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Instability of solutions for nonlinear functional differential equations of fifth order with n-deviating arguments

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Abstract. In this paper, we study the instability properties of solutions of a class of nonlinear functional differential equations of the fifth order with n-constant deviating arguments. By using the Lyapunov-Krasovskii functional approach, we obtain some interesting sufficient conditions ensuring that the zero solution of the equations is unstable.

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1 Introduction

In 1990, Li and Duan [6] proved some instability theorems for the nonlinear differential equation of the fifth order without delay,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x'') + f_2(x') + f_1(x) = 0. \tag{1}$$

Later, in a recent paper, Tunç [15] improved the results obtained for Eq. (1) to the nonlinear differential equation of the fifth order with a constant delay r,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-r), ..., x^{(4)}, x^{(4)}(t-r))x''$$
$$+ f_2(x'(t-r)) + f_1(x(t-r)) = 0.$$

In this paper, instead of these equations, we consider the nonlinear differential equations of the fifth order n-constant deviating arguments τ_i,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-\tau_1), ..., x(t-\tau_n), ..., x^{(4)}, ..., x^{(4)}(t-\tau_n))x''$$

$$+ \sum_{i=1}^{n} g_i(x'(t-\tau_i)) + \sum_{i=1}^{n} h_i(x(t-\tau_i)) = 0. \tag{2}$$

We write Eq. (2) in the system form as follows

$$x' = y, y' = z, z' = w, w' = u,$$

$$
\begin{aligned}
u' \quad = \quad & -f_5(w)u - f_4(z)w - f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n))z \\
& -\sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} h_i(x) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \\
& +\sum_{i=1}^{n} \int_{t-\tau_i}^{t} h_i'(x(s))y(s)ds, \qquad (3)
\end{aligned}
$$

where τ_i are positive constants, n – fixed deviating arguments, the primes in Eq. (2) denote differentiation with respect to $t, t \in \Re_+, \ \Re_+ = [0, \infty); f_5, \quad f_4, f_3, g_i$ and h_i are continuous functions on $\Re, \Re, \Re^{2n+2}, \Re$and $\Re$, respectively, with $h_i(0) = g_i(0) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Èl'sgol'ts [1], pp. 14, 15). We assume in what follows that the functions f_i and g_i are also differentiable, and $x(t), \ y(t), \ z(t), \ w(t)$ and $u(t)$are abbreviated as $x, \ y, \ z, \ w$ and u, respectively.

To the best of our knowledge from the literature, so far, the instability of solutions for nonlinear differential equations of the fifth order with multiple deviating arguments has not been investigated. However, since 1978 up to now, the instability of solutions of various nonlinear scalar and vector differential equations of the fifth order without or with a delay has been investigated and is still being studied by researchers. In particular, for some results proceeded on this topic related to these type equations, the reader can refer to the papers of Ezeilo [2]-[4], Li and Duan [6], Li and Yu [7], Sadek [8], Sun and Hou [9], Tiryaki [10], Tunç [11]-[17], Tunç and Erdogan [18], Tunç and Karta [19], Tunç and Sevli [20]. In all these papers, the authors used some suitable Lyapunov functions or functionals as basic tool to achieve their proposed goal in the works. They also based on the Krasovskii's properties (see Krasovskii [5]) to study the instability of solutions of the equations considered therein. In this paper, we employ the Lyapunov-Krasovskii functional approach to investigate the subject for Eq. (2) by defining two new appropriate Lyapunov functionals. In fact, when we take into consideration the differential equations of the fifth order discussed in the above mentioned papers and the literature, it can be seen that all the equations studied there do not include or include only a deviating argument. However, this paper includes n-deviating arguments and is a continuation of the instability results related to the scalar nonlinear differential equations of the fifth order mentioned above (see Ezeilo [2]-[4], Li and Duan [6], Li and Yu [7], Sun and Hou [9], Tiryaki [10], Tunç [14]-[17]). The researches related to the instability of solutions are also very important in the theory and applications of differential equations, and the investigation of this topic for nonlinear differential equations of the fifth order with multiple deviating arguments takes an important place for the researchers work in this area. This work makes a contribution to the existing studies made in the literature.

