BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(67), 2011, Pages 91–107
ISSN 1024–7696

On LCA groups whose rings of continuous endomorphisms have at most two non-trivial closed ideals. I

Valeriu Popa

Abstract. We describe the torsion, locally compact abelian (LCA) groups X for which the ring $E(X)$ of continuous endomorphisms of X, endowed with the compact-open topology, has no more than two non-trivial closed ideals.

Mathematics subject classification: 16W80.
Keywords and phrases: LCA groups, rings of continuous endomorphisms, compact-open topology, closed ideals.

1 Introduction

Motivated by the work of F. Perticani [6], the author discussed in [10] general topological rings with identity having at most two non-trivial closed ideals. The main results of [10] characterize the mentioned rings in terms of ideal extensions of topological rings. In the present paper, we are interested in a more concrete class of topological rings of the mentioned type, namely, those which occur as rings of continuous endomorphisms of LCA groups. Precisely speaking, we are dealing with the problem of determining the LCA groups X with the property that the ring $E(X)$ of all continuous endomorphisms of X, taken with the compact-open topology, has no more than two non-trivial closed ideals.

In the following, we establish some bounds for the class of groups in question and solve completely the considered problem in the case of torsion LCA groups.

2 Notation

We use without explanations some terminology and notations introduced in [10]. In addition, we denote by $\mathbb{P}$ the set of primes and by $\mathcal{L}$ the class of LCA groups. For $p \in \mathbb{P}$, we denote by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group corresponding to p and by $\mathbb{Z}(p^n)$, where n is a positive integer, the cyclic group of order p^n (both with the discrete topology). For $X \in \mathcal{L}$, we let 1_X, $t(X)$, X^*, and $E(X)$, denote, respectively, the identity map on X, the torsion subgroup of X, the character group of X, and the ring of continuous endomorphisms of X, endowed with the compact-open topology. Recall that the compact-open topology on $E(X)$ is generated by the sets

$$\Omega(K, U) = \{u \in E(X) \mid u(K) \subset U\},$$

where $K, U \subset X$, K is compact and U is open. For a positive integer n, we let $X[n] = \{x \in X \mid nx = 0\}$ and $nX = \{nx \mid x \in X\}$. Also, $o(a)$ denotes the order of a in X, $\langle S \rangle$ the subgroup of X generated by S, and (M) the ideal of $E(X)$ generated by M. Further, given a family $(X_i)_{i \in I}$ of groups in $\mathcal{L}$, we write $\prod_{i \in I} X_i$ for its topological direct product. In case each X_i coincides with a fixed $X \in \mathcal{L}$, we use X^I for $\prod_{i \in I} X_i$. For a discrete $X \in \mathcal{L}$, we let $X^{(I)}$ denote the discrete direct sum of I copies of X. Finally, $\oplus$ stands for topological direct sum and $\cong$ for topological isomorphism.

3 Some necessary conditions

In this section we shall reduce the study of groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ has no more than two non-trivial closed ideals to the case of some more special groups.

We begin with the following preparatory lemma.

Lemma 1. *Let X be a group in $\mathcal{L}$ such that $\overline{mnE(X)} = \overline{nE(X)}$ for some positive integers m and n. Then $\overline{nX} = \overline{mnX}$ and $X[mn] = X[n]$.*

Proof. In view of our hypothesis, $n1_X \in \overline{mnE(X)}$, and hence there exists a net $(u_\lambda)_{\lambda \in L}$ of elements of $E(X)$ such that $n1_X = \lim_{\lambda \in L} mnu_\lambda$ [4, Proposition 1.6.3]. Pick any $x \in X$, and define $\delta_x : E(X) \to X$ by setting $\delta_x(u) = u(x)$ for all $u \in E(X)$. Then δ_x is a continuous group homomorphism, so

$$\begin{aligned} nx &= \delta_x(n1_X) = \delta_x(\lim_{\lambda \in L} mnu_\lambda) \\ &= \lim_{\lambda \in L} \delta_x(mnu_\lambda) = \lim_{\lambda \in L} \big((mnu_\lambda)(x)\big) \in \overline{mnX}. \end{aligned}$$

Since $x \in X$ was arbitrary, it follows that $nX \subset \overline{mnX}$, which gives $\overline{nX} = \overline{mnX}$. Further, since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic [8, (1.1)], we also have $\overline{mnE(X^*)} = \overline{nE(X^*)}$, so as above $\overline{nX^*} = \overline{mnX^*}$, and hence $X[n] = X[mn]$ by [5, (24.22)]. $\square$

Next we recall two definitions.

Definition 1. Let n be a positive integer. A group $X \in \mathcal{L}$ is said to be of finite exponent n if n is the least positive integer satisfying $nX = \{0\}$.

