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Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic

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Abstract. For cubic differential systems with a bundle of two invariant straight lines and one invariant conic it is proved that a weak focus is a center if and only if the first four Liapunov quantities $L_j,\ j=\overline{1,4}$, vanish.

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1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned}\dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y),\\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y),\end{aligned} \tag{1}$$

in which all variables and coefficients are assumed to be real. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a weak focus. The purpose of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that the origin is a center for system (1) if and only if it has in some neighborhood of $O(0,0)$ a holomorphic integrating factor of the form

$$\mu = 1 + \sum \mu_j(x,y).$$

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

The quantities $L_j,\ j=\overline{1,\infty}$, are polynomials in the coefficients of system (1) and are called the Liapunov quantities. The order of the weak focus $O(0,0)$ is r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ but $L_r \neq 0$.

The origin is a center for (1) if and only if $L_j = 0,\ j=\overline{1,\infty}$. By the Hilbert's basis theorem there exists a natural number N such that the infinite system $L_j = 0,\ j=\overline{1,\infty}$, is equivalent with a finite system $L_j = 0,\ j=\overline{1,N}$. The number N is known only for quadratic systems $N=3$ [13] and for cubic systems with only homogeneous cubic nonlinearities $N=5$ [18, 22]. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see for instance [1, 2, 4, 6–12, 15, 16, 19, 20]).

In this paper we solve the problem of the center for cubic differential system (1) assuming that (1) has two invariant straight lines and one invariant conic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. The results concerning relation between integrability, invariant algebraic curves and Liapunov quantities are presented in Section 2. In Section 3 we find seventeen sufficient sets of conditions for the existence of a bundle of two invariant straight lines and one invariant conic. In Section 4 we obtain sufficient conditions for the existence of a center and finally we give the proof of the main result: a weak focus $O(0,0)$ is a center for cubic system (1) with a bundle of two invariant straight lines and one invariant conic if and only if the first four Liapunov quantities vanish.

2 Invariant algebraic curves, Liapunov quantities, center

An algebraic curve $\Phi(x,y)=0$ (real or complex) is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P\frac{\partial \Phi}{\partial x}+Q\frac{\partial \Phi}{\partial y}=\Phi K.$$

The polynomial K is called the cofactor of the invariant algebraic curve $\Phi=0$. We shall consider only algebraic curves $\Phi=0$ with Φ irreducible.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x,y)=0,\ j=1,\dots,q$, then in most cases an integrating factor can be constructed in the Darboux form

$$\mu=\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}. \tag{2}$$

A function (2), with $\alpha_j\in\mathbb{C}$ not all zero, is an integrating factor for (1) if and only if

$$\sum_{j=1}^{q}\alpha_j K_j\equiv-\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}.$$

System (1) is called Darboux integrable if the system has a first integral or an integrating factor of the form (2).

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These last years, interesting results which relate algebraic solutions, Liapunov quantities and Darboux integrability have been published (see, for example, [3, 5, 6, 9–12, 17, 21]). The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

Definition 1. We shall say that ($\Phi_j,\ j=\overline{1,M};\ L=N$) is *ILC* ($I$ – invariant algebraic curves, L – Liapunov quantities, C – center) for (1) if the existence of M algebraic curves $\Phi_j(x,y)=0$ and the vanishing of the focal values $L_\nu,\ \nu=\overline{1,N}$, implies the origin $O(0,0)$ to be a center for (1).

The works [6–9, 19, 20] are dedicated to the investigation of the problem of the center for cubic differential systems with invariant straight lines. In these papers, the problem of the center was completely solved for cubic systems with at least three invariant straight lines. The principal results of these works are gathered in the following two theorems:

Theorem 1. *$(\Phi_j(x,y),\ \Phi_j(0,0)\neq 0,\ j=\overline{1,4};\ \ L=1)$ is ILC for system (1).*

Theorem 2. *$(a_jx+b_jy+c_j,\ j=\overline{1,4};\ \ L=2)$ and $(a_jx+b_jy+c_j,\ j=\overline{1,3};\ \ L=7)$ are ILC for cubic system (1).*

The problem of the center was solved for cubic systems (1) with two parallel invariant straight lines and one invariant conic [10]; for cubic systems (1) with two homogeneous invariant straight lines and one invariant conic [11] and for a class of cubic systems (1) with a bundle of two invariant straight lines and one invariant conic [12]. The following theorem was proved:

Theorem 3. *$(x\pm iy,\ \Phi;\ \ L=2)$ and $(l_j=1+a_jx+b_jy,\ j=1,2,\ l_1||l_2,\ \Phi;\ \ L=3)$, where $\Phi=0$ is an irreducible invariant conic, are ILC for system (1).*

In this paper we shall prove that $(l_j = 1+a_jx-y,\ j=1,2,\ l_1\cap l_2\cap\Phi = (0,1);\ \ L=4)$, where $\Phi=0$ is an irreducible invariant conic, is ILC for system (1).

3 Conditions for the existence of a bundle of two invariant straight lines and one invariant conic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a point (x_0,y_0). The intersection point (x_0,y_0) is a singular point for (1) and has real coordinates. By rotating the system of coordinates $(x\to x\cos\varphi - y\sin\varphi, y\to x\sin\varphi + y\cos\varphi)$ and rescaling the axes of coordinates $(x\to\alpha x,\ y\to\alpha y)$, we obtain $l_1\cap l_2=(0,1)$. In this case the invariant straight lines can be written as

$$l_j = 1+a_jx-y,\ a_j\in\mathbb{C},\ j=1,2;\ \Delta_{12}=a_2-a_1\neq 0. \tag{3}$$

The straight lines (3) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{array}{l} k=(a-1)(a_1+a_2)+g,\ \ l=-b,\ \ s=(1-a)a_1a_2,\\ m=-a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2,\ \ r=-f-1,\\ n=a_1a_2(-f-2)-(d+1),\ \ p=(f+2)(a_1+a_2)+b-c,\\ q=(a_1+a_2-c)a_1a_2-g,\ \ (a-1)^2+(f+2)^2\neq 0. \end{array} \tag{4}$$

Let the conditions (4) be satisfied and assume that $f=-2$ (the case $f+2\neq 0$ was considered in [12]), then $a\neq 1$ and the cubic system (1) looks:

$$\begin{array}{l} \dot{x} = y+ax^2+cxy-y^2+[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]x^2y+\\ \qquad [(a-1)(a_1+a_2)+g]x^3+(b-c)xy^2+y^3\equiv P(x,y),\\ \dot{y} = -x-gx^2-dxy-by^2+[g+a_1a_2(c-a_1-a_2)]x^2y+\\ \qquad (a-1)a_1a_2x^3+(d+1)xy^2+by^3\equiv Q(x,y). \end{array} \tag{5}$$

Next for cubic system (5) we find conditions for the existence of one invariant conic passing through the same singular point $(0,1)$, i.e. forming a bundle. Let the conic curve be given by the equation

$$\Phi(x,y) \equiv a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0 \tag{6}$$

with $(a_{20}, a_{11}, a_{02}) \neq 0$ and $a_{20}, a_{11}, a_{02}, a_{10}, a_{01} \in \mathbb{R}$.

For every conic curve (6) the following quantities [14]:

$$I_1 = a_{02} + a_{20}, \quad I_2 = (4a_{02}a_{20} - a_{11}^2)/4,$$
$$I_3 = (4a_{02}a_{20} - a_{01}^2 a_{20} + a_{01}a_{10}a_{11} - a_{02}a_{10}^2 - a_{11}^2)/4$$

are invariants with respect to the translation and rotation of axes. These invariants will be taken into account classifying conics. A conic (6) is reducible into two straight lines if and only if $I_3 = 0$. If $I_2 > 0$, then (6) is an ellipse, if $I_2 < 0$ – a hyperbola and if $I_2 = 0$ – a parabola.

