On cyclically-interval edge colorings of trees

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Abstract. For an undirected, simple, finite, connected graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge t-coloring of a graph G if adjacent edges are colored differently and each of t colors is used. An arbitrary nonempty subset of consecutive integers is called an interval. If φ is a proper edge t-coloring of a graph G and $x \in V(G)$, then $S_G(x, \varphi)$ denotes the set of colors of edges of G which are incident with x. A proper edge t-coloring φ of a graph G is called a cyclically-interval t-coloring if for any $x \in V(G)$ at least one of the following two conditions holds: a) $S_G(x, \varphi)$ is an interval, b) $\{1, 2, \ldots, t\} \setminus S_G(x, \varphi)$ is an interval. For any $t \in \mathbb{N}$, let $\mathfrak{M}_t$ be the set of graphs for which there exists a cyclically-interval t-coloring, and let $\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t$. For an arbitrary tree G, it is proved that $G \in \mathfrak{M}$ and all possible values of t are found for which $G \in \mathfrak{M}_t$.

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1 Introduction

We consider undirected, simple, finite, and connected graphs. For a graph G we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. The set of edges of G incident with a vertex $x \in V(G)$ is denoted by $J_G(x)$. The set of vertices of G adjacent to a vertex $x \in V(G)$ is denoted by $I_G(x)$. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex x in G. For a graph G, we denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree of a vertex of G and the chromatic index of G [32], respectively. The distance in a graph G between its vertices $x \in V(G)$ and $y \in V(G)$ is denoted by $\rho_G(x, y)$. For any vertex $x_0 \in V(G)$ and an arbitrary subset V_0 of the set $V(G)$, we define the distance $\rho_G(x_0, V_0)$ in a graph G between x_0 and V_0 as follows:

$$\rho_G(x_0, V_0) \equiv \min_{z \in V_0} \rho_G(x_0, z)$$

For any integer $n \geq 3$, we denote by C_n a simple cycle with n vertices. The terms and concepts that we do not define can be found in [35].

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$. An interval D is called an h-interval if $|D| = h$.

For any $t \in \mathbb{N}$ and arbitrary integers i_1, i_2 satisfying the conditions $i_1 \in [1,t]$, $i_2 \in [1,t]$, we define [22] the sets $intcyc_1((i_1,i_2),t)$, $intcyc_1[(i_1,i_2),t]$, $intcyc_2((i_1,i_2),t)$, $intcyc_2[(i_1,i_2),t]$ and the number $dif((i_1,i_2),t)$ as follows:

$$intcyc_1[(i_1,i_2),t] \equiv [\min\{i_1,i_2\}, \max\{i_1,i_2\}],$$

$$intcyc_1((i_1,i_2),t) \equiv intcyc_1[(i_1,i_2),t] \backslash (\{i_1\} \cup \{i_2\}),$$

$$intcyc_2((i_1,i_2),t) \equiv [1,t] \backslash intcyc_1[(i_1,i_2),t],$$

$$intcyc_2[(i_1,i_2),t] \equiv [1,t] \backslash intcyc_1((i_1,i_2),t),$$

$$dif((i_1,i_2),t) \equiv \min\{|intcyc_1[(i_1,i_2),t]|, |intcyc_2[(i_1,i_2),t]|\} - 1.$$

If $t \in \mathbb{N}$ and Q is a non-empty subset of the set $\mathbb{N}$, then Q is called a t-cyclic interval if there exist integers i_1, i_2, j_0 satisfying the conditions $i_1 \in [1,t]$, $i_2 \in [1,t]$, $j_0 \in \{1,2\}$, $Q = intcyc_{j_0}[(i_1,i_2),t]$.

A function $\varphi : E(G) \to [1,t]$ is called a proper edge t-coloring of a graph G if adjacent edges are colored differently and each of t colors is used.

If φ is a proper edge t-coloring of a graph G and $E_0 \subseteq E(G)$, then $\varphi[E_0] \equiv \{\varphi(e)/e \in E_0\}$.

