \underset{A}{\prec} x' \quad \Leftrightarrow \ \exists p \in N_s \ (g_p(x, x', A_p) < 0 \quad \& \quad p = \max\{k \in N_s : \ g_k(x, x', A_k) \neq 0\}),$$

$$g_k(x, x', A_k) = f_k(x, A_k) - f_k(x', A_k) = \max_{i' \in N_m} \min_{i \in N_m} (A_{ik}x - A_{i'k}x'), \ k \in N_s. \quad (1)$$

Evidently, the set $L^s(A)$ is a non-empty subset of the Pareto set for any matrix $A \in \mathbf{R}^{m \times n \times s}$. It is also well-known (see e.g. [11]), that the lexicographic set $L^s(A)$ can be determined as a result of sequential solving of s scalar problems:

$$L^s_k(A) := \operatorname{Arg\,min}\{f_k(x, A_k) : \ x \in L^s_{k-1}(A)\}, \qquad k \in N_s,$$

where $L^s_0(A) = X$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all individual solutions of the corresponding scalar minimization problem. Thus, we have the chain of inclusions

$$X \supseteq L^s_1(A) \supseteq L^s_2(A) \supseteq \ldots \supseteq L^s_s(A) = L^s(A).$$

Therefore, the problem $Z^s(A)$ of fining the lexicographic set $L^s(A)$ can be seen as a problem of sequential minimization of partial objective functions $f_k(x, A_k)$, $k \in N_s$.

The following properties are obvious.

Property 1. *If for a portfolio $x^0 \in X$ it holds that*

$$\forall x \in X \setminus \{x^0\} \quad (g_1(x, x^0, A_1) > 0),$$

then $x^0 \in L^s(A)$.

Property 2. *If for a portfolio $x^0 \in X$ it holds that*

$$\exists x^* \in X \setminus \{x^0\} \quad (g_1(x^*, x^0, A_1) < 0),$$

then $x^0 \notin L^s(A)$.

In portfolio space $\mathbf{R}^n$, market state space $\mathbf{R}^m$ and efficiency (criteria) space $\mathbf{R}^s$, we define the linear metric l_1, i.e.

$$\|A_{ik}\| = \sum_{j \in N_n} |a_{ijk}|, \quad i \in N_m, \quad k \in N_s,$$

$$\|A_k\| = \sum_{i \in N_m} \|A_{ik}\| = \sum_{i \in N_m} \sum_{j \in N_n} |a_{ijk}|, \quad k \in N_s,$$

$$\|A\| = \sum_{k \in N_s} \|A_k\| = \sum_{i \in N_m} \sum_{j \in N_n} \sum_{k \in N_s} |a_{ijk}|.$$

The following inequalities are evident

$$\|A\| \geq \|A_k\| \geq \|A_{ik}\|, \quad i \in N_m,\ k \in N_s. \tag{2}$$

Apart from that, it is easy to see that for any x and x' the following inequalities hold

$$A_{ik}x - A_{i'k}x' \geq -\|A_k\|, \qquad i,\ i' \in N_m, \quad k \in N_s. \tag{3}$$

As usual [9, 13], the stability radius of portfolio $x^0 \in L^s(A)$ is defined as the number

$$\rho^s(x^0, A) = \begin{cases} \sup\ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where $\Xi = \{\varepsilon > 0 :\ \forall A' \in \Omega(\varepsilon) \quad (x^0 \in L^s(A + A'))\}$, $\Omega(\varepsilon) = \{A' \in \mathbf{R}^{m \times n \times s} : \|A'\| < \varepsilon\}$ is the set of perturbing matrices, $L^s(A + A')$ is the set of lexicographically optimal portfolios in the perturbed problem $Z^s(A + A')$.

Thus, the stability radius defines an extreme level of problem initial data perturbations (elements of matrix A) preserving lexicographic optimality of the portfolio.

2 Stability radius bounds

For $x^0 \in L^s(A)$ and $Z^s(A)$, denote

$$\varphi = \min_{x \in X \setminus \{x^0\}} \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x).$$

Evidently, $\varphi \geq 0$.

