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The stochastic optimal growth problem

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Abstract. This paper concerns the formulation of economic optimal control problem in a stochastic form. Equilibrium growth rate for this problem was obtained on the base of the stochastic maximum principle following the new approach [1] to the solution of optimal control stochastic problem, in which the stochastic dynamic programming formulation is transformed into formulation of the maximum principle. This approach was applied to the solution of the stochastic optimal growth problem.

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1 Problem formulation

Let us consider optimal control problem [2,3] in stochastic formulation

$$\max_{(C,L_Y,L_R)} L \;=\; E\left[\int_0^\infty e^{(\beta_k-\rho)t}\frac{C^{1-\vartheta}}{1-\vartheta}dt\right],$$

subject to

$$\dot{K} = Y - C = K^\alpha A L_Y^{1-\alpha} N^{1-\alpha} - C, \quad K(0) = 0, \tag{1}$$

$$dN = b_1 L_R N dt + g dz, \quad N(0) = 0, \tag{2}$$

$$L_Y + L_R - L = 0, \tag{3}$$

here E is an expectation operator $x = (K, N)$ and $F = e^{(\beta_1-\rho)t}\dfrac{C^{1-\vartheta}}{1-\vartheta}$, the utility function with constant elasticity of substitution ϑ, ρ is the subjective rate of discount, β_k is the subsidy for capital accumulation stimulation. K is the capital stock observed in economic activity, N is the stock of innovation elaborated by $R\&D$ sector, A is the productivity parameter in the final goods production sector, L_Y is the labor force enrolled in the final goods production sector, L_R is the number of employers in $R\&D$ sector, C is the final consumption.

$$dN \;=\; b_1 L_R N dt \;+\; g dz \tag{4}$$

here dz is the stochastic Wiener process, $f = (Y - C, b_1 L_R N)$, $g = \sigma$ is a constant while gdz is normally distributed with mean zero E[gdz]=o, Var(gdz)= $\sigma_z^2 dt$, $dz = \sqrt{dt}$.

2 Problem solution

The respective optimality conditions now are:

$$0 \ = \ \max_{C,L_Y,L_R}\left[F+\frac{E(dL)}{dt}\right] \tag{5}$$

and the corresponding HJB (Hamilton − Jacoby − Bellman) equation becomes:

$$0 \ = \ \max_{C,L_Y,L_R}\left[F+\frac{\partial L}{\partial t}+\frac{\partial L}{\partial x}f+\frac{1}{2}\frac{g^2\partial^2 L}{(\partial x)^2}\right]=$$

$$=\max_{C,L_Y,L_R}\left[F\ +\ L_t+L_x f+\frac{1}{2}g^2 L_{xx}\right]. \tag{6}$$

The Hamiltonian function H for the stochastic case is presented below:

$$H \ = \ F \ + \ L_x f+\tfrac{1}{2}g^2 L_{xx} = e^{(\beta_1-\rho)t}\frac{C^{1-\vartheta}}{1-\vartheta}+$$

$$+L_{x_1}\left(K^{\alpha}AL_Y^{1-\alpha}N^{1-\alpha}-C\right)+L_{x_2}b_1L_RN+\nu\left(L_Y+L_R-L\right)+\tfrac{1}{2}\sigma^2 L_{xx}.$$

Let's mention that the second order term in (6) is explained by the fact that state variable N being an Ito process (Lemma *Ito's*). Taking derivative of the equation (6) with respect to x gives:

$$L_{xt}+F_x+L_{xx}f \ + \ f_x L_x+\frac{1}{2}g^2 L_{xxx}+\ \frac{1}{2}\left(g^2\right)_x L_{xx}= \ 0 \tag{7}$$

and, therefore,

$$L_{xt}+L_{xx}f+\frac{1}{2}g^2 L_{xxx}= \ -F_x-f_x L_x-\frac{1}{2}\left(g^2\right)_x L_{xx}. \tag{8}$$

Applying chain rule and considering second order contribution of the derivatives with respect to x (Lemma *Ito's*), result in:

$$dLx \ = \frac{\partial L_x}{\partial t}dt+\frac{\partial Lx}{\partial x}\frac{dx}{dt}dt+\frac{1}{2}\frac{\partial^2 L_x}{(\partial x)^2}\ dx^2\,.$$