A proper edge t-coloring φ of a graph G is called an interval t-coloring of G [8,9,20] if for any $x \in V(G)$, the set $\varphi[J_G(x)]$ is a $d_G(x)$-interval. For any $t \in \mathbb{N}$, we denote by $\mathfrak{N}_t$ the set of graphs for which there exists an interval t-coloring. Let us also define the set $\mathfrak{N}$ of all interval colorable graphs:

$$\mathfrak{N} \equiv \bigcup_{t \geq 1} \mathfrak{N}_t.$$

For any $G \in \mathfrak{N}$, we denote by $w_{int}(G)$ and $W_{int}(G)$ the minimum and the maximum possible value of t, respectively, for which $G \in \mathfrak{N}_t$. For a graph G, let us set $\theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{N}_t\}$.

The problem of deciding whether a regular graph G belongs to the set $\mathfrak{N}$ is NP-complete [8,9,20]. Nevertheless, for graphs G of some classes the relation $G \in \mathfrak{N}$ was proved and investigations of the set $\theta(G)$ were fulfilled [8,9,19,20,26,27]. The concept of interval colorability of a graph represents an especially high interest for a bipartite graph, because in this case it can be used for mathematical modelling of timetable problems with compactness requirements (i.e. the lectures of each teacher and each group must be scheduled at consecutive periods) [1, 7, 20, 29]. Unfortunately, for an arbitrary bipartite graph G the problem keeps the complexity of a general case [3,13,31]. Some positive results were obtained for "small" bipartite graphs [14,15,25], for bipartite graphs with the "small" maximum degree of a vertex [13, 16, 28], and for biregular bipartite graphs [2–6,11,16–18,24,30,36]. Very interesting approaches for biregular bipartite graphs were developed in [6,11,30]. The examples of interval non-colorable bipartite graphs were given in [7,15,18,31].

Remark 1. It is not difficult to see that for any integer $k \geq 2$, $C_{2k} \in \mathfrak{N}$ and $\theta(C_{2k}) = [2, k+1]$.

A proper edge t-coloring φ of a graph G is called a cyclically-interval t-coloring of G if for any $x \in V(G)$, the set $\varphi[J_G(x)]$ is a t-cyclic interval. For any $t \in \mathbb{N}$, we denote by $\mathfrak{M}_t$ the set of graphs for which there exists a cyclically-interval t-coloring. Let us also define the set $\mathfrak{M}$ of all cyclically-interval colorable graphs:

$$\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t.$$

For any $G \in \mathfrak{M}$, we denote by $w_{cyc}(G)$ and $W_{cyc}(G)$ the minimum and the maximum possible value of t, respectively, for which $G \in \mathfrak{M}_t$. For a graph G, let us set $\Theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{M}_t\}$.

Remark 2. The concept of cyclically-interval colorability of a graph generalizes that of interval colorability. Clearly, for an arbitrary graph $G \in \mathfrak{N}$, and for any $t \in \theta(G)$, an arbitrary interval t-coloring of the graph G is also a cyclically-interval t-coloring of G, therefore, for any $t \in \mathbb{N}$, $\mathfrak{N}_t \subseteq \mathfrak{M}_t$. $\mathfrak{N}_2 = \mathfrak{M}_2$. For any integer $t \geq 3$, $\mathfrak{N}_t \subset \mathfrak{M}_t$ (it is enough to consider the simple cycle C_t). $\mathfrak{N} \subset \mathfrak{M}$ (it is enough to consider the simple cycle C_3). For an arbitrary graph G, $\theta(G) \subseteq \Theta(G)$.

Remark 3. For any $G \in \mathfrak{N}$, the following inequality is true:

$$\Delta(G) \leq \chi'(G) \leq w_{cyc}(G) \leq w_{int}(G) \leq W_{int}(G) \leq W_{cyc}(G) \leq |E(G)|.$$

Remark 4. It is not difficult to note that there exist examples G_1 and G_2 of graphs from $\mathfrak{N}$ for which $w_{cyc}(G_1) < w_{int}(G_1)$, $W_{int}(G_2) < W_{cyc}(G_2)$. Let us set $G_1 = K_{3,2}$ and $G_2 = K_{2,2}$. In this case, evidently, $w_{cyc}(G_1) = 3$, $w_{int}(G_1) = 4$ [19], $W_{int}(G_2) = 3$ [19], $W_{cyc}(G_2) = 4$.