In order the conic (6) pass through a singular point $(0,1)$ and form a bundle with the invariant straight lines (3), we shall assume $a_{01} = -a_{02} - 1$. In this case

$$\Phi(x,y) \equiv a_{20}x^2 + a_{11}xy + a_{10}x + (a_{02}y - 1)(y - 1) = 0. \tag{7}$$

The conic (7) is an invariant conic for (5) if and only if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$, where $c_{10} = -a_{01}$, $c_{01} = a_{10}$ such that

$$P(x,y)\frac{\partial \Phi}{\partial x} + Q(x,y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x,y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + (a_{02}+1)x + a_{10}y). \tag{8}$$

Identifying the coefficients of $x^i y^j$ in (8), we reduce this identity to three systems of equations $\{F_{ij} = 0\}$ for the unknowns a_{20}, a_{11}, a_{02}, a_{10}, c_{20}, c_{11}, c_{02} :

$$\begin{aligned}
F_{40} &\equiv (a-1)(a_1a_2a_{11} + 2a_1a_{20} + 2a_2a_{20}) + a_{20}(2g - c_{20}) = 0,\\
F_{31} &\equiv (a-1)(2a_1a_2a_{02} + a_1a_{11} + a_2a_{11}) - (a_2a_{11} + 2a_{20})a_1^2 -\\
&\quad - (a_1a_{11} + 2a_{20})a_2^2 + (ca_{11} - 2a_{20})a_1a_2 + (2ca_1 + 2ca_2 - 2a -\\
&\quad - c_{11} + 2d + 4)a_{20} + (2g - c_{20})a_{11} = 0,\\
F_{22} &\equiv 2(c - a_1 - a_2)a_1a_2a_{02} + (2g - c_{20})a_{02} + [c(a_1 + a_2) - a_1^2 -\\
&\quad - a_2^2 - a_1a_2 - a - c_{11} + 2d + 3]a_{11} + (2b - 2c - c_{02})a_{20} = 0,\\
F_{13} &\equiv (2 + 2d - c_{11})a_{02} + (2b - c - c_{02})a_{11} + 2a_{20} = 0,\\
F_{04} &\equiv (2b - c_{02})a_{02} + a_{11} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
F_{30} &\equiv (a-1)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - ga_{11} +\\
&\quad + (2a - 1 - a_{02})a_{20} + (g - c_{20})a_{10} = 0,\\
F_{21} &\equiv [g - c_{20} + ca_1a_2 - (a_1 + a_2)a_1a_2](-a_{02} - 1) +\\
&\quad + [c(a_1 + a_2) - a_1^2 - a_1a_2 - a_2^2 - a + d + 2 - c_{11}]a_{10} +\\
&\quad + (2c - a_{10})a_{20} + (a - d + 1 + a_{02})a_{11} - 2ga_{02} = 0,\\
F_{12} &\equiv -(d + 1 - c_{11})(a_{02} + 1) - (a_{02} + 2d + 1)a_{02} +\\
&\quad + (b - c - c_{02})a_{10} + (c - b - a_{10})a_{11} - 2a_{20} = 0,\\
F_{03} &\equiv (b - c_{02})(a_{02} + 1) + (a_{10} + 2b)a_{02} - a_{10} + 2a_{11} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned} F_{20} &\equiv (a-a_{02}-1)a_{10}+g(a_{02}+1)-a_{11}-c_{20}=0,\\ F_{11} &\equiv (a_{02}+d+1)a_{01}+(a_{10}-c)a_{10}+2a_{02}-2a_{20}+c_{11}=0,\\ F_{02} &\equiv c_{02}-(a_{10}+b)(a_{02}+1)+2a_{10}-a_{11}=0. \end{aligned} \tag{11}$$

Let us denote

$$j_1=(a_1+a_2-c)a_{02}-a_{11},\ \ j_2=a_{02}a_1^2+a_{11}a_1+a_{20},$$
$$j_3=a_{02}a_2^2+a_{11}a_2+a_{20},\ \ j_4=4a_{02}a_{20}-a_{11}^2.$$

We shall study the compatibility of $\{(9), (10), (11)\}$ when $I_3\neq 0,\ \Delta_{12}\neq 0,\ a\neq 1$ and divide the investigation into five subcases: $\{j_1=0\},\ \{j_1\neq 0, j_2=0\},\ \{j_1j_2\neq 0,\ j_3=0\},\ \{j_1j_2j_3\neq 0, j_4=0\},\ \{j_1j_2j_3j_4\neq 0\}$.

3.1 Case $j_1=0$

In this case $a_{11}=a_{02}(a_1+a_2-c)$. We express c_{02}, c_{11} and c_{20} from (9), then we obtain

$$\begin{aligned} F_{40} &\equiv h_1[a_1a_2(a_1+a_2-c)a_{02}+(3a_1+3a_2-c)a_{20}]=0,\\ F_{31} &\equiv h_1[((a_2-c+3a_1)(2a_2-c)+2a_1^2)a_{02}-2a_{20}]=0, \end{aligned} \tag{12}$$

where $h_1=(a-1)a_{02}+a_{20}$.

3.1.1. $h_1=0$. In this case $a_{20}=(1-a)a_{02}$ and

$$F_{02}\equiv F_{03}=(a_{02}-1)(a_{10}+a_1+a_2+b-c)=0.$$

Let $a_{02}=1$, then express a_{10} from $F_{11}=0$, a_1 from $F_{12}=0$, a_2^2 from $F_{20}+F_{21}=0$ and g from $F_{30}=0$. We obtain the following conditions

1)

$$\begin{aligned} &g=[b^2(4a-d-6)+2bc(1-a)-2a(8a+c^2-4d-24)+\\ &\quad+(c^2-8)(d+4)]/(4b),\ \ 2a(2b+c)-(b+c)(d+4)=0,\\ &2a_2^2+a_2(b-c)-6a+2d+10=0,\ \ a_1=(c-b-2a_2)/2 \end{aligned}$$

for the existence of a conic $2(a-1)x^2+(b+c)xy+(b-c)x-2(y-1)^2=0$.

Assume $a_{02}\neq 1$, then $F_{02}=0$ yields $c=a_1+a_2+a_{10}+b$. Express d and g from (11). If $b=0$, then we obtain the following conditions

2)

$$\begin{aligned} &a=[a_{02}(a_{02}-a_1a_2+1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]/(2a_{02}),\ \ b=0,\\ &c=a_1+a_{10}+a_2,\ \ d=[a_{10}(a_1+a_2-a_{10})-a_1a_2-a_{02}(a_1a_2+2)]/a_{02},\\ &g=[a_{10}^3-3(a_1+a_2)a_{10}^2+a_{10}((2a_1+a_2)(a_1+2a_2)-a_{02}^2+(3a_1a_2+1)a_{02})\\ &\quad+2(a_{02}^2-a_1a_2-(a_1a_2+1)a_{02})(a_1+a_2)]/[2a_{02}(a_{02}-1)] \end{aligned}$$

for the existence of a conic $[a_{02}(a_{02}-a_1a_2-1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]x^2+2(a_{02}y-1)a_{10}x-2(y-1)(a_{02}y-1)=0$.

If $b\neq 0$, then express a from $F_{21}=0$ and we get the following conditions

3)

$$\begin{aligned} &a=[a_{10}(c-3b-2a_{10})+b(c-b)]/(a_{02}-1),\ \ a_1=c-a_{10}-a_2-b,\\ &d=[2a_{10}(c-4b-2a_{10})-a_{02}(a_{02}+2)+3bc-3b^2+3]/(a_{02}-1),\\ &g=[(a_{10}+b)a_1a_2+2(a-1)(a_1+a_2)+(1-a)a_{10}-ba_{02}]/(a_{02}-1),\\ &F_{30}\equiv a_{02}(a_{02}-a_1a_2-1)+a_1a_2+a_{10}(a_{10}-a_1-a_2)=0 \end{aligned}$$

for the existence of a conic

$$[(1-a)x^2-(a_{10}+b)xy+y^2]a_{02}+a_{10}x-(a_{02}+1)y+1=0.$$

3.1.2. $h_1\neq 0$. In this case we express a_{20} from $F_{31}=0$ of (12) and obtain

$$F_{40}\equiv g_1g_2g_3=0,$$

where $g_1=a_1+3a_2-c,\ g_2=a_2+3a_1-c$ and $g_3=2a_1+2a_2-c$.

3.1.2.1. $g_1=0$. If $a_{02}=1$, then we obtain the following conditions

4)

$$a=2,\ d=(-3b^2-4bc-c^2-16)/8,\ g=-b,\ a_1=(c-3b)/4,\ a_2=(b+c)/4.$$

The invariant conic is $(b+c)^2x^2-8(b+c)xy-8(b-c)x+16(y-1)^2=0$.

If $a_{02}\neq 1$, then from $F_{02}=F_{03}=0$ we have $a_{10}=2a_2-b$. We express g from $F_{20}=0$, c from $F_{30}=0$ and reduce the equations $\{F_{21}=0,F_{12}=0\}$ by d from $F_{11}=0$, then $F_{12}\equiv 0$ and $F_{21}\equiv b(a-2)I_3=0$.