Since, from *Ito's* Lemma $E[d(x^2)] \ = \ g^2\ dt$, the previous equation is reduced to:

$$\frac{dL_x}{dt}=\frac{\partial L_x}{\partial t}+\frac{\partial L_x}{\partial x}\frac{dx}{dt}+\frac{1}{2}\frac{\partial^2 L_x}{(\partial x)^2}\ g^2$$

$$\frac{dL_x}{dt}=L_{xt}+L_{xx}f+\frac{1}{2}L_{xxx}g^2\,. \tag{9}$$

Substituting (8) in (9) we obtain:

$$\frac{dL_x}{dt} = -F_x - f_x L_x - \frac{1}{2}\,(g^2)_x L_{xx}\,.$$

Equating adjoint variable μ to the first derivatives of the objective function L with respect to x $\mu_1 = L_{x_1}$, $\mu_2 = L_{x_2}$ and ω to the second derivatives of the objective function with respect to state variables $\omega_1 = L_{x_1x_1}$, the system of ordinary differential equation with respect to state variable is as follows:

$$\frac{dL_{x_1}}{dt} = -F_{x_1} - f_{x_1} L_{x_1} - \frac{1}{2}\,(\sigma^2)_{x_1} L_{x_1x_1} \Rightarrow \tag{10}$$

$$\Rightarrow \frac{d\mu_1}{dt} = -\mu_1 \alpha \frac{Y}{K},$$

$$\frac{dL_{x_2}}{dt} = -F_{x_2} - f_{x_2} L_{x_2} - \frac{1}{2}\,(\sigma^2)_{x_2} L_{x_2x_2} \Rightarrow \tag{11}$$

$$\Rightarrow \frac{d\mu_2}{dt} = -\mu_1(1-\alpha)\frac{Y}{N} - \mu_2 b_1 L_R\,,$$

$$\frac{dL_{x_2x_2}}{dt} = -F_{x_2x_2} - 2f_{x_2} L_{x_2x_2} - \frac{1}{2}\,(\sigma^2)_{x_2x_2} L_{x_2x_2} \Rightarrow \tag{12}$$

$$\Rightarrow \frac{d\omega}{dt} = -2\omega b_1 L_R - \omega \frac{\sigma^2_{x_2x_2}}{2}\,.$$

From the functional maximization with respect to C, L_Y, L_R we obtain:

$$\mu_1 = C^{-\vartheta} e^{-(\beta_1+\rho)t}\,, \tag{13}$$

$$\upsilon = -\mu_1(1-\alpha)\frac{Y}{N}\,, \tag{14}$$

$$\nu = -\mu_2 b_1 N\,. \tag{15}$$

and the resulting system of conjugate equations becomes:

$$\frac{d\mu_1}{dt} = -\mu_1 \alpha \frac{Y}{K}\,, \tag{16}$$

$$\frac{d\mu_2}{dt} = -\mu_1(1-\alpha)\frac{Y}{N} - \mu_2 b_1 L\,, \tag{17}$$

$$\mu_1 = C^{-\vartheta} e^{-(\beta_1+\rho)t}\,, \tag{18}$$

$$\nu = -\mu_1(1-\alpha)\frac{Y}{N}\,, \tag{19}$$

$$\nu = -\mu_2 b_1 N\,, \tag{20}$$

$$\frac{d\omega}{dt} = -2\omega b_1 L_R - \omega \frac{{\sigma_{x_2x_2}}^2}{2}\,. \tag{21}$$

From the conjugate equations for variables μ we obtain:

$$\mu_1(1-\alpha)\frac{Y}{N} = \mu_2 b_1 N \Rightarrow \frac{\mu_1}{\mu_2} = \frac{b_1 N}{(1-\alpha)\frac{Y}{N}}\,,$$

while from (17) it results

$$\frac{\dot{\mu}_2}{\mu_2} = -\frac{\mu_1}{\mu_2}(1-\alpha)Y/N - b_1 L_R.$$

If in the previous equation to introduce the ratio between variables μ_1 and μ_2 then

$$-\frac{\dot{\mu}_2}{\mu_2} = b_1 L_Y + b_1 L_R = b_1 L.$$

From (16) it becomes $\frac{\dot{\mu}_1}{\mu_1} = -\alpha\frac{Y}{K}$, while in equilibrium the growth rates of the conjugate variables are the same, then $\frac{\dot{\mu}_1}{\mu_1} = \frac{\dot{\mu}_2}{\mu_2}$, and considering [2] that

$$g_{opt} = g_C = \frac{\dot{C}}{C} = \frac{1}{\vartheta}\left(\frac{\alpha K}{Y} + \beta_k - \rho\right),$$

and taking advantage of the last equality, we obtain

$$g_{opt} = \frac{1}{\vartheta}\left(b_1 L + \beta_k - \rho\right).$$

In conclusion, there are the same balanced growth rates of the conjugate variables μ for stochastic problem formulation as for the deterministic problem formulation [2].