Theorem 1. *Given $Z^s(A)$, the stability radius $\rho^s(x^0,\ A)$, $\ s \geq 1$, of a lexicographically optimal portfolio x^0 has the following lower and upper bounds*

$$\varphi \leq \rho^s(x^0, A) \leq 2\varphi.$$

Proof. Let $x^0 \in L^s(A)$. First we will prove that $\rho^s(x^0, A) \geq \varphi$, which is evident if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for every portfolio $x \neq x^0$ the following inequality holds

$$\max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) \geq \varphi. \tag{4}$$

Let A' be an arbitrary perturbing matrix belonging to $\Omega(\varphi)$. Then, taking into account (1)–(4), we obtain

$$g_1(x^0, x, A_1 + A'_1) = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x + A'_{i'1}x^0 - A'_{i1}x) \geq$$

$$\geq \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) - \|A'_1\| \geq \varphi - \|A'_1\| \geq \varphi - \|A'\| > 0.$$

Therefore, due to Property 1, the portfolio x^0 preserves lexicographic optimality in any perturbed problem $Z^s(A + A')$, $A' \in \Omega(\varphi)$. Hence, $\rho^s(x^0, A) \geq \varphi$.

Further we show that $\rho^s(x^0, A) \leq 2\varphi$. Let $x^* \neq x^0$ be a portfolio such that the following equalities hold

$$g_1(x^0, x^*, A_1) = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x^*) = \varphi. \tag{5}$$

The existence of such portfolio comes from the definition of φ.

Let us prove that

$$\forall \varepsilon > 2\varphi \quad \exists A^0 \in \Omega(\varepsilon) \qquad (x^0 \notin L^s(A + A^0)). \tag{6}$$

For this in accordance with Property 2 it is sufficient to construct a perturbing matrix A^0 with cut A_1^0 such that the following conditions hold

$$2\varphi < \|A^0\| < \varepsilon, \tag{7}$$

$$g_1(x^0, x^*, A_1 + A_1^0) < 0. \tag{8}$$

Let

$$i(x^0) = \arg\min\{A_{i1}x^0 : \ i \in N_m\}$$

and consider two possible cases.

Case 1. There exists an index $l \in N_n$ such that $x_l^0 = 1$ and $x_l^* = 0$. We define the elements of the cut $A_1^0 = [a_{ij1}^0] \in \mathbf{R}^{m \times n}$ of the perturbing matrix $A^0 = [a_{ijk}^0] \in \mathbf{R}^{m \times n \times s}$ as follows:

$$a_{ij1}^0 = \begin{cases} -\delta = & \text{if } \ i = i(x^0), \ j = l, \\ 0 & \text{otherwise,} \end{cases}$$

where $2\varphi < \delta < \varepsilon$. The elements of the remaining cuts A_k^0, $k \neq 1$, of the perturbing matrix A^0 set equal to zero. Hence we have

$$A^0_{i(x^0)1}x^0 = -\delta, \qquad A^0_{i(x^0)1}x^* = 0, \tag{9}$$

$$A^0_{i1}x^0 = A^0_{i1}x^* = 0, \qquad i \in N_m \setminus \{i(x^0)\}, \tag{10}$$

$$\|A^0\| = \|A^0_1\| = \delta.$$

Therefore, the inequality (7) is true.

As a result we have

$$f_1(x^0, A_1 + A^0_1) = \min\Big\{(A_{i(x^0)1} + A^0_{i(x^0)1})x^0, \min_{i \neq i(x^0)}(A_{i1} + A^0_{i1})x^0\Big\} = f_1(x^0, A_1) - \delta,$$

$$f_1(x^*, A_1 + A^0_1) = \min\Big\{(A_{i(x^0)1} + A^0_{i(x^0)1})x^*, \min_{i \neq i(x^0)}(A_{i1} + A^0_{i1})x^*\Big\} = f_1(x^*, A_1).$$

Thus, from (5) and $\delta > \varphi$ we verify the validity of the inequality (8).

Case 2. $x^0 \leq x^*$. Then in view of the inequalities $x^0 \neq x^* \neq \mathbf{0}$ there exists a pair of indexes $(p \times q) \in N_n \times N_n$ such that $x^0_p = 0$, $x^*_p = 1$, $x^0_q = x^*_q = 1$. The elements of the cut $A^0_1 = [a^0_{ij1}] \in \mathbf{R}^{m \times n}$ we define as follows:

$$a^0_{ij1} = \begin{cases} -\delta & \text{if } \; i = i(x^0), \; j = q, \\ \delta & \text{if } \; i = i(x^0), \; j = p, \\ 0 & \text{otherwise,} \end{cases}$$

where $2\varphi < 2\delta < \varepsilon$. The elements of the remaining cuts A^0_k, $k \neq 1$ of the perturbing matrix A^0 set equal to zero. Then the equations (9), (10) and $\|A^0_1\| = \|A^0\| = 2\delta$ hold, i.e. (7) holds. Further, repeating the reasoning of the case 1 and taking into account $\delta > \varphi$, we see that the inequality (8) is true.