Let $r \geqslant 0$ be given, and let $C = C([-r, 0],\ \Re^n)$ with

$$\|\phi\| = \max_{-r \leqslant s \leqslant 0} |\phi(s)|\,,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r,\ A) \to \Re^n$ is continuous, $0 < A \leqslant \infty$,then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leqslant s \leqslant 0, \quad t \geqslant 0.$$

Let Gbe an open subset of $C$and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), \quad x_t = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \quad t \geqslant 0,$$

where $F(0) = 0, F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on Fthat each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G,$$

has a unique solution defined on some interval $[0, A), 0 < A \leqslant \infty$. This solution will be denoted by $x(\phi)(.)$so that $x_0(\phi) = \phi$.

Definition 1. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geqslant 0$. The zero solution is said to be unstable if it is not stable.

2 Main results

The first result of this paper is the following theorem.
Let $\tau = \max \tau_i,\ (i = 1, 2, ..., n)$.

Theorem 1. *In addition to all the assumptions imposed on the functions* f_5, f_4, f_3, g_i *and* h_i *appearing in Eq. (2), we assume that there exist positive constants* $a_3, \bar{b}_i$, b_i *and* c_i *such that the following conditions hold:*

$$h_i(0) = g_i(0) = 0, \quad h_i(x) \neq 0, \quad (x \neq 0), g_i(y) \neq 0, \quad (y \neq 0),$$

$$-\bar{b}_i \leqslant h_i'(x) \leqslant -b_i, \quad 0 \leqslant \left|g_i'(y)\right| \leqslant c_i, f_5(w) \leqslant 0$$

for all $x,\ y,\ w$ *and*

$$f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n)) \geqslant a_3$$

for all $x, ...,\ u,\ ...,u(t-\tau_n)$.

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i+2c_i)}\right\},$$

then the zero solution of Eq. (2) is unstable.

Proof. Define the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$:

$$V = \frac{1}{2}w^2 - zu - z\int\limits_0^w f_5(s)ds - \int\limits_0^z f_4(s)sds - \int\limits_0^y \sum_{i=1}^n g_i(s)ds$$

$$-y\sum_{i=1}^n h_i(x) - \sum_{i=1}^n \lambda_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t y^2(\theta)d\theta ds - \sum_{i=1}^n \mu_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t z^2(\theta)d\theta ds, \qquad (4)$$

where s is a real variable such that the integrals $\sum\limits_{i=1}^n \lambda_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t y^2(\theta)d\theta ds$ and $\sum\limits_{i=1}^n \mu_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t z^2(\theta)d\theta ds$are non-negative, and $\lambda_i$and $\mu_i$are some positive constants which will be determined later in the proof.

It is clear that

$$V(0,0,0,\varepsilon,0) = \frac{1}{2}\varepsilon^2 > 0$$

for all sufficiently small ε. Hence, in every neighborhood of the origin,$(0,0,0,0,0)$, there exists a point $(0,0,0,\varepsilon,0)$ such that $V(0,0,0,\varepsilon,0) > 0$,which shows that V has the property (K_1), (see [5]).

By a direct computation from (3) and (4), we obtain

$$\frac{d}{dt}V = -\sum_{i=1}^n h_i'(x)y^2 + f_3(x, ..., x(t-\tau_n), u, ..., u(t-\tau_n))z^2$$

$$-w\int\limits_0^w f_5(s)ds - z\sum_{i=1}^n \int\limits_{t-\tau_i}^t g_i'(y(s))z(s)ds$$

$$-z\sum_{i=1}^n \int\limits_{t-\tau_i}^t h_i'(x(s))y(s)ds - \sum_{i=1}^n (\lambda_i\tau_i)y^2 - \sum_{i=1}^n (\mu_i\tau_i)z^2$$

$$+\sum_{i=1}^n \lambda_i \int\limits_{t-\tau_i}^t y^2(s)ds + \sum_{i=1}^n \mu_i \int\limits_{t-\tau_i}^t z^2(s)ds.$$