If $b=0$, then $F_{11}=0$ yields $a_{02}=-(1+d+2a_2^2)$ and we get the following conditions

5)

$$b=g=0,\quad c=[a_2(2a_2^2+5a+d-4)]/(a-1),\quad a_1=c-3a_2$$

for the existence of a conic

$$a_2(1+d+2a_2^2)(a_2x-2y)x-2a_2x+(1+dy+y+2a_2^2y)(y-1)=0.$$

If $b\neq 0$ and $a=2$, then $F_{11}=0$ yields $a_{02}=-(1+d+ba_2+2a_2^2)$ and we obtain the following conditions

6)

$$a=2,\ c=2a_2^3+ba_2^2+(d+6)a_2-b,\ g=-b,\ a_1=c-3a_2$$

for the existence of a conic

$$(2a_2^2+ba_2+d+1)(a_2x-y)^2+(b-2a_2)x-(2a_2^2+ba_2+d)y-1=0.$$

3.1.2.2. The case $g_2=0$ can be reduced to $g_1=0$ if we replace a_2 by a_1.

3.1.2.3. $g_1g_2\neq 0, g_3=0$. If $a_{02}=1$, then $F_{11}=I_3\neq 0$. Let $a_{02}\neq 1$. In this case we express a_{10} from $F_{02}=F_{03}=0$, g from $F_{20}=0$, a_{02} from $F_{30}=0$, d from $F_{12}=0$ and b from $F_{21}=0$. We obtain

7)

$$b=g=0,\quad c=2(a_1+a_2),\quad d=-(a+2a_1a_2).$$

The invariant conic is

$$a_1a_2(a-1)x^2-(ay-y-1)(a_1x+a_2x-y+1)=0.$$

3.2 Case $j_1\neq 0, j_2=0$

In this case $a_{20}=-a_1(a_{02}a_1+a_{11})$. If $a_{02}=0$, then $F_{04}=j_1\neq 0$. Assume $a_{02}\neq 0$ and express c_{02}, c_{11} and c_{20} from (9), then we obtain

$$F_{40}\equiv F_{31}=u_1u_2u_3=0,$$

where $u_1=2a_{02}a_1+a_{11},\ u_2=a_{02}(a_1+a_2)+a_{11},\ u_3=(a_1a_2-ca_1+a-1)a_{02}^2+(a_2-a_1-c)a_{02}a_{11}-a_{11}^2$.

3.2.1. $u_1=0$, i.e. $a_{11}=-2a_{02}a_1$. If $a_{02}=1$, then $F_{02}\equiv F_{03}\equiv 0$. We express a and a_{10} from (11); a_1, a_2 and d from (10). In this case we get the following conditions

8)
$$a=2,\ d=[2cg-(b+c)^2-2(4g^2+3bg+8)]/8,$$
$$a_1=(b+c)/4,\ a_2=(c+4g+b)/4$$

for the existence of a hyperbola
$$(b+c)^2x^2-8(b+c)xy+8(c-b)x+16(y-1)^2=0.$$

If $a_{02}\neq 1$, then $F_{02}\equiv F_{03}=0$ yields $a_{10}=2a_1-b$. Reduce the equations of (10) by g from $F_{20}=0$ and express d from $F_{11}\equiv F_{12}=0$, a_2 from $F_{30}=0$ and c from $F_{21}=0$. Then we obtain the following conditions

9)
$$c=[a_1(b-bh+5hv-v)+v(v-b-hv)]/(hv),\quad h=a-1,$$
$$d=[a_1v(v-b)-2a_1^2hv-bh^2+bh+bv^2-2hv-v^3]/(hv),$$
$$a_2=[(ab+2ag-b-3g)a_1+gv]/(hv),\ v=b+g$$

for the existence of a hyperbola
$$((2a_1-b)x+1)v-(ab+2g)y-(b-g-ab)(a_1x-y)^2=0.$$

3.2.2. $u_1\neq 0,\ u_2=0$. In this case $a_{11}=-a_{02}(a_1+a_2)$. If $a_{02}=1$, then express a and a_1 from (11); g and a_{10} from $\{F_{30}=0,F_{12}=0\}$. We get the following conditions

$$a=2,\quad g=0,\quad a_1=(b+c-2a_2)/2,\quad 2a_2^2-(b+c)a_2-d-2=0 \tag{13}$$

for the existence of a hyperbola $a_2(b+c-2a_2)x^2-(b+c)xy+(c-b)x+2(y-1)^2=0$.

If $a_{02}\neq 1$, then $F_{02}\equiv F_{03}=0$ yields $a_1=a_{10}-a_2+b$. Reduce the equations of (10) by d from $F_{11}=0$ and express g, a_{02}, a_{10} from $\{F_{20}=0,F_{30}=0,F_{21}=0\}$, respectively. Then we obtain the following conditions

10)
$$f=-2,\quad g=0,\quad a_1=(b+c-2a_2)/2,\quad 2a_2^2-(b+c)a_2-a-d=0$$

for the existence of a hyperbola
$$(a-1)[(a+d)x^2+(b+c)xy]+(b-c)x-2(ay-y-1)(y-1)=0.$$

It is easy to check that (13) are contained in **11)**.

3.2.3. $u_1u_2\neq 0,\ u_3=0$. If $a_{02}=1$, then express a from $u_3=0$ and c,a_{10},d,g from $\{F_{12}=0,F_{11}=0,F_{21}=0,F_{30}=0\}$, respectively. Then we obtain the following conditions

11)
$$a=1-a_{11}^2-(a_1+a_2+b)a_{11}-a_1a_2-ba_1,\quad c=-2a_{11}-b,$$
$$d=(a_1-2a_2-b)a_{11}-a_{11}^2+2a_1^2-a_1a_2-a_2^2-b(a_1+a_2)-2,$$
$$g=a_1[2a_{11}^2+(2a_1+3a_2+2b)a_{11}+2a_1a_2+2ba_1+a_2^2+ba_2+1],$$
$$F_{20}\equiv a_{11}^3+(b+a_2)(2a_{11}+b+a_2)a_{11}+(2a_1+3a_{11}+2a_2+b)\times$$
$$(b+a_{11}+a_2)a_1+b=0$$

for the existence a hyperbola $a_1(a_1+a_{11})x^2+bx-a_{11}x(y-1)-(y-1)^2=0$.

Assume $a_{02}\neq 1$, then express c from $u_3=0$ and b,d,g from (11). If $a_{11}+a_{10}+(a_{02}-1)a_2=0$, then $F_{21}=0$ yields $a_{11}=-a_{02}a_{10}$ and we get the following conditions

12)
$$b=0,\quad c=(2a_2^2-2a_1a_2-a+1)/(a_2-a_1),\quad g=a_1,$$
$$d=(2a_1^3-4a_1^2a_2+2a_1a_2^2+a_2-2aa_1+aa_2)/(a_1-a_2)$$

for the existence of a conic for system (5):

$$[(2a-1)a_1-aa_2](a_1x-a_2x+y)(a_1x-y)+(2aa_1-aa_2-a_2)y-(a_1-a_2)(a_2x+1)=0.$$

If $a_{11}+a_{10}+(a_{02}-1)a_2 \neq 0$, then express a from $F_{21}=0$ and denote $v = a_{10}-a_2$, $s_1 = a_{02}^2+2a_{02}a_1v-v^2$, $s_2 = a_{02}^2a_2+2a_{02}a_1^2v+a_{02}a_1a_2v+a_{02}v-a_1v^2$, then $F_{30} \equiv s_1a_{11}+s_2a_{02}=0$.

If $s_1=0$, then we obtain the following conditions

13)

$$\begin{aligned}
&a = [v^3a_2(1-a_2^4)+v^2(1-a_2^4-2a_2^3z-2a_2z)+4vz(a_2^2-1)+4z^2]/(4va_2^2z),\\
&b = (z-v^2a_2-v)/(va_2),\quad d = [v^3a_2(a_2^2+1)+v^2(2a_2^3z+a_2^2-2a_2z+1)\\
&\quad -vz(a_2^4+a_2^2+4)+2z^2(2-a_2^2)]/(2va_2^2z),\quad a_1=(a_2^2-1)/(2a_2),\\
&c = [v^3a_2(a_2^2+1)+v^2(a_2^2+1)+4vz(a_2^2-1)+2z^2]/(2va_2z),\\
&g = [v^3a_2(1-a_2^4)+v^2(1-a_2^4+2a_2^3z-2a_2z)+2vz(a_2^4+a_2^2-2)\\
&\quad +4z^2(1-a_2^2)]/(4va_2^3z),\quad z=a_{11}+v-a_2^2v,\quad v=a_{10}-a_2
\end{aligned}$$

for the existence a hyperbola $x^2(2a_2^2v-a_2^4v-2a_2^2z-v+2z)+4xya_2(a_2^2v-v+z)+4xa_2(a_2+v)-4(a_2vy+1)(y-1)a_2=0$.