As a result we construct in the first and second case the perturbing matrix A^0 such that the formula (6) is true. Hence, $\rho^s(x^0, A) \leq 2\varphi$. □

3 Lower bound attainability

We show that the lower bound of the stability radius $\rho^s(x^0, A)$, indicated in Theorem 1, is attainable.

Theorem 2. *There exists a class of investment problems $Z^s(A)$, $s \geq 1$, such that the stability radius of any lexicographically optimal portfolio x^0 is expressed by the formula $\rho^s(x^0, A) = \varphi$.*

Proof. To prove the equality $\rho^s(x^0, A) = \varphi$, where $\varphi > 0$, it is sufficient to identify a class of problems with $\rho^s(x^0, A) \leq \varphi$.

Assume x^* be such that the equality (5) holds. Since $x^0 \neq x^*$, there exists an index $l \in N_n$ such that $x^0_l \neq x^*_l$. We will assume that $x^0_l = 1$ and $x^*_l = 0$ (this is the actual specific of the class of problems we would like to identify).

Assuming $\varepsilon > \varphi$, we define the elements of the cut $A_1^0 = [a_{ij1}^0] \in \mathbf{R}^{m\times n}$ of the perturbing matrix $A^0 = [a_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ as follows

$$a_{ij1}^0 = \begin{cases} -\delta & \text{if } i = i(x^0),\ j = l, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varphi < \delta < \varepsilon, \tag{11}$$

$$i(x^0) = \arg\min\{A_{i1}x^0 : \quad i \in N_m\}. \tag{12}$$

All elements in the remaining cuts A_k^0, $k \in N_s \setminus \{1\}$, of the perturbing matrix A^0 set equal to zero. As a result we get

$$A_{i(x^0)1}^0 x^0 = -\delta, \quad A_{i1}^0 x^* = 0, \quad i \in N_m,$$

$$A_{i1}^0 x^0 = 0, \quad i \in N_m \setminus \{i(x^0)\},$$

$$\|A^0\| = \|A_1^0\| = \delta, \quad A^0 \in \Omega(\varepsilon).$$

Now due to (12) it is easy to see that

$$f_1(x^*, A_1 + A_1^0) = \min_{i\in N_n}(A_{i1} + A_{i1}^0)x^* = \min_{i\in N_n} A_{i1}x^* = f_1(x^*, A_1),$$

$$f_1(x^0, A_1 + A_1^0) = \min\Big\{(A_{i(x^0)1} + A_{i(x^0)1}^0)x^0, \min_{i\neq i(x^0)}(A_{i1} + A_{i1}^0)x^0\Big\} =$$

$$= \min\Big\{f_1(x^0, A_1) - \delta, \min_{i\neq i(x^0)} A_{i1}x^0\Big\} = f_1(x^0, A_1) - \delta.$$

Therefore, based on (5) and (11), we obtain

$$g_1(x^0, x^*, A_1 + A_1^0) = g_1(x^0, x^*, A_1) - \delta = \varphi - \delta < 0.$$

The last together with Property 2 imply that for any $\varepsilon > \varphi$ there exists $A^0 \in \Omega(\varepsilon)$ such that $x^0 \notin L^s(A + A^0)$. Hence, $\rho^s(x^0, A) \leq \varphi$. □

Consider a short numerical example illustrating Theorem 2.

Example. Let $m = 2$, $n = 3$, $s = 1$, $X = \{x^0, x^*\}$, $x^0 = (0, 1, 1)^T$, $x^* = (1, 1, 0)^T$,

$$A = \begin{pmatrix} -6 & 5 & -1 \\ 2 & -2 & 3 \end{pmatrix}.$$

Then $f(x^0, A) = 1$, $f(x^*, A) = -1$, i.e. x^0 is an optimal portfolio of $Z^1(A)$. Since $\varphi = 2$ then according to Theorem 1 $\rho^1(x^0, A) \geq 2$. If we define the perturbing matrix as follows

$$A^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix}, \quad \delta > 2,$$

then we have $\|A^0\| = \delta$ and $f(x^0, A + A^0) = 1 - \delta < -1 = f(x^*, A + A^0)$. Therefore, $x^0 \notin L^1(A + A^0)$, and hence $\rho^1(x^0, A) \leq 2$. Finally, $\rho^1(x^0, A) = 2 = \varphi$.

4 Upper bound attainability

Before proving upper bound attainability 2φ we consider one of the properties of the matrixes by size $m \times 2$, $m \geq 2$.