The assumptions of Theorem 1 and the estimate $2|mn| \leqslant m^2+n^2$ imply

$$-\sum_{i=1}^{n} h_i'(x)y^2 \geqslant \sum_{i=1}^{n} b_i y^2,$$

$$f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))z^2 \geqslant a_3 z^2,$$

$$\begin{aligned}
-z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} h_i'(x(s))y(s)ds &\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|h_i'(x(s))\right| \ |y(s)|\,ds \\
&\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \bar{b}_i \ |y(s)|\,ds \\
&\geqslant -\frac{1}{2}\sum_{i=1}^{n}(\bar{b}_i\tau_i)z^2 - \frac{1}{2}\sum_{i=1}^{n}\bar{b}_i\int_{t-\tau_i}^{t} y^2(s)ds
\end{aligned}$$

and

$$\begin{aligned}
-z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds &\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|g_i'(y(s))\right| \ |z(s)|\,ds \\
&\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} c_i \ |z(s)|\,ds \\
&\geqslant -\frac{1}{2}\sum_{i=1}^{n}(c_i\tau_i)z^2 - \frac{1}{2}\sum_{i=1}^{n}c_i\int_{t-\tau_i}^{t} z^2(s)ds
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}V &\geqslant \sum_{i=1}^{n}(b_i-\lambda_i\tau_i)y^2 + \{a_3 - 2^{-1}\sum_{i=1}^{n}(c_i+\bar{b}_i+2\mu_i)\tau_i\}z^2 \\
&+\sum_{i=1}^{n}(\lambda_i-2^{-1}\bar{b}_i)\int_{t-\tau_i}^{t} y^2(s)ds + \sum_{i=1}^{n}(\mu_i-2^{-1}c_i)\int_{t-\tau_i}^{t} z^2(s)ds.
\end{aligned}$$

Let $\lambda_i = \frac{1}{2}\bar{b}_i,\quad \mu_i=\frac{1}{2}c_i$ and $\tau = \max\tau_i,\ (i=1,2,...,n)$. Hence, we have

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i-2^{-1}\bar{b}_i\tau)y^2 + \{a_3-2^{-1}\sum_{i=1}^{n}(\bar{b}_i+2c_i)\tau\}z^2.$$

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i+2c_i)}\right\},$$

then

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - 2^{-1}\bar{b}_i\tau)y^2 + \{a_3 - 2^{-1}\sum_{i=1}^{n}(\bar{b}_i + 2c_i)\tau\}z^2 > 0,$$

which verifies that V has the property (K_2), (see [5]).

On the other hand, $\dfrac{d}{dt}V = 0$ if and only if $y = z = 0$, which implies that

$$y = z = w = u = 0.$$

Besides, by $h_i(0) = g_i(0) = 0, h_i(x) \neq 0$ for all $x \neq 0, g_i(y) \neq 0$for all $y \neq 0$and the system (3), we can conclude that $\frac{d}{dt}V = 0$ if and only if $x = y = z = w = u = 0$. Thus, the property (K_3), (see [5]), holds. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of Theorem 1 is completed. □

Example 1. We consider the nonlinear differential equation of the fifth order with two deviating arguments,

$$x^{(5)} - \frac{1}{1+(x''')^4}x^{(4)} + 9x''' + \{2 + \exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2))\}x''$$

$$+ \sin x'(t-\tau_1) + \sin x'(t-\tau_2) - x(t-\tau_1) - x(t-\tau_2)$$

$$-4arctgx(t-\tau_1) - arctgx(t-\tau_2) = 0. \tag{5}$$

We write Eq. (5) in system form as follows

$$x' = y, \quad y' = z, z' = w, w' = u,$$

$$u' = \frac{u}{1+w^4} - 9w - \{2 + \exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\}z$$

$$-2\sin y + 2x + 5arctgx - \int_{t-\tau_1}^{t} y(s)ds - \int_{t-\tau_2}^{t} y(s)ds$$

$$+ \int_{t-\tau_1}^{t} \cos y(s)z(s)ds + \int_{t-\tau_2}^{t} \cos y(s)z(s)ds$$

$$-4\int_{t-\tau_1}^{t} \frac{1}{1+x^2(s)}y(s)ds - \int_{t-\tau_2}^{t} \frac{1}{1+x^2(s)}y(s)ds.$$