3 Mayer form presentation

If the problem is represented in the Mayer linear form, $F = 0$, then:

$$\frac{dL_x}{dt} = -f_x L_x - \frac{1}{2}(g^2)_x L_{xx}. \tag{22}$$

Equation (22) describes dynamics of the conjugate variables in the stochastic case. The presence of the second order term in the equation follows from the fact that the state variable is the stochastic variable which is an *Ito* process.

From the equation (22) one concludes that the calculation of L_{xx} is necessary. In order to obtain some expression for the L_{xx} dynamics, the same derivation as earlier will be utilized. Resulting equation will be called a conjugate equation. Let's differentiate again equation (7) with respect to x:

$$L_{xxt} + F_{xx} + L_{xx}f_x + L_{xxx}f + L_{xx}f_x + L_x f_{xx} + \frac{1}{2}g^2 L_{xxxx} +$$
$$+\frac{1}{2}\left(g^2\right)_x L_{xxx} + \frac{1}{2}\left(g^2\right)_x L_{xxx} + \frac{1}{2}\left(g^2\right)_{xx} L_{xx} = 0 \tag{23}$$

and, therefore,

$$L_{xxt} + L_{xxx}f + \frac{1}{2}g^2 L_{xxxx} =$$

$$= -F_{xx} - 2L_{xx}f_x - L_x f_{xx} - (g^2)_x L_{xxx} - \frac{1}{2}(g^2)_{xx} L_{xx} \; . \tag{24}$$

Using chain rule and considering the second order contribution with respect to x we obtain:

$$dLxx = \frac{\partial L_{xx}}{\partial t}dt + \frac{\partial L_{xx}}{\partial x}\frac{dx}{dt}dt + \frac{1}{2}\frac{\partial^2 L_{xx}}{(\partial x)^2}dx^2 \; . \tag{25}$$

Using *Ito's* Lemma $E[d(x^2)] = g^2 \, dt$ we obtain:

$$\frac{dL_{xx}}{dt} = \frac{\partial L_{xx}}{\partial t} + \frac{\partial L_{xx}}{\partial x}\frac{dx}{dt} + \frac{1}{2}\frac{\partial^2 L_{xx}}{(\partial x)^2}g^2,$$

$$\frac{dL_{xx}}{dt} = L_{xxt} + L_{xxx}f + \frac{1}{2}L_{xxxx}g^2 \; . \tag{26}$$

Inserting equation (24) in equation (26) we obtain:

$$\frac{dL_{xx}}{dt} = -F_{xx} - 2L_{xx}f_x - L_x f_{xx} - \left(g^2\right)_x L_{xxx} - \frac{1}{2}(g^2)_{xx} L_{xx}$$

and, finally, if $F = 0$ and the third order contribution is not considered (hypotheses accepted by *Ito*Lemma), it follows:

$$\frac{dL_{xx}}{dt} = -2L_{xx}f_x - L_x f_{xx} - \frac{1}{2}(g^2)_{xx} L_{xx} \, . \tag{27}$$

Equating conjugate variable, μ, to the prime derivative from the objective function L with respect to the state variable x and equating conjugate variable ω to the second derivative, we rewrite equations (9) and (27) in the following way:

$$d\mu/dt = -f_x \mu - 1/2(g^2)_x, \tag{28}$$

$$d\omega/dt = -2\omega f_x - \mu f_{xx} - 1/2(g^2)_{xx}\omega \, . \tag{29}$$

Summarizing the results for the stochastic case ($F = 0$), Hamiltonian function and conjugate equations shall be solved in the stochastic principle maximum formulation:

$$H = \mu f + 1/2 g^2 \omega \, ,$$

$$\frac{d\mu}{dt} = -f_x \mu - \frac{1}{2}(g^2)_x \omega \quad \mu(T) = c \, ,$$

$$\frac{d\omega}{dt} = -2\omega f_x - \mu f_{xx} - \frac{1}{2}(g^2)_{xx}\omega \quad \omega(T) = 0 \, .$$

It must be mentioned that the resulting problem is two-dimensional.

4 Conclusions

In the present article the solution of the economic optimal control problem in the stochastic formulation is obtained. In order to obtain the solution of this problem the derivation of the respective Hamilton – Jacobi – Bellman equation was applied. This method contributed to obtaining solution in the stochastic maximum principle form containing the first order system of the conjugate differential equations. Note that the growth rate reached in stable condition for the examined problem is the same as for deterministic, with one difference – there is an additional ordinary equation characterizing additional conjugate variable (shadow price of the stochastic restriction). More complete stochastic optimal control problem with the shock above all economy and with the shocks under productivity in intermediate good sectors will be studied in the future.

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