When $\varphi > 0$ the matrix $W = [u, v] \in \mathbf{R}^{m\times 2}$, $m \geq 2$ with $u = (u_1, u_2, \ldots, u_m)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ is called φ-special if the inequality holds

$$\min_{i\in N_m} (u_i + v_i) - \min_{i\in N_m} u_i < \varphi.$$

Lemma. *The matrix $W = [u, v] \in \mathbf{R}^{m\times 2}$, $m \geq 2$, with the norm $\|W\| < 2\varphi$, where $\varphi > 0$, is φ-special.*

Proof. The proof is by induction on $m \geq 2$.

First we proof the lemma for $m = 2$. Let

$$W = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

Let us show that the inequality

$$\min\{u_1 + v_1, u_2 + v_2\} - \min\{u_1, u_2\} < \varphi \tag{13}$$

follows from the inequality $\|W\| < 2\varphi$, i.e from the inequality

$$|u_1| + |u_2| + |v_1| + |v_2| < 2\varphi. \tag{14}$$

Without loss of generality we assume that

$$u_1 + v_1 \leq u_2 + v_2. \tag{15}$$

We consider two possible cases.

Case 1. $u_1 \leq u_2$. Then the inequality (13) in view of (15) takes the form $\varphi > v_1$. We give the proof by contradiction. Let

$$\varphi \leq v_1. \tag{16}$$

From (15) and (16) we have

$$\varphi \leq -u_1 + u_2 + v_2,$$

and from (14) and (16) we derive

$$\varphi > |u_1| + |u_2| + |v_2|.$$

These inequalities lead to the contradiction

$$0 \leq |u_2| - u_2 + |v_2| - v_2 < -(u_1 + |u_1|) \leq 0.$$

Case 2. $u_1 > u_2$. Then the inequality (13) in view of (15) transform into inequality $\varphi > u_1 + v_1 - u_2$. Suppose the contrary

$$\varphi \leq u_1 + v_1 - u_2. \tag{17}$$

Therefore, taking into account (15) we have $\varphi \leq v_2$. Hence in view of (14) we find

$$\varphi > |u_1| + |u_2| + |v_1|.$$

This inequality with (17) leads to the contradiction

$$0 \leq |u_1| - u_1 + |v_1| - v_1 < -(u_2 + |u_2|) \leq 0.$$

Further we assume that the lemma is true for $m \geq 2$ and we show that the matrix $W = [u, v] \in \mathbf{R}^{(m+1)\times 2}$ with column $u = (u_1, u_2, \dots, u_{m+1})^T$, $v = (v_1, v_2, \dots, v_{m+1})^T$ and norm $\|W\| < 2\varphi$ is φ-special.

Let

$$i_1 = \arg\min\{u_i + v_i : \ i \in N_{m+1}\},$$

$$i_2 = \arg\min\{u_i : \ i \in N_{m+1}\}$$

and let the index $l \in N_{m+1}$ is such that

$$l \neq i_1 \ \ \& \ \ l \neq i_2. \tag{18}$$

Doped from the matrix W the l-th row, we have a matrix $W' \in \mathbf{R}^{m\times 2}$ with the norm $\|W'\| \leq \|W\| < 2\varphi$. Then by induction the matrix W' is φ-special, i.e. the following inequality is true:

$$\min_{i \in N_{m+1}\setminus\{l\}} (u_i + v_i) - \min_{i \in N_{m+1}\setminus\{l\}} u_i < \varphi.$$

In addition, according to (18) we have the equalities:

$$\min_{i \in N_{m+1}} (u_i + v_i) = u_{i_1} + v_{i_1} = \min_{i \in N_{m+1}\setminus\{l\}} (u_i + v_i),$$

$$\min_{i \in N_{m+1}} u_i = u_{i_2} = \min_{i \in N_{m+1}\setminus\{l\}} u_i.$$

Hence, the matrix W is φ-special. □

Theorem 3. *For $\varphi > 0$ there exists a class of investment problems $Z^s(A)$, $s \geq 1$, such that the stability radius of a lexicographically optimal portfolio x^0 is expressed by the formula*

$$\rho^s(x^0, A) = 2\varphi.$$

Proof. Due to Theorem 1 it is sufficient to identify a class of problems with $\rho^s(x^0, A) \geq 2\varphi$. Let us show that there exists a class when $m \geq 2$ and $X = \{x^0, x^*\}$, $x^0 \in L^s(A)$, $x^* \neq x^0$.

According to the definition of φ the following equality holds

$$\max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^* - A_{i1}x^0) = \varphi. \tag{19}$$

Further we assume that the cut A_1 of the matrix A and portfolios x^0 and x^* satisfy the following conditions:

(a) $\forall i, i' \in N_m \quad \forall x \in X \qquad (A_{i1}x = A_{i'1}x)$,

(b) $x^0 \leq x^*$.