It follows that Eq. (5) is a special case of Eq. (2) and

$$f_5(w) = -\frac{1}{1+w^4} \leqslant 0,$$

$$f_4(z) = 9,$$
$$f_3(.) = 2 + \exp\{-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\} \geqslant 2 = a_3,$$
$$f_2(y) = \sin y, -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2},$$
$$f_2(0) = 0, g_1'(y) = \cos y, |\cos y| \leqslant 1 = c_1,$$
$$g_2(y) = \sin y, -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2},$$
$$g_2(0) = 0, g_2'(y) = \cos y, |\cos y| \leqslant 1 = c_2,$$
$$h_1(x) = x + 4arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$
$$h_1(0) = 0, h_1'(x) = 1 + \frac{4}{1+x^2},$$
$$\bar{b}_1 = 5 \geqslant 1 + \frac{4}{1+x^2} \geqslant 1 = b_1,$$
$$h_2(x) = x + arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$
$$h_2(0) = 0, h_2'(x) = 1 + \frac{1}{1+x^2},$$
$$\bar{b}_2 = 2 \geqslant 1 + \frac{1}{1+x^2} \geqslant 1 = b_2,$$
$$\tau < 2\min\{\frac{b_i}{\bar{b}_i}, \ \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i + 2c_i)}\} = \frac{4}{11}.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 1 hold. Hence, if $\tau < \frac{4}{11}$, then the zero solution of Eq. (5) is unstable.

Second, we consider the special case of Eq. (2) with $g_i(x'(t-\tau_i)) = f_2(x')$, namely, the differential equation of the fifth order n-constant deviating arguments τ_i,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-\tau_1), ..., x(t-\tau_n), ..., x^{(4)}, ..., x^{(4)}(t-\tau_n))x''$$

$$+f_2(x') + \sum_{i=1}^{n} h_i(x(t-\tau_i)) = 0. \tag{6}$$

We write Eq. (6) in the system form as follows

$$x' = y, \quad y' = z, \quad z' = w, w' = u,$$

$$u' \quad = \quad -f_5(w)u - f_4(z)w - f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n))z$$

$$-f_2(y)-\sum_{i=1}^{n}h_i(x)+\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t}h_i'(x(s))y(s)ds. \qquad (7)$$

The second result of this paper is the following theorem.

Let $\tau=\max\tau_i,\ (i=1,2,...,n)$.

Theorem 2. *In addition to all the assumptions imposed to the functions* $f_5,\quad f_4,f_3,f_2$ *and* h_i *that appearing in Eq. (6), we assume that there exist positive constants* a_3,b_i *and* $\bar{b}_i$ *such that the following conditions hold:*

$$h_i(0)=f_2(0)=0,\quad h_i(x)\neq 0,\quad (x\neq 0),f_2(y)\neq 0,\quad (y\neq 0),$$

$$\bar{b}_i\geqslant h_i'(x)\geqslant b_i, f_5(w)\geqslant 0$$

for arbitrary $x,\ y,\ w$ *and*

$$f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))\leqslant -a_3$$

for all $x,...,\ u,\ ...,u(t-\tau_n)$.

If

$$\tau<2\min\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}\bar{b}_i}\},$$

then the zero solution of Eq. (6) is unstable.

Proof. Define the Lyapunov functional $V_1=V_1(x_t,y_t,z_t,w_t,u_t):$

$$V_1=\quad -\frac{1}{2}w^2+y\sum_{i=1}^{n}h_i(x)+zu+z\int\limits_0^w f_5(s)ds$$

$$+\int\limits_0^z f_4(s)sds+\int\limits_0^y f_2(s)ds-\sum_{i=1}^{n}\gamma_i\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t}y^2(\theta)d\theta ds, \qquad (8)$$

where s is a real variable such that the integrals $\sum\limits_{i=1}^{n}\gamma_i\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t}y^2(\theta)d\theta ds$ are non-negative, and γ_i are positive constants which will be determined later in the proof.

Let $M=\max\limits_{|z|\leqslant 1}|f_4(z)|$, there exists a positive constant $e$such that $Me<1$ and $0<e<1$.