If $s_1 \neq 0$, then express a_{11} from $F_{30}=0$ and obtain

14)

$$\begin{aligned}
&a = [(a_1-a_2)a_{02}^4+(2a_1^2v-3a_1a_2v+a_1-v)a_{02}^3+v(4a_1^2-2a_1^2a_2v-2a_1a_2-\\
&\quad -3a_1v+a_2^2+a_2v)a_{02}^2+v^2(a_1a_2^2+a_1a_2v-4a_1+2a_2+v)a_{02}+v^3]/(s_1s_3),\\
&b = [v^2(v-a_1+a_2)-va_{02}^2+va_{02}(2a_1^2-a_1a_2-2va_1+1)]/s_1,\\
&c = [a_{02}^4+(4a_1a_2+4va_1-2a_2^2-1)a_{02}^3+(4a_1^2(a_1+a_2+v)-3a_1a_2^2-4a_2-\\
&\quad -2v)va_{02}^2+v^2(a_2^2-4a_1^2-6a_1a_2-4va_1)a_{02}+v^3(a_1+2a_2+v)]/(s_1s_3),\\
&d = [a_{02}^4(a_2-2a_1)+a_{02}^3(4a_1^3-6a_1^2a_2-4va_1^2+2a_1a_2^2+2va_1a_2-2a_1+a_2)+\\
&\quad +v(2a_1^2a_2^2-4a_1^3a_2-8a_1^2+5a_1a_2+2va_1-2va_2+2)a_{02}^2+v^2(4a_1^2a_2-\\
&\quad -a_1a_2^2-2va_1a_2+5a_1+a_2)a_{02}+v^3(a_2^2-a_1a_2+va_2-1)]/(s_1s_3),\\
&g = [(a_{02}^2(2a_1-a_2+v)+2va_{02}(a_1v-1)-a_2v^2-v^3)(a_{02}+a_2v)a_1]/(s_1s_3),\\
&v=a_{10}-a_2,\quad s_1=a_{02}^2+2a_{02}a_1v-v^2,\quad s_3=2a_{02}a_1-a_{02}a_2-v.
\end{aligned}$$

The invariant hyperbola is of the form (7) with $a_{10}=a_2+v$, $a_{01}=-a_{02}-1$, $a_{20}=[a_{02}^2a_1(a_{02}(a_2-a_1)+va_1a_2+v)]/s_1$, $a_{11}=(-s_2a_{02})/s_1$.

3.3 Case $j_1 \cdot j_2 \neq 0$, $j_3 = 0$

In this case we also obtain the sets of conditions **8)–14)**.

3.4 Case $j_1 \cdot j_2 \cdot j_3 \neq 0$, $j_4 = 0$

In this case $a_{20}=a_{11}^2/(4a_{02})$, $I_2=0$ and the conic is a parabola. We express c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and a_1 from $F_{31}\equiv F_{40}=0$.

3.4.1. $a_{02}=1$. In this subcase from the equations $F_{11}=0$ and $F_{12}=0$, we get respectively that $a_{10}=a_{11}+c$, $c=-(2a_{11}+b)$. We find g from $F_{30}=0$, d from $F_{21}=0$ and a_2^2 from $F_{20}=0$, then obtain the following conditions

15)

$$\begin{aligned}
&c=-2a_{11}-b,\ \ g=(-a_{11}^2-2ba_{11}-4h^2)/(2a_{11}),\\
&d=(-2ha_{11}^4+b(1-h)a_{11}^3-4h(h+2)a_{11}^2-8bh^2a_{11}-16h^3)/(4ha_{11}^2),\\
&a_1=(-3a_{11}^2-2a_{11}(a_2+b)-4h)/(2a_{11}),\ \ h=a-1,\\
&2ha_{11}^3+a_{11}^2(6a_2h+bh+b)+4a_{11}h(a_2^2+ba_2+h)+8a_2h^2=0
\end{aligned}$$

for the existence of a parabola $a_{11}^2x^2+4a_{11}xy-4a_{11}x-4bx+4(y-1)^2=0$.

3.4.2. $a_{02}\neq 1$. In this case the equations $F_{02}\equiv F_{03}=0$ yields $a_{11}=-(a_{10}+b)a_{02}$. We reduce the equations of (10) by d from $F_{11}=0$ and a_2^2 from $F_{20}=0$. Next we find c from $F_{30}=0$ and denote $u=a_{10}+b,\ h=a-1,\ v=2(b+g)-u$, then the equation $F_{21}=0$ becomes $F_{21}\equiv r_1a_{02}+r_2=0$, where

$$r_1=(u^2+4h)v,\ \ r_2=2buv-8bh^2-4hv-u^2v.$$

Let $r_1=0$. If $u^2+h=0$, then $j_1=0$. Suppose $u^2+h\neq 0$ and $v=0$, then $b=0$ and $F_{20}\equiv f_1f_2=0$, where $f_1=a_2-u,\ f_2=a_{02}u+4ha_2-2hu-u$.

If $f_1=0$ or $f_2=0$, then we get

16)

$$\begin{aligned}
&a=h+1,\ \ b=0,\ \ c=(dg^2+2g^4+8g^2h+2g^2-2h^2)/(2gh),\\
&a_1=[g(d+2g^2+2h+2)]/(2h),\ \ a_2=2g.
\end{aligned}$$

The invariant parabola is $(1+d+2g^2)(gx-y)^2-2gx-(d+2g^2)y-1=0$.

Assume $r_1\neq 0$ and express a_{02} from $F_{21}=0$, then we obtain the following conditions

17)

$$\begin{aligned}
&c=[u^4v(16h^2-bv)+4u^3hv(b+v-bh)+8u^2h(6h^2v-2bh^2-bv^2)\\
&\quad+16uh^2v^2-64h^4v]/[8h^2uv(4h+u^2)],\\
&d=[u^5v(bv-8h^2)-4u^4hv(b+v)+4u^3hv(3bv-8h^2-8h)\\
&\quad+16u^2h^2v(b-2v)+32uh^2(bv^2-4bh^2-4hv)-64h^3v^2]/[16h^2uv(4h+u^2)],\\
&a_1=[u^3v(12h^2-bv)+4u^2hv(v-2a_2h-bh+b)+8uh(6h^2v-2bh^2-bv^2)\\
&\quad+16h^2v(v-2a_2h)]/[8h^2v(4h+u^2)],\\
&u^4v(bv-8h^2)+2u^3v(12a_2h^2-a_2bv-2bh-2hv)\\
&+4u^2h(bv^2-4a_2^2hv-2a_2bhv+2a_2bv+2a_2v^2+8bh^2-8h^2v)\\
&+16uh(bhv-2a_2bh^2-a_2bv^2+6a_2h^2v-hv^2)+32a_2h^2v(v-2a_2h)=0
\end{aligned}$$

for the existence of a parabola $2[((ux-2y+2)(ux-2y)u+8hx)v-4((ux-4y)ux+4(y-1)y)h^2]b-(4h+u^2)(ux-2y+2)^2v=0$

3.5 Case $j_1\cdot j_2\cdot j_3\cdot j_4\neq 0$

In this case we express c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and substitute into the equations $\{F_{40}=0,\ F_{31}=0\}$ of (7). Calculating the resultant of the equations $\{F_{40}=0,\ F_{31}=0\}$ by a we obtain

$$Res(F_{40},F_{31},a)=j_1j_2j_3j_4\neq 0.$$

In this case the system of equations $\{(9),(10),(11)\}$ is not compatible.

Remark 1. For cubic differential system (1) seventeen sets of conditions for the existence of at least two invariant straight lines and one invariant conic passing through the same singular point (0, 1) were obtained.

4 Sufficient conditions for the existence of a center

Lemma 1. *The following eighteen sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$\begin{aligned}&a=[a_{02}(a_{02}-a_1a_2+1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]/(2a_{02}),\quad b=0,\\&c=a_1+a_{10}+a_2,\quad d=[a_{10}(a_1+a_2-a_{10})-a_1a_2-a_{02}(a_1a_2+2)]/a_{02},\\&g=[a_{10}^3-3(a_1+a_2)a_{10}^2+a_{10}((2a_1+a_2)(a_1+2a_2)-a_{02}^2+(3a_1a_2+1)a_{02})\\&\quad+2(a_{02}^2-a_1a_2-(a_1a_2+1)a_{02})(a_1+a_2)]/[2a_{02}(a_{02}-1)];\end{aligned}$$

ii)

$$\begin{aligned}&a=2,\quad b=[(c(9-2c^2)]/[3(c^2+9)],\quad d=[(2(-4c^2-9)]/(c^2+9),\\&g=[c(2c^2-9)]/[3(c^2+9)],\quad a_1=c/3,\quad a_2=0;\end{aligned}$$

iii)

$$b=c=g=0,\quad d=2a_1^2-a,\quad a_2=-a_1;$$

iv)

$$b=g=0,\quad d=a-2,\quad a_2=(1-a)/a_1,\quad 2a_1^2-ca_1-2a+2=0;$$

v)

$$\begin{aligned}&a=2,\quad d=-[b^3g+7b^2g^2+2b^2+14bg^3-8bg+8g^4+8g^2]/(b-2g)^2,\\&c=[(b+2g)a_2]/g,\quad a_1=[2(b+g)g]/(2g-b),\quad a_2=[(b+4g)g]/(2g-b)\\&(b+4g)(b+2g)(b+g)^2bg+(b^2+bg+6g^2)(b-2g)^2=0;\end{aligned}$$

vi)

$$\begin{aligned}&c=[((b+4g)(b+g)-a(b+2g)^2)(g-b)]/[bg(b+g)^2],\\&d=[b^2(1-a)+bg(2a-5)+2g^2(a-1)]/[b(b+g)],\quad a_1=[(b+g)g]/(g-b),\\&bg(b+g)^2+(a-1)(b-g)^2=0,\quad a_2=g[a(b+2g)-2(b+g)]/[(a-1)(g-b)];\end{aligned}$$

vii)

$$a=4g^2+2,\quad b=-3g,\quad c=-5g,\quad d=-2(7g^2+1),\quad a_1=-2g,\ a_2=-g;$$

viii)