The condition *(a)* shows that $A_{i1}x$ for any portfolio $x \in X$ does not depend from index i. Denoting it by $\sigma(x)$ we have the following form of the equality (19)

$$\sigma(x^0) - \sigma(x^*) = \varphi.$$

From that equality for any matrix $A'_1 \in \mathbf{R}^{m \times n}$ we derive

$$g_1(x^0, x^*, A_1 + A'_1) = \min_{i \in N_m} (A_{i1} + A'_{i1})x^0 - \min_{i \in N_m} (A_{i1} + A'_{i1})x^* =$$

$$= \sigma(x^0) - \sigma(x^*) + \min_{i \in N_m} A'_{i1}x^0 - \min_{i \in N_m} A'_{i1}x^* = \varphi - \gamma, \tag{20}$$

where

$$\gamma = \min_{i \in N_m} (A'_{i1}x^0 + A'_{i1}(x^* - x^0)) - \min_{i \in N_m} A'_{i1}x^0.$$

Now let the perturbing matrix $A' = [a'_{ijk}] \in \Omega(2\varphi)$. Let us consider the matrix $W = [u, v] \in \mathbf{R}^{m \times 2}$ with column $u = A'_1x^0$ and $v = A'_1(x^* - x^0)$, where $A'_1 = [a'_{ij1}]$. Then for portfolio $x = (x_1, x_2, \ldots, x_n)^T$ we introduce the following notations: let

$$N(x) = \{j \in N_n : \quad x_j = 1\},$$

and also, taking into account (b) and $x^* \neq x^0$, we have

$$\|W\| = \|A'_1x^0\| + \|A'_1(x^* - x^0)\| = \sum_{i \in N_m} \mid \sum_{j \in N(x^0)} a'_{ij1} \mid + \sum_{i \in N_m} \mid \sum_{j \in N(x^* - x^0)} a'_{ij1} \mid \leq$$

$$\leq \sum_{i \in N_m} \sum_{j \in N(x^*)} |a'_{ij1}| \leq \|A'_1\| \leq \|A'\| < 2\varphi.$$

Therefore due to the lemma the matrix W is φ-special, i.e. the inequality $\gamma < \varphi$ holds, which with (20) gives us

$$g_1(x^0, x^*, A_1 + A'_1) > 0.$$

Hence due to Property 1 we conclude that for any perturbing matrix $A' \in \Omega(2\varphi)$ the inclusion $x^0 \in L^s(A + A')$ holds, i.e. $\rho^s(x^0, A) \geq 2\varphi$. □

5 Stability conditions

The portfolio $x^0 \in L^s(A)$ is called stable if $\rho^s(x^0, A) > 0$. Additionally, we introduce the set of strict lexicographically optimal portfolios of $Z^s(A)$:

$$S^s(A) = \{x \in X : \quad \forall x' \in X \setminus \{x\} \qquad (f_1(x, A_1) > f_1(x', A_1))\}.$$

Obviously, $S^s(A) \subseteq L^s(A)$ for any $A \in \mathbf{R}^{m \times n \times s}$. Apart from that it is clear that $S^s(A)$ can be empty.

Theorem 4. *For a lexicographically optimal portfolio x^0 of $Z^s(A)$ the following statements are equivalent:*

(i) $x^0 \in S^s(A)$,

(ii) portfolio x^0 is stable,

(iii) $\varphi > 0$.

Proof. $(i) \Rightarrow (ii)$. Let $x^0 \in L^s(A)$ be a strict lexicographically optimal portfolio, i. e. $x^0 \in S^s(A)$. Then for every $x \in X \setminus \{x^0\}$ we have

$$\xi(x) = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) = g_1(x^0, x, A_1) > 0.$$

Thus, due to Theorem 1 we conclude $\rho^s(x^0, A) \geq \varphi = \min\Big\{\xi(x) : \quad x \in X \setminus \{x^0\}\Big\} > 0$, i. e. $x^0 \in L^s(A)$ is stable.

$(ii) \Rightarrow (iii)$. Assume $x^0 \in L^s(A)$ be stable. Then according to Theorem 1 $2\varphi \geq \rho^s(x^0, A) > 0$, i. e. $\varphi > 0$.

$(iii) \Rightarrow (i)$. According to the definition of φ for any portfolio $x \neq x^0$ the inequality $\varphi \leq f_1(x, A_1) - f_1(x^0, A_1)$ is true. Hence from the inequality $\varphi > 0$ we have $x^0 \in S^s(A)$. □

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