The problem of cyclically-interval colorability of a graph has been completely investigated as yet only for simple cycles [21, 23] and trees [22]. Some interesting results on this and related topics were obtained in [10, 12, 33, 34].

For a tree H with $V(H) = \{b_1, ..., b_p\}$, $p \geq 1$, we denote by $P(b_i, b_j)$ the simple path connecting the vertices b_i and b_j, $1 \leq i \leq p$, $1 \leq j \leq p$. The sets of vertices and edges of the path $P(b_i, b_j)$ are denoted by $VP(b_i, b_j)$ and $EP(b_i, b_j)$, respectively, $1 \leq i \leq p$, $1 \leq j \leq p$.

Let us also define:

$$intVP(b_i, b_j) \equiv VP(b_i, b_j) \backslash (\{b_i\} \cup \{b_j\});$$

$$\tilde{V}P(b_i, b_j) \equiv VP(b_i, b_j) \cup \left(\bigcup_{x \in intVP(b_i,b_j)} I_H(x) \right);$$

$$TP(b_i, b_j) \equiv \begin{cases} \bigcup\limits_{x \in intVP(b_i,b_j)} J_H(x), & \text{if } intVP(b_i, b_j) \neq \varnothing \\ EP(b_i, b_j), & \text{if } intVP(b_i, b_j) = \varnothing; \end{cases}$$

$$1 \leq i \leq p, 1 \leq j \leq p.$$

Assume:
$$M(H) \equiv \max \left\{ |TP(b_i, b_j)| / 1 \leq i \leq p, 1 \leq j \leq p \right\}.$$

In [19] the following result was obtained.

Theorem 1 (see [19])**.** *Let H be an arbitrary tree. Then*

1. *$H \in \mathfrak{N}$,*

2. *$w_{int}(H) = \Delta(H)$,*

3. *$W_{int}(H) = M(H)$,*

4. *$\theta(H) = [\Delta(H), M(H)]$.*

Corollary 1. *For any tree H, $H \in \mathfrak{M}$, $w_{cyc}(H) = \Delta(H)$, $W_{cyc}(H) \geq M(H)$, $[\Delta(H), M(H)] \subseteq \Theta(H)$.*

In this paper, for any tree H, we show that $W_{cyc}(H) = M(H)$ and $\Theta(H) = [\Delta(H), M(H)]$.

2   Results

Lemma 1. *If $Q_1, ..., Q_n$ ($n \geq 2$) are t-cyclic intervals, and for any $j \in [1, n-1]$, $Q_j \cap Q_{j+1} \neq \varnothing$, then $\bigcup\limits_{i=1}^{n} Q_i$ is a t-cyclic interval.*

Proof can be easily accomplished by induction on n.

Lemma 2. *Let α be a cyclically-interval t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ with a vertex $x_k \in V(G)$, $k \geq 2$. Then $\alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right]$ is a t-cyclic interval.*

Proof. If $k = 2$, then the statement follows from the definition of the cyclically-interval t-coloring. Now assume that $k \geq 3$. It is clear that the sets $\alpha[J_G(x_1)], ..., \alpha[J_G(x_{k-1})]$ are t-cyclic intervals with

$$\alpha[J_G(x_j)] \cap \alpha[J_G(x_{j+1})] \neq \varnothing \text{ for any } j \in [1, k-2].$$

Lemma 1 implies that $\alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right]$ is a t-cyclic interval. $\square$

Lemma 3. *Let α be a cyclically-interval t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ with a vertex $x_k \in V(G)$, $k \geq 2$. Then at least one of the following statements is true:*

1. $intcyc_1((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right]$,

2. $intcyc_2((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right]$.