$$\begin{aligned}&a=[2(b^2+bg-6g^2)a_1^2-(b^3-19bg^2-18g^3)a_1-6g^2(b+g)^2]/z,\\&c=[a_1(b-bh+5hv-v)+v(v-b-hv)]/(hv),\\&d=[a_1v(v-b)-2a_1^2hv-bh^2+bh+bv^2-2hv-v^3]/(hv),\\&a_2=[a_1(b-bh+2hv-v)+v(v-b)]/(hv),\quad h=a-1,\quad v=b+g,\\&z=2a_1^2(8bv-4v^2-3b^2)+v(5b^2-16bv+10v^2)a_1-(b^2-3bv+2v^2)v^2,\\&(2a_1-v)(2a_1-v-g)z+b(2a_1-b+g)((2b-4g)a_1+4g^2+3bg-b^2)=0;\end{aligned}$$

ix)

$$\begin{aligned}&c=[2(680a-877)]/[b(245a-316)],\quad d=(1178-913a)/(55a-71),\\&g=5b(1-a),\quad a_1=[b(35a-44)]/(5a-8),\quad a_2=[b(100a-129)]/(10a-13),\\&5a^2-8a+2=0,\quad b^2(245a-316)-65a+84=0;\end{aligned}$$

x)

$$\begin{aligned}&a=4/3,\quad c=5b,\quad g=0,\quad 36b^2d+12b^2+9d^2+12d+4=0,\\&a_1=3b-a_2,\quad 6a_2^2-18ba_2-3d-4=0;\end{aligned}$$

xi)
$$c=b,\ d=2(1-a),\ g=0,\ a_1=b-a_2,\ 2a_2^2-2ba_2+a-2=0;$$

xii)
$$\begin{aligned}&d=(6b+2c-7ab-ac)/(5b-c),\ \ a_1=(c+b-2a_2)/2,\\ &g=0,\ \ 36a^2-10ab^2-8abc+2ac^2-96a+5b^2+14bc-3c^2+64=0,\\ &(5b-c)(2a_2^2-(b+c)a_2)+2(ab+ac-3b-c)=0;\end{aligned}$$

xiii)
$$\begin{aligned}&a=(3a_2^2+2ba_2-1)/(2a_2^2),\ \ d=(4ba_2-a_2^4-2ba_2^3-3a_2^2-2)/(2a_2^2),\\ &c=2a_2+b-2a_2^{-1},\ \ g=(a_2^4-2ba_2^3+2ba_2-1)/(2a_2^3),\ \ a_1=(a_2^2-1)/(2a_2);\end{aligned}$$

xiv)
$$\begin{aligned}&a=1-2a_1^2+3a_1a_2-a_2^2,\ \ b=0,\ \ c=3a_2-2a_1,\\ &d=6a_1^2-6a_1a_2+a_2^2-2,\ \ g=a_1;\end{aligned}$$

xv)
$$\begin{aligned}&a=[v^3a_2(1-a_2^4)+v^2(1-a_2^4-2a_2^3z-2a_2z)+4vz(a_2^2-1)+4z^2]/(4va_2^2z),\\ &b=(z-v^2a_2-v)/(va_2),\ \ d=[v^3a_2(a_2^2+1)+v^2(2a_2^3z+a_2^2-2a_2z+1)\\ &\quad -vz(a_2^4+a_2^2+4)+2z^2(2-a_2^2)]/(2va_2^2z),\ \ a_1=(a_2^2-1)/(2a_2),\\ &c=[v^3a_2(a_2^2+1)+v^2(a_2^2+1)+4vz(a_2^2-1)+2z^2]/(2va_2z),\\ &g=[v^3a_2(1-a_2^4)+v^2(1-a_2^4+2a_2^3z-2a_2z)+2vz(a_2^4+a_2^2-2)\\ &\quad +4z^2(1-a_2^2)]/(4va_2^3z),\ \ z=a_{11}+v-a_2^2v,\ v=a_{10}-a_2;\end{aligned}$$

xvi)
$$\begin{aligned}&a=2,\ \ b=[2(a_{11}^2+4)]/[a_{11}(a_{11}^2-4)],\ \ g=[a_{11}(a_{11}^2+4)]/[2(4-a_{11}^2)],\\ &c=[2(a_{11}^4-3a_{11}^2+4)]/[a_{11}(4-a_{11}^2)],\ \ d=[(a_{11}^2+4)(a_{11}^2-2)]/[2(4-a_{11}^2)],\\ &a_1=a_{11}^3/(4-a_{11}^2),\ \ a_2=(-a_{11})/2;\end{aligned}$$

xvii)
$$\begin{aligned}&a=h+1,\ \ b=0,\ \ c=(8g^2-3h)/(2g),\ \ a_2=2g,\\ &d=(-2g^4-2g^2-h^2)/g^2,\ \ a_1=(2g^2-h)/(2g);\end{aligned}$$

xviii)
$$\begin{aligned}&a=(uv+4)/4,\ b=[(u+v)v]/[2(v-u)],\ g=[(u+v)u]/[2(u-v)],\\ &c=((2u-v)^2+4)/[2(u-v)],\ d=[(2u^2-2uv-v^2+12)u]/[4(v-u)],\\ &a_1=u/2,\ a_2=(2u^2-uv+4)/[2(u-v)].\end{aligned}$$

Proof. In each of the cases **i) − xviii)** the system (5) has two invariant straight lines of the form (3) and one invariant conic $\Phi=0$. The system (5) has a Darboux integrating factor of the form
$$\mu=l_1^{\alpha_1}l_2^{\alpha_2}\Phi^{\alpha_3}.$$

In the case i): $\Phi=[a_{02}(a_{02}-a_1a_2-1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]x^2+2(a_{02}y-1)a_{10}x-2(y-1)(a_{02}y-1)=0$ and $\alpha_1=\alpha_2=\alpha_3=-1$.

In the case ii): $\Phi=(4c^2+9)[(cx-3y)^2+3cx]-9y(5c^2+18)+9(c^2+9)$ and $\alpha_1=3,\ \alpha_2=-(10c^2+9)/(4c^2+9),\ \alpha_3=-2(5c^2+18)/(4c^2+9)$.

In the case iii): $\Phi=(a-1)(y-a_1x)(y+a_1x)-(ay-1)$ and $\alpha_1=\alpha_2=(2a_1^2)/(2-a),\ \alpha_3=(2a_1^2-3a+6)/(a-2)$.

In the case iv): $\Phi = (a-1)(2ax^2+cxy-2x^2-2y^2)-cx+2ay-2$ and $\alpha_1=\alpha_2=\alpha_3=-1$.

In the case v): $\Phi=[b(2gx+y)+2g(gx-y)]^2+(8g^3-b^3)x+(b-2g)^2(1-2y)$ and $\alpha_1=3,\ \alpha_2=[-2(b^2+bg+3g^2)]/[b(b+g)],\ \alpha_3=[-3(b+2g)(b-g)]/[(b+g)b]$.

In the case vi): $\Phi=(a-1)[((b+g)gx+2(b-g)y)(ab-b+g)g-(b^2+bg+2g^2)(b-g)]x-bg(b+g)(aby-by-b+gy-g)(y-1)$ and $\alpha_1=2,\ \alpha_2=(b^2-ab^2-2ag^2-bg+2g^2)/[b(ab-b+g)],\ \alpha_3=(3b^2-3ab^2+2ag^2-3bg-2g^2)/[b(ab-b+g)]$.

In the case vii): $\Phi=(a-2)(2(3a-4)(gx+y)-1)x+2g(3ay-4y-2)(y-1)$ and $\alpha_1=3,\ \alpha_2=[2(3-2a)]/(3a-4),\ \alpha_3=(14-11a)/(3a-4)$.

In the case viii): $\Phi=((2a_1-b)x+1)v+(2b-ab-2v)y-(2b-v-ab)(a_1x-y)^2$ and $\alpha_1=3,\ \alpha_2=[2(3a_1g^2-3bg^2-3g^3-u)]/u,\ \alpha_3=[3(2g^3-2a_1g^2+2bg^2-u)]/u$, where $u=(4b-8g)a_1^2-(4b^2-12bg-10g^2)a_1+b^3-4b^2g-7bg^2-2g^3$.

In the case ix): $\Phi=(11874abx-4620ay+6725a-15316bx+5960y-8675)x+5b(338ay-545a-436y+703)(y-1)$ and $\alpha_1=3,\ \alpha_2=(667a-860)/[2(17a-22)],\ \alpha_3=[27(40-31a)]/[2(17a-22)]$.

In the case x): $\Phi=[(3d+4)x+18b(y-2)]x-6(y-1)(y-3)$ and $\alpha_1=(6bd-b-3da_2)/(2a_2-3b),\ \alpha_2=(b+3bd-3da_2)/(2a_2-3b),\ \alpha_3=3d-2$.

In the case xi): $\Phi=(a-1)[(a-2)x-2by]x+2(ay-y-1)(y-1)$ and $\alpha_1=1,\ \alpha_2=1,\ \alpha_3=-4$.