Then, it follows that

$$V_1(0,0,e^2,0,e)=e^3+\int\limits_0^{e^2}f_4(s)sds\geqslant e^3-\frac{1}{2}Me^4>0$$

for all sufficiently small e. Hence, in every neighborhood of the origin,$(0,0,0,0,0)$, there exists a point $(0,0,e^2,0,e)$ such that $V_1(0,0,e^2,0,e)>0$.

By an elementary differentiation, time derivative of the functional V_1 in (8) along the solutions of (7) yields

$$\frac{d}{dt}V_1=\sum_{i=1}^{n}h_i'(x)y^2-f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))z^2$$

$$+w\int\limits_0^w f_5(s)ds+z\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t}h_i'(x(s))y(s)ds$$

$$-\sum_{i=1}^{n}(\gamma_i\tau_i)y^2+\sum_{i=1}^{n}\gamma_i\int\limits_{t-\tau_i}^{t}y^2(s)ds.$$

The assumptions $\bar{b}_i\geqslant h_i'(x)\geqslant b_i$, $f_3(.)\leqslant -a_3$and the estimate $2\left|mn\right|\leqslant m^2+n^2$ imply that

$$\sum_{i=1}^{n}h_i'(x)y^2\geqslant\sum_{i=1}^{n}b_iy^2,$$

$$-f_3(x,...,u(t-\tau_n))z^2\geqslant a_3z^2,$$

$$z\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t}h_i'(x(s))y(s)ds\geqslant-\left|z\right|\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t}\left|h_i'(x(s))\right|\ \left|y(s)\right|ds$$

$$\geqslant-\left|z\right|\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t}\bar{b}_i\ \left|y(s)\right|ds$$

$$\geqslant-\frac{1}{2}\sum_{i=1}^{n}(\bar{b}_i\tau_i)z^2-\frac{1}{2}\sum_{i=1}^{n}\bar{b}_i\int\limits_{t-\tau_i}^{t}y^2(s)ds$$

so that

$$\frac{d}{dt}V\geqslant\sum_{i=1}^{n}(b_i-\gamma_i\tau_i)y^2+(a_3-2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau_i)z^2$$

$$+\sum_{i=1}^{n}(\gamma_i-2^{-1}\bar{b}_i)\int\limits_{t-\tau_i}^{t}y^2(s)ds.$$

Let $\gamma_i=\frac{1}{2}\bar{b}_i$ and $\tau=\max\tau_i,\ (i=1,2,...,n)$.Hence

$$\frac{d}{dt}V\geqslant\sum_{i=1}^{n}(b_i-2^{-1}\bar{b}_i\tau)y^2+(a_3-2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau)z^2.$$

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}\bar{b}_i}\right\},$$

then

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - 2^{-1}\bar{b}_i\tau)y^2 + (a_3 - 2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau)z^2 > 0.$$

The remainder of the proof follows as before, Theorem 1. □

Example 2. We consider nonlinear differential equation of the fifth order with two deviating arguments,

$$x^{(5)} + \frac{1}{1+(x''')^4}x^{(4)} + x''' - \{3 + \exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\}x''$$

$$+x'(t) - x(t-\tau_1) - 4arctgx(t-\tau_1)$$

$$-x(t-\tau_2) - arctgx(t-\tau_2) = 0. \tag{9}$$

We write Eq. (9) in system form as follows

$$x' = \quad y, \quad y' = \quad z, z' = \quad w, w' = \quad u,$$

$$u' = \quad -\frac{u}{1+w^4} - w + \{3 + \exp(-x^2 - x^2(t-\tau_1) - u^2(t-\tau_2)\}z$$

$$-2y + x + 5arctgx$$

$$-\int_{t-\tau_1}^{t} y(s)ds - 4\int_{t-\tau_1}^{t} \frac{1}{1+x^2(s)}y(s)ds$$

$$-\int_{t-\tau_2}^{t} y(s)ds - \int_{t-\tau_2}^{t} \frac{1}{1+x^2(s)}y(s)ds.$$

It follows that Eq. (9) is a special case of Eq. (2) and

$$f_5(w) = \frac{1}{1+w^4} \geqslant 0,$$

$$f_4(z) = 1,$$

$$f_3(.) = -3 - \exp\{-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\} \leqslant -3 = -a_3,$$

$$f_2(y) = y, f_2(0) = 0,$$

$$h_1(x) = x + 4arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_1'(x) = 1 + \frac{4}{1+x^2},$$

$$\bar{b}_1 = 5 \geqslant 1 + \frac{4}{1+x^2} \geqslant 1 = b_1,$$

$$h_2(x) = x + arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_2'(x) = 1 + \frac{1}{1+x^2},$$

$$\bar{b}_2 = 2 \geqslant 1 + \frac{1}{1+x^2} \geqslant 1 = b_2.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 2 hold. Hence, we conclude that if $\tau < \frac{2}{5}$,then the zero solution of Eq. (9) is unstable.