Proof. Without loss of generality we may assume that $dif((\alpha(e_1), \alpha(e_k)), t) \geq 2$.

Let us assume that none of the statements 1) and 2) is true. Then there are τ_1, τ_2 such that

$$\tau_1 \in intcyc_1((\alpha(e_1), \alpha(e_k)), t), \tau_1 \notin \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right],$$

$$\tau_2 \in intcyc_2((\alpha(e_1), \alpha(e_k)), t), \tau_2 \notin \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right],$$

therefore $\{\tau_1, \tau_2\} \cap \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right] = \varnothing$.

Lemma 2 implies that $\alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right]$ is a t-cyclic interval with

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right].$$

It is not hard to see that the relations

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right] \text{ and } \{\tau_1, \tau_2\} \cap \alpha \left[\bigcup\limits_{i=1}^{k-1} J_G(x_i) \right] = \varnothing$$

are incompatible.   □

Lemma 4. *If α is a cyclically-interval t-coloring of a tree H, $t \in \Theta(H)$, $V(H) = \{b_1, ..., b_p\}$, $p \geq 1$, then there are vertices $b' \in V(H)$, $b'' \in V(H)$ such that $[1, t] = \alpha[TP(b', b'')]$.*

Proof. Assume the contrary. Suppose that for an arbitrary $b_i \in V(H)$, $b_j \in V(H)$, $\alpha[TP(b_i, b_j)] \subset [1, t]$. Set: $\max \left\{ |\alpha[TP(b_i, b_j)]| / 1 \leq i \leq p, 1 \leq j \leq p \right\} \equiv m_0$. It is clear that $m_0 < t$. Without loss of generality we may assume that $m_0 \geq 2$. Consider the simple path $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ of the tree H with $|\alpha[TP_0]| = m_0$. Clearly, without loss of generality, we may assume that $k \geq 2$.

Lemma 2 implies that there are $i' \in [1, t]$, $i'' \in [1, t]$, and $j' \in \{1, 2\}$, for which $\alpha \left[\bigcup\limits_{i=1}^{k-1} J_H(x_i) \right] = intcyc_{j'}[(i', i''), t]$. As $m_0 < t$, there is $\tau_0 \in [1, t]$ such that $\tau_0 \notin intcyc_{j'}[(i', i''), t]$.

Consider an edge $e^1 \in E(H)$ for which $\alpha(e^1) = \tau_0$, and assume that $e^1 = (u_0, u_1)$. Clearly, $e^1 \notin TP_0(x_0, x_k)$.

Without loss of generality we may assume that $\rho_H(u_1, \tilde{V}P_0(x_0, x_k)) < \rho_H(u_0, \tilde{V}P_0(x_0, x_k))$. Let $z_0 \in \tilde{V}P_0(x_0, x_k)$ be the vertex with $\rho_H(u_1, z_0) = \rho_H(u_1, \tilde{V}P_0(x_0, x_k))$. It is not hard to see that $z_0 \in \tilde{V}P_0(x_0, x_k) \setminus intVP_0(x_0, x_k)$ and for any $z' \in \tilde{V}P_0(x_0, x_k) \setminus intVP_0(x_0, x_k)$, $z' \neq z_0$, $\rho_H(u_1, z_0) < \rho_H(u_1, z')$.

Case 1. $z_0 = x_0$. Clearly, $\big|\alpha[TP(u_0, x_k)]\big| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 2. $z_0 = x_k$. This case is considered similarly as the case 1.

Case 3. $z_0 \neq x_0$, $z_0 \neq x_k$.

Clearly, there is $\tilde{x} \in intVP_0(x_0, x_k)$ such that $z_0 \in I_H(\tilde{x})$. Suppose that $\alpha((z_0, \tilde{x})) = \tau'$. Clearly, $i' \neq i''$.