In the case xii): $\Phi=(a-1)[2(ab+ac-3b-c)x-(5b-c)(b+c)y]x-(5b^2-6bc+c^2)x+2(5b-c)(ay-y-1)(y-1)$ and $\alpha_1=[2(7ab+ac-6b-2c)a_2-3ab^2-4abc-ac^2+4b^2+2bc+2c^2]/[(b+c-4a_2)(1-a)(5b-c)],\ \alpha_2=[2(7ab+ac-6b-2c)a_2+2b(b+3c-2ab-2ac)]/[(b+c-4a_2)(1-a)(5b-c)],\ \alpha_3=(ac-17ab+16b)/[(a-1)(5b-c)]$.

In the case xiii): $\Phi=[(a_2^2+2ba_2-1)(a_2^2-1)x-4(a_2^2+ba_2-1)a_2y+4a_2(a_2^2-1)]x+4a_2^2(y-1)^2$ and $\alpha_1=1,\ \alpha_2=(a_2^2+2ba_2-1)/2,\ \alpha_3=(-a_2^2-2ba_2-5)/2$.

In the case xiv): $\Phi=(2a_1-a_2+1)(2a_1-a_2-1)(a_1x-a_2x+y)(a_1x-y)+a_2x+(4a_1^2-4a_1a_2+a_2^2-2)y+1$ and $\alpha_1=2,\ \alpha_2=(2a_1^2)/(4a_1a_2-4a_1^2-a_2^2+1),\ \alpha_3=(12a_1a_2-10a_1^2-3a_2^2+3)/(4a_1^2-4a_1a_2+a_2^2-1)$.

In the case xv): $\Phi=x^2(2a_2^2v-a_2^4v-2a_2^2z-v+2z)+4xya_2(a_2^2v-v+z)+4xa_2(a_2+v)-4(a_2vy+1)(y-1)a_2$ and $\alpha_1=1,\ \alpha_2=(a_2^3v^3+a_2^2v^2-a_2^2vz+a_2v^3-2a_2v^2z+v^2-vz-2z^2)/(2a_2v^2z),\ \alpha_3=(2z^2-a_2^3v^3-a_2^2v^2+a_2^2vz-a_2v^3-4a_2v^2z-v^2+vz)/(2a_2v^2z)$.

In the case xvi): $\Phi=a_{11}(a_{11}^2-4)(a_{11}^2x^2+4a_{11}xy+4(y-1)^2)-4x(a_{11}^4-2a_{11}^2+8)$ and $\alpha_1=(a_{11}^4-2a_{11}^2+8)/[2(a_{11}^2-4)],\ \alpha_2=2,\ \alpha_3=(a_{11}^4+6a_{11}^2-24)/[2(4-a_{11}^2)]$.

In the case xvii): $\Phi=(g^2+h^2)(xg-y)^2+2g^3x-(2g^2+h^2)y+g^2$ and $\alpha_1=-3,\ \alpha_2=(2g^2+h)^2/[2(g^2+h^2)],\ \alpha_3=-(4g^4+4g^2h+g^2+2h^2)/[2(g^2+h^2)]$.

In the case xviii): $\Phi=u(v^2-4)(ux-2y)^2+8(uv-2u^2-v^2)x+4(8u-uv^2-4v)y+16(v-u)$ and $\alpha_1=(4-2u^2-v^2)/(v^2-4),\ \alpha_2=2,\ \alpha_3=(2u^2-3v^2+12)/(v^2-4)$. □

Lemma 2. *The following three sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)
$$b=g=0,\quad d=(9-18a-2c^2)/9,\quad a_2=c/3,\quad a_1=0;$$

ii)
$$a=1-6b^2,\quad c=11b,\quad d=-(54b^2+5)/3,\quad g=0,$$
$$a_1=6b-a_2,\quad 3a_2^2-18ba_2+36b^2+1=0;$$

iii)
$$a = (9-2b^2)/9,\ c = (5b)/3,\ d = (3-2b^2)/3,\ g = (-4b)/3,$$
$$a_1 = (2b-3a_2)/3,\ 9a_2^2 - 6ba_2 + 4b^2 + 27 = 0.$$

Proof. In each of the cases **i)–iii)** the first Liapunov quantities vanish $L_1 = 0$. The system (5) along with invariant straight lines (3) has also one more invariant straight line and one invariant conic.

In the case i): $l_3 = 2(9a+c^2-9)y-9,\ \ \Phi = 2(a-1)(cx-3y)^2+6cx+9y(1-2a)+9.$

In the case ii): $l_3 = 6bx-2y+3,\ \ \Phi = 2b^2(36b^2+1)x^2 - 36b^3xy - 5bx + (6b^2y+1)(y-1).$

In the case iii): $l_3 = 2bx-6y-3,\ \Phi = 2(b^2+9)(2bx-3y)^2-27bx-9(2b^2+9)y-81.$

By Theorem 1 in each of these cases the origin is a center. □

Lemma 3. *The following four sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)
$$a = [(b^2+5bg+2g^2)(2b+g)]/u,\ \ c = [g(4g^2-3b^2+5bg)]/[u(2b+g)],$$
$$d = -(3b^3+11b^2g+16bg^2+6g^3)/u,\ \ (2b+g)u - g^2 = 0,$$
$$u = (b^2+4bg+2g^2)(2b+g),\ \ a_1 = (g^2+bg-b^2)/g,\ \ a_2 = 3b+2g;$$

ii)
$$a = 4/7,\ c = -6b,\ d = (-48)/7,\ g = -3b,\ 7b^2 - 9 = 0,$$
$$a_1 = (-3a_2-14b)/3,\ 7ba_2^2 + 42a_2 + 45b = 0;$$

iii)
$$a = -1,\ c = 16b,\ d = (-37)/7,\ g = (-b)/4,\ 49b^2 - 8 = 0,$$
$$a_1 = (29b-4a_2)/4,\ 196a_2^2 - 1421ba_2 + 380 = 0;$$

iv)
$$a = h+1,\ b = [2(h+u^2)^2(2-u^2)]/[u(h+2)(8h+7u^2+2)],$$
$$c = [2(h+u^2)(19u^2-4h^2-hu^2+10h-6)]/[u(h+2)(8h+7u^2+2)],$$
$$d = (u^6-52hu^2-88h-50u^4-84u^2-24)/[4(h+2)(8h+7u^2+2)],$$
$$a_1 = [2(h+u^2)(13u^2-6h^2-4hu^2+8h-2)]/[u(h+2)(8h+7u^2+2)] - a_2,$$
$$g = [-2b(h+u^2)]/(u^2-2),\ u^4 - 2u^2 + 8h^2 + 8hu^2 = 0,$$
$$[2(8h+7u^2+2)(h+2)^2a_2^2 + (5u^2-2h^2-hu^2+2h-2)(4h+u^2+6)(h+u^2)]u - 4(13u^2-6h^2-4hu^2+8h-2)(h+u^2)(h+2)a_2 = 0.$$

Proof. In each of the cases **i)–iv)** the system (5) along with invariant straight lines (3) has also two more invariant straight lines and one invariant conic:

In the case i): $l_3 = ((2b+g)x-y)(b+g)+2b+g,\ \ l_4 = 2(b(2b+g)x+gy)(b+g) - g(2b+g),\ \Phi = ((2a_1-b)x+1)(b+g) - (ab+2g)y - (b-g-ab)(a_1x-y)^2.$

In the case ii): $l_3 = 2bx+2y-1,\ l_4 = 4bx+12y-3,\ \Phi = 144bx^2 + 288xy + 112by^2 - 189x - 161by + 49b.$

In the case iii): $l_3 = 21bx-12y+28,\ l_4 = 42bx-6y+7,\ \Phi = 27x^2 - 189bxy + 35bx + 54y^2 - 68y + 14.$

In the case iv): $l_3 = 2(20h^2+17hu^2-10h-15u^2-2)(2hx+uy)+u(8h^2+7hu^2+18h+14u^2+4),\ l_4 = 2(34h^3+29h^2u^2-30h^2-36hu^2+8u^2)x+yu(20h^2+17hu^2-10h-15u^2-2)+u(8h^2+7hu^2+18h+14u^2+4),\ \Phi = a_{02}(ux-2y)^2+4(u-b)x-4(a_{02}+1)y+4,$

where $a_{02} = (8bh^2 - 2buv + 4hv + u^2v)/(v(4h + u^2))$, $v = [4(h + u^2)^2(u^2 - 2)]/[(8h + 7u^2 + 2)(h + 2)u]$.
By Theorem 1 in each of these cases the origin is a center. □

Theorem 4. *($l_j = 1 + a_jx - y$, $j = 1, 2$, $l_1 \cap l_2 \cap \Phi = (0, 1)$; $L = 4$), where $\Phi = 0$ is an invariant conic of the form (7) is ILC for system (1), i.e. the order of a weak focus is at most four.*

Proof. To prove the theorem, we compute the first four Liapunov quantities L_j, $j = \overline{1, 4}$, in each of the following sets of conditions **1)–17)** using the algorithm described in [21]. In the expressions for L_j we will neglect denominators and non-zero factors.