References

[1] Èl'sgol'ts L. È. *Introduction to the theory of differential equations with deviating arguments.* Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1966.

[2] Ezeilo J. O. C. *Instability theorems for certain fifth-order differential equations.* Math. Proc. Cambridge Philos. Soc., 1978, **84**, No. 2, 343–350.

[3] Ezeilo J. O. C. *A further instability theorem for a certain fifth-order differential equation.* Math. Proc. Cambridge Philos. Soc., 1979, **86**, No. 3, 491–493.

[4] Ezeilo J. O. C. *Extension of certain instability theorems for some fourth and fifth order differential equations.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 1979, **66**, No. 4, 239–242.

[5] Krasovskii N. N. *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay.* Translated by J. L. Brenner Stanford University Press, Stanford, Calif. 1963.

[6] Li Wen-jian, Duan Kui-chen. *Instability theorems for some nonlinear differential systems of fifth order.* J. Xinjiang Univ. Natur. Sci., 2000, **17**, No. 3, 1–5.

[7] Li W. J., Yu Y. H. *Instability theorems for some fourth-order and fifth-order differential equations.* (Chinese) J. Xinjiang Univ. Natur. Sci., 1990, **7**, No. 2, 7–10.

[8] Sadek A. I. *Instability results for certain systems of fourth and fifth order differential equations.* Appl. Math. Comput., 2003, **145**, No. 2-3, 541–549.

[9] Sun W. J., Hou X. *New results about instability of some fourth and fifth order nonlinear systems.* (Chinese) J. Xinjiang Univ. Natur. Sci., 1999, **16**, No. 4, 14–17.

[10] Tiryaki A. *Extension of an instability theorem for a certain fifth order differential equation.* National Mathematics Symposium (Trabzon, 1987), J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math.-Phys, 1988, **11**, 225–227.

[11] Tunç C. *On the instability of solutions of certain nonlinear vector differential equations of fifth order.* Panamer. Math. J., 2004, No. 4, 25–30.

[12] Tunç C. *An instability result for a certain non-autonomous vector differential equation of fifth order.* Panamer. Math. J., 2005, **15**, No. 3, 51–58.

[13] Tunç C. *Further results on the instability of solutions of certain nonlinear vector differential equations of fifth order*. Appl. Math. Inf. Sci., 2008, **2**, No. 1, 51–60.

[14] Tunç C *Recent advances on instability of solutions of fourth and fifth order delay differential equations with some open problems*, World Scientific Review, **9**, World Scientific Series on Nonlinear Science Series B (Book Series), 2011, 105–116.

[15] Tunç C. *On the instability of solutions of some fifth order nonlinear delay differential equations*. Appl. Math. Inf. Sci., AMIS, 2011, **5**, No. 1, 112–121.

[16] Tunç C. *An instability theorem for a certain fifth-order delay differential equation*. Filomat., 2011, **25**, No. 3, 145–151.

[17] Tunç C. *On the instability of solutions of nonlinear delay differential equations of fourth and fifth order*. Sains Malaysiana, 2011, **40**, No. 12, 1455–1459.

[18] Tunç C., Erdogan F. *On the instability of solutions of certain non-autonomous vector differential equations of fifth order*. SUT J. Math. 2007, **43**, No. 1, 35–48.

[19] Tunç C., Karta M. *A new instability result to nonlinear vector differential equations of fifth order*. Discrete Dyn. Nat. Soc. 2008, Art. ID 971534, 6 p.

[20] Tunç C., Sevli H. *On the instability of solutions of certain fifth order nonlinear differential equations*. Mem. Differential Equations Math. Phys., 2005, **35**, 147–156.

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