Case 3a. $\tau' = i'$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = i'$, and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1, 2\}$ such that $intcyc_{j_1}[(\tau_0, i'), t] \subseteq$
$$\alpha\left[\bigcup_{x \in intVP(u_0, \tilde{x})} J_H(x)\right].$$ Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality we may assume that $\rho_H(z_0, x') < \rho_H(z_0, x'')$. It is not hard to check that $TP(z_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha[TP(z_0, x'')]$. Lemma 2 implies that $\alpha[TP(z_0, x'')]$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1, 2\}$ such that $\tau_0 \in intcyc_{j_2}((i', i''), t)$, and, therefore, $intcyc_{j_2}((i', i''), t) \nsubseteq \alpha[TP(z_0, x'')]$.

This conclusion, the equalities $\alpha((z_0, \tilde{x})) = i'$, $\alpha(\tilde{e}) = i''$, and Lemma 3 imply that $intcyc_{3-j_2}[(i', i''), t] \subseteq \alpha[TP(z_0, x'')]$, hence $\big|\alpha[TP(u_0, x'')]\big| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 3b. $\tau' = i''$. This case is considered similarly as the case 3a with interchanging of the roles of i' and i''.

Case 3c. $\tau' \notin \{i', i''\}$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = \tau'$, and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1, 2\}$ such that $intcyc_{j_1}[(\tau_0, \tau'), t] \subseteq$
$$\alpha\left[\bigcup_{x \in intVP(u_0, \tilde{x})} J_H(x)\right].$$ This implies that at least one of the following statements is true:

1. $i' \in intcyc_{j_1}[(\tau_0, \tau'), t]$,

2. $i'' \in intcyc_{j_1}[(\tau_0, \tau'), t]$.

Without loss of generality let us assume that the statement 1) is true. Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality we may assume that $\rho_H(z_0, x') < \rho_H(z_0, x'')$. It is not hard to check that

$TP(z_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha[TP(z_0, x'')]$. Lemma 2 implies that $\alpha[TP(z_0, x'')]$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1, 2\}$ such that $\tau_0 \in intcyc_{j_2}((\tau', i''), t)$, and, therefore, $intcyc_{j_2}((\tau', i''), t) \nsubseteq \alpha[TP(z_0, x'')]$. This conclusion, the equalities $\alpha((z_0, \tilde{x})) = \tau'$, $\alpha(\tilde{e}) = i''$, and Lemma 3 imply that $intcyc_{3-j_2}[(\tau', i''), t] \subseteq \alpha[TP(z_0, x'')]$, hence $\left|\alpha[TP(u_0, x'')]\right| \geq m_0 + 1$, which contradicts the choice of P_0. $\qquad\square$

Corollary 2. *If α is a cyclically-interval t-coloring of a tree H, where $t \in \Theta(H)$, then there are vertices $x' \in V(H)$, $x'' \in V(H)$ such that $t \leq |TP(x', x'')|$.*

Proof. Since the inequality $\left|\alpha[TP(x, y)]\right| \leq |TP(x, y)|$ holds for arbitrary vertices $x \in V(H)$, $y \in V(H)$, it is not difficult to notice that our statement follows from Lemma 4. $\qquad\square$

Corollary 3. *If α is a cyclically-interval $W_{cyc}(H)$-coloring of a tree H, then there are vertices $x' \in V(H)$, $x'' \in V(H)$ such that $W_{cyc}(H) \leq |TP(x', x'')|$.*

Corollary 4. *For any tree H, $W_{cyc}(H) \leq M(H)$.*

Theorem 2. *For any tree H, $W_{cyc}(H) = M(H)$.*

Proof follows from Corollaries 1 and 4.

Corollary 5. *[22] Let H be an arbitrary tree. Then*

1. *$H \in \mathfrak{M}$,*

2. *$w_{cyc}(H) = \Delta(H)$,*

3. *$W_{cyc}(H) = M(H)$,*

4. *$\Theta(H) = [\Delta(H), M(H)]$.*

Corollary 6. *For an arbitrary tree H and any positive integer t, $H \in \mathfrak{M}_t$ if and only if $H \in \mathfrak{N}_t$.*

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