In the case **1)** we calculate L_1. If $c = -b$, then $L_1 \equiv b^2 + (d + 4)^2 \neq 0$ and if $c \neq -b$, then $L_1 \equiv (a - 1)[4a^2 + (b + c)^2]I_3 \neq 0$. Therefore the origin is a focus.

In the case **2)** the first Liapunov quantity vanishes. We are in the conditions of Lemma 1, i).

In the case **3)** we have $L_1 = f_1f_2$, where $f_1 = (a_{02} - 1)^2 + (b - c + 2a_{10})^2 \neq 0$ and $f_2 = a_{02} + 2a_{10}^2 + 3ba_{10} - ca_{10} + b^2 - bc - 1$. If $f_2 = 0$, then $a = 1$. Therefore the origin is a focus.

In the case **4)** the vanishing of the first Liapunov quantity gives $c = 3b$. Then $L_2 \neq 0$ and the origin is a focus.

In the case **5)** the vanishing of the first Liapunov quantity gives $d = 1 - 2a_2^2 - 2a$ and we are in the conditions of Lemma 2, i).

In the case **6)** the vanishing of the first Liapunov quantity gives $a_2 = 0$. Then $L_2 = f_1f_2$, where $f_1 = a_{02} - 2$, $f_2 = (a_1^2 + 1)a_{02} - 4a_1^2 - 1$. If $f_1 = 0$, then Lemma 2, i) ($a = 2$), and if $f_2 = 0$, then Lemma 1, ii).

In the case **7)** the first Liapunov quantity is $L_1 = g_1g_2$, where $g_1 = a_1 + a_2$, $g_2 = a_1a_2 - 1 + a$. If $g_1 = 0$, then Lemma 1, iii) and if $g_2 = 0$, then Lemma 1, iv).

In the case **8)** the first Liapunov quantity is $L_1 = (5\beta^3 + 9\beta^2\gamma + 34\beta^2\delta + 3\beta\gamma^2 + 4\beta\gamma\delta + 64\beta\delta^2 - \gamma^3 + 2\gamma^2\delta + 32\delta^3)t^2 + 16(\beta - \gamma - 2\delta)$, where $\beta = b/t$, $\gamma = c/t$, $\delta = g/t$ and t is a real parameter. From $L_1 = 0$ we find t^2 and substituting into the expression for L_2, we obtain $L_2 = f_1f_2f_3$, where $f_1 = \beta^2 + \beta\gamma + 6\beta\delta - 2\gamma\delta + 8\delta^2$, $f_2 = \beta^2 + \beta\gamma + 3\beta\delta - \gamma\delta + 4\delta^2$, $f_3 = 11\beta - \gamma + 4\delta$. If $f_1 = 0$, then Lemma 1, v) and if $f_2 = 0$, then Lemma 1, xii).

Assume $f_1f_2 \neq 0$ and let $f_3 = 0$. Then $\gamma = 11\beta + 4\delta$ and $L_3 = h_1h_2$, where $h_1 = \beta + 2\delta$, $h_2 = 5\beta + \delta$. If $h_1 = 0$, then $\beta = -2\delta$ and $L_1 \equiv 18\delta^2 + 1 \neq 0$, therefore the origin is a focus. If $h_2 = 0$, then $L_3 = 0$, $L_4 \neq 0$ and the origin is a focus.

In the case **9)** we denote $b = \beta t$, $g = \gamma t$, $a_1 = \alpha t$ and calculate the first Liapunov quantity. From $L_1 = 0$ we find t^2 and substituting into the expression for L_2, we obtain $L_2 = f_1f_2f_3$, where $f_1 = \alpha\beta - \alpha\gamma + \beta\gamma + \gamma^2$, $f_2 = 2[(\beta + 3\gamma - (\beta + 2\gamma)a)(2\gamma - \beta)]\alpha^2 + [(\beta^2 - \beta\gamma - 18\gamma^2 - (\beta^2 - 4\beta\gamma - 10\gamma^2)a)(\beta + \gamma)]\alpha - [(\beta + 2\gamma)a - 6\gamma](\beta + \gamma)^2\gamma$, $f_3 = (a\beta + 2a\gamma - \beta - 3\gamma)\alpha - (3a\beta^2 + 5a\beta\gamma + 2a\gamma^2 - 3\beta^2 - 6\beta\gamma - 3\gamma^2)$.

If $f_1 = 0$, then Lemma 1, vi), if $f_2 = 0$ and $\alpha = -2\gamma$, then Lemma 1, vii); if $f_2 = 0$ and $\alpha \neq -2\gamma$, then Lemma 1, viii).

Assume $f_1 f_2 \neq 0$ and $f_3 = 0$. Then express α from $f_3 = 0$ and substituting in L_3 we obtain $L_3 = h_1 h_2$, where $h_1 = 5\beta(a-1) + \gamma,\ h_2 = a\beta^2 + 4a\beta\gamma + 2a\gamma^2 - \beta^2 - 5\beta\gamma - 2\gamma^2$. Let $h_1 = 0$, then $\gamma = 5\beta(1-a)$ and $L_4 = 5a^2 - 8a + 2$. If $L_4 = 0$, then Lemma 1, ix). Let now $h_1 \neq 0$ and $h_2 = 0$, then Lemma 3, i).

In the case **10)** the first Liapunov quantity is $L_1 = 7ab + ac + 5bd - 6b - cd - 2c$. If $L_1 = 0$ and $c = 5b$, then $a = 4/3$ and $L_2 = 36b^2 d + 12b^2 + 9d^2 + 12d + 4$. Let $L_2 = 0$, then Lemma 1, x).

Assume $L_1 = 0$ and $c \neq 5b$, then express d and calculate $L_2 = f_1 f_2 f_3$, where $f_1 = c - b,\ f_2 = 36a^2 - 10ab^2 - 8abc + 2ac^2 - 96a + 5b^2 + 14bc - 3c^2 + 64,\ f_3 = 11b - c$.

If $f_1 = 0$, then Lemma 1, xi) and if $f_2 = 0$, then Lemma 1, xii). Let $f_1 f_2 \neq 0$ and $f_3 = 0$, then $c = 11b$ and $L_3 = 1 - a - 6b^2$. If $L_3 = 0$, then Lemma 2, ii).

In the case **11)** we denote $a_1 = \gamma_1 t,\ a_2 = \gamma_2 t,\ b = \beta t$ and $a_{11} = \gamma_3 t$. From $F_{20} = 0$ we find t^2 and substituting into the expression for L_1, we obtain $L_1 = g_1 g_2 g_3$, where $g_1 = \gamma_1 + \gamma_3,\ g_2 = 2\gamma_1 + \gamma_3 + \beta,\ g_3 = 2\gamma_1^2 + 2\gamma_1\gamma_3 + \gamma_1\gamma_2 - \gamma_3\gamma_2 - \gamma_2^2 - \gamma_2\beta$.

If $g_1 = 0$, then $a = 1$; if $g_2 = 0$, then Lemma 1, xiii) and if $g_3 = 0$, then express β and substituting in L_2, we get $L_2 = 3\gamma_1 - 2\gamma_2$. If $L_2 = 0$, then $L_3 = 2\gamma_1^2 + 19\gamma_1\gamma_3 + 8\gamma_3^2$ and $F_{20} = (49\gamma_1^2 + 56\gamma_1\gamma_3 + 16\gamma_3^2)t^2 + 3$. The system of equations $\{L_3 = 0,\ F_{20} = 0\}$ has no real solutions. In this case the origin is a focus.

In the case **12)** the vanishing of the first Liapunov quantity gives $a = 1 - 2a_1^2 + 3a_1 a_2 - a_2^2$, then Lemma 1, xiv).

In the case **13)** the first Liapunov quantity vanishes, Lemma 1, xv).

In the case **14)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = (a_1 a_2 + 1)v - a_{02}(a_1 - a_2),\ g_2 = a_{02}^3 + (4a_1^2 - 4a_1 a_2 + 2va_1 + a_2^2 + va_2 - 1)a_{02}^2 + v(2a_1 a_2 v - 4a_1 - v)a_{02} - v^2(a_2^2 + a_2 v - 1)$. If $g_1 = 0$, then $a = 1$. Let $g_1 \neq 0,\ g_2 = 0$ and denote $a_{02} = \alpha t,\ v = \beta t$, then express t from $g_2 = 0$ and calculate $L_2 = 3(2\alpha a_1 - \alpha a_2 - \beta)^2 + (\beta a_2 + \alpha)^2$. The equation $L_2 = 0$ has no real solutions.

In the case **15)** the vanishing of the first Liapunov quantity gives $b = [2h^2(2ha_{11}^2 - a_{11}^2 + 4h^2)(a_{11}^2 + 4h)]/[a_{11}(a_{11}^4 - 2h^3 a_{11}^2 - 4h^2 a_{11}^2 + 6ha_{11}^2 - 16h^4)]$. The second one looks $L_2 = f_1 f_2 f_3$, where $f_1 = h - 1,\ f_2 = 2ha_{11}^2 - a_{11}^2 + 4h^2,\ f_3 = (8h^2 - 9h + 3)a_{11}^4 + 2h(5h^3 + 31h^2 - 29h + 9)a_{11}^2 + 16h^4(5h + 1)$.

If $f_1 = 0$, then Lemma 1, xvi) and if $f_1 \neq 0,\ f_2 = 0$, then $I_3 = 0$.

Assume $f_1 f_2 \neq 0$ and $f_3 = 0$. Then we calculate L_3 and the resultant of polynomials f_3 and L_3 by h. We obtain that $Res(f_3, L_3, h) = g_1 g_2 g_3$, where $g_1 = a_{11}^4 + 48a_{11}^2 + 144,\ g_2 = a_{11}^4 + 184a_{11}^2 + 16,\ g_3 = 12a_{11}^4 + 41a_{11}^2 + 36,\ g_4 = 100a_{11}^8 + 1225a_{11}^6 + 4380a_{11}^4 + 3440a_{11}^2 + 576$. It is easy to verify that $g_k = 0,\ k = \overline{1,4}$, have not real solutions and therefore the origin is a focus.

In the case **16)** the vanishing of the first Liapunov quantity gives $d = (-2g^4 - 2g^2 - h^2)/g^2$, then Lemma 1, xvii).

In the case **17)** the first Liapunov quantity is $L_1 = bf_1 - f_2$, where $f_1 = u(32h^3 u - 32h^3 v + 48h^2 v - 8hv^3 - u^2 v^3),\ f_2 = 4hv(4h^2 - 2huv - v^2)(4h + u^2)$.

Let $L_1 = 0$ and assume $f_1 = 0$. Then $L_1 \equiv f_2 = 0$ yields $u = (4h^2 - v^2)/(2hv)$ and $f_1 \equiv 64h^4 - 32h^3v^2 + 12h^2v^2 - v^4 = 0$. The equation $f_1 = 0$ admits the following parametrization $h = [(4\alpha^2 + \beta^2)(16\alpha^2 - \beta^2)]/(32\alpha^2\beta^2)$, $v = [(4\alpha^2 + \beta^2)(16\alpha^2 - \beta^2)]/(32\alpha^3\beta)$.

The vanishing of the second Liapunov quantity gives $b = [(768\alpha^6 - 432\alpha^4\beta^2 - 8\alpha^2\beta^4 + 5\beta^6)(96\alpha^4 - 4\alpha^2\beta^2 + \beta^4)(\beta^2 - 16\alpha^2)]/[256\alpha^3\beta(1152\alpha^6 - 136\alpha^4\beta^2 + 2\alpha^2\beta^4 - \beta^6)(4\alpha^2 - \beta^2)]$. Then $L_3 = g_1g_2$, where $g_1 = 28\alpha^2 - \beta^2$, $g_2 = 3840\alpha^6 - 304\alpha^4\beta^2 - 40\alpha^2\beta^4 + 5\beta^6$. If $g_1 = 0$, then we are in the conditions of Lemma 3, ii). The equation $g_2 = 0$ has not real solutions.

Let now $L_1 = 0$ and assume $f_1 \neq 0$, then $b = f_2/f_1$. The second Liapunov quantity is $L_2 = g_1g_2$, where $g_1 = 4h - uv$, $g_2 = 384h^4 + 96h^3(2u^2 - 3uv - 2) + 16h^2(21uv - 3u^3v - 6u^2 - 2v^2) + 4huv(6u^2 - 13uv - 10v^2 - 24) + u^2v^2(8 - 4u^2 - 5uv)$.

If $g_1 = 0$, then Lemma 1, xviii). Assume $g_1 \neq 0$ and calculate L_3. The resultant of the polynomials g_2 and L_3 by v is

$$Res(g_2, L_3, v) = h_1h_2h_3h_4h_5h_6h_7,$$

where $h_1 = 2h + u^2$, $h_2 = 8h + u^2$, $h_3 = 16h + u^2 - 4$, $h_4 = 8h^2 + 8hu^2 + u^4 - 2u^2$, $h_5 = 64h^2 - 24hu^2 - 96h - 4u^4 - 7u^2 + 36$, $h_6 = 1536h^6 + 768h^5(2u^2 - 1) + 8h^4u^2(63u^2 - 148) + 6h^3u^2(9u^4 - 84u^2 - 64) + 4h^2u^2(72 - 15u^4 - 2u^2) + 18hu^4(u^2 + 4) + u^6(u^2 + 4)$, $h_7 = 800h^6(3u^2 + 4) + 40h^5(15u^4 - 212u^2 - 96) + 8h^4(144 - 245u^4 + 1200u^2) - 2h^3u^2(15u^4 - 1160u^2 + 2352) + 8h^2u^2(10u^4 - 135u^2 + 108) + 2hu^4(84 - 25u^2) + u^6(4 - u^2)$.

Let $h_1 = 0$, then $h = (-u^2)/2$ and the system of equations $\{g_2 = 0,\ L_3 = 0\}$ has no real solutions.

Assume $h_1 \neq 0$, $h_2 = 0$, then $h = (-u^2)/8$ and $g_2 = e_1e_2$, where $e_1 = 3u + 2v$, $e_2 = 3u^3 - 32v$. If $e_1 = 0$, then Lemma 2, iii) and if $e_2 = 0$, then $L_3 \neq 0$.

Let $h_1h_2 \neq 0$, $h_3 = 0$, then $h = (4 - u^2)/16$ and $g_2 = e_1e_2$, where $e_1 \equiv 3(u^2 - 4)^2 + (8v)^2 \neq 0$, $e_2 = 7u^2 + 20uv + 4$. If $e_2 = 0$, then $v = (-7u^2 - 4)/(20u)$ and $L_3 \neq 0$.

Assume $h_1h_2h_3 \neq 0$ and $h_4 = 0$. If $h = -2$, then $h_4 = 0$ yields $u^2 = 2$. In this subcase the system of equations $\{g_2 = 0,\ L_3 = 0\}$ has solutions if $v = -8/(7u)$, then Lemma 3, iii). If $h \neq -2$, the equation $h_4 = 0$ admits the following parametrization $h = (-2\alpha\beta)/(\alpha^2 - 8\alpha\beta + 8\beta^2)$, $u^2 = (16\beta^2)/(\alpha^2 - 8\alpha\beta + 8\beta^2)$. In this case $g_2 = e_1e_2$, where $e_1 = (\alpha^2 - 8\alpha\beta + 8\beta^2)(\alpha - \beta)uv + 8\alpha\beta^2$, $e_2 = 5(\alpha^2 - 8\alpha\beta + 8\beta^2)^2v^2 + 12\alpha(\alpha^2 - 4\alpha\beta + 8\beta^2)(\alpha - 4\beta)$. If $e_1 = 0$, then Lemma 3, iv); if $e_1 \neq 0$, $e_2 = 0$, then reduce $L_3 = 0$ by v^2 from $e_2 = 0$. We express v from $L_3 = 0$, then the equation $e_2 = 0$ has no real solutions.

Let $h_1h_2h_3h_4 \neq 0$. The case $h_5 = 0$ or $h_6 = 0$ implies $f_1 = 0$, in contradiction with assumption that $f_1 \neq 0$. Therefore the origin is a focus.

Assume $h_1h_2h_3h_4h_5h_6 \neq 0$ and $h_7 = 0$. In this case from the system of equations $\{g_2 = 0,\ L_3 = 0\}$ we express v and calculate L_4. The resultant of the polynomials h_7 and L_4 by h is

$$Res(h_7, L_4, h) = e_1e_2e_3e_4e_5e_6e_7e_8e_9,$$

where $e_1 = u - 2$, $e_2 = u + 2$, $e_3 = 3u^2 - 4$, $e_4 = 5u^4 - 20u^3 - 80u^2 - 240u - 144$, $e_5 = 5u^4 + 20u^3 - 80u^2 + 240u - 144$, $e_6 = 5u^2 - 4u + 4$, $e_7 = 5u^2 + 4u + 4$, $e_8 = 15u^4 + 40u^2 + 128$, $e_9 = 300u^6 + 1105u^4 - 1080u^2 + 1296$.

If $e_1 = 0$, then $g_2 = L_3 = 0$ yields $15v^3 + 34v^2 + 36v - 72 = 0$ and $L_4 \neq 0$; if $e_2 = 0$, then $g_2 = L_3 = 0$ yields $15v^3 - 34v^2 + 36v + 72 = 0$ and $L_4 \neq 0$.

If $e_3 = 0$, then $b = 0$ and $a_{02} = 1$, in contradiction with assumption 3.4.2.

If $e_4 = 0$ or $e_5 = 0$, then $f_1 = 0$. The equations $e_6 = 0$, $e_7 = 0$, $e_8 = 0$, $e_9 = 0$ have no real solutions. □

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