Definition 2. A subgroup F of a group $X \in \mathcal{L}$ is said to be topologically fully invariant in X if $u(F) \subset F$ for all $u \in E(X)$.

Let $X \in \mathcal{L}$. Then X can be viewed as a left topological module over $E(X)$. It is clear that the topologically fully invariant subgroups of X are just the $E(X)$-submodules of X. Now, if F is a topologically fully invariant subgroup of X, then $ann_{E(X)}(F)$, the annihilator of F in $E(X)$, is a closed ideal of $E(X)$ because X is Hausdorff. Further, if F is in addition closed in X, then X/F is a Hausdorff topological $E(X)$-module, so that $ann_{E(X)}(X/F)$ is a closed ideal of $E(X)$ as well. In fact

$$ann_{E(X)}(X/F) = \{u \in E(X) \mid \mathrm{im}(u) \subset F\}.$$

We now state the main result of this section.

Theorem 1. *Let X be a non-zero group in $\mathcal{L}$ such that $E(X)$ has no more than two non-trivial closed ideals. Then exactly one of the following conditions holds:*

(i) $X \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$ *for some $p \in \mathbb{P}$ and some cardinal numbers α, β satisfying $\alpha + \beta \geq 1$.*

(ii) $X \cong \mathbb{Z}(p)^{(\alpha_p)} \times \mathbb{Z}(p)^{\beta_p} \times \mathbb{Z}(q)^{(\alpha_q)} \times \mathbb{Z}(q)^{\beta_q}$ *for some distinct $p, q \in \mathbb{P}$ and some cardinal numbers $\alpha_p, \beta_p, \alpha_q, \beta_q$ satisfying $\alpha_p + \beta_p \geq 1$ and $\alpha_q + \beta_q \geq 1$.*

(iii) *X is a group of finite exponent p^2 for some $p \in \mathbb{P}$.*

(iv) *X is a group of finite exponent p^3 for some $p \in \mathbb{P}$.*

(v) *X is densely divisible and torsion-free.*

(vi) *There exists $p \in \mathbb{P}$ such that $t(X) = X[p]$, $\overline{pX}$ is non-zero and densely divisible, and $\overline{pE(X)} \subsetneq E(X)$.*

(vii) *There exist $p, q \in \mathbb{P}$ such that $t(X) = X[pq]$, $\overline{pqX}$ is non-zero and densely divisible, and $\overline{pqE(X)} \subsetneq \overline{pE(X)} \subsetneq E(X)$.*

Proof. If there exists $p \in \mathbb{P}$ such that $pE(X) = \{0\}$, then $pX = (p1_X)(X) = 0$, and so, by [2, Ch. 2, §4, Theorem 2], $X \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$ for some cardinal numbers α and β satisfying $\alpha + \beta \geq 1$. Consequently, in this case we are led to (i).

Suppose $mE(X) \neq \{0\}$ for all $m \in \mathbb{P}$. If there exist $p, q \in \mathbb{P}$ such that $pqE(X) = \{0\}$, then $pqX = (pq1_X)(X) = \{0\}$, and hence $X = X[pq]$. Now, if $q \neq p$, we conclude from [1, Theorem 3.13] that $X = X[p] \oplus X[q]$, where the primary components $X[p]$ and $X[q]$ are non-zero. Thus, again appealing to [2, Ch. 2, §4, Theorem 2], in this case we get (ii). Further, in the remaining case when $q = p$, we have $X = X[p^2] \neq X[p]$, which gives us (iii).

Next suppose $lmE(X) \neq \{0\}$ for all $l, m \in \mathbb{P}$. If there exist $p, q, r \in \mathbb{P}$ such that $pqrE(X) = \{0\}$, then $pqrX = (pqr1_X)(X) = \{0\}$, and hence $X = X[pqr]$. We claim that $p = q = r$. Indeed, if the numbers p, q, and r were distinct, we could write $X = X[p] \oplus X[q] \oplus X[r]$. Since, in view of our assumption, the topologically fully invariant subgroups $X[p]$, $X[q]$, and $X[r]$ of X are non-zero, it would follow, as can be seen by considering the endomorphisms $p1_X$, $q1_X$, $r1_X$, and 1_X, that the annihilators $ann_{E(X)}(X[p])$, $ann_{E(X)}(X[q])$, and $ann_{E(X)}(X[r])$ are distinct, non-trivial, closed ideals of $E(X)$. This contradicts the hypothesis. Similarly, if only two of the numbers p, q, and r coincided, say $p \neq q = r$, we would have $X = X[p^2] \oplus X[q]$, where $X[p^2] \neq X[p]$ and $X[q] \neq \{0\}$. By invoking the endomorphisms $p^2 1_X$, $p1_X$, $q1_X$, and 1_X, we would then conclude that $ann_{E(X)}(X[p^2])$, $ann_{E(X)}(X[p])$, and

$ann_{E(X)}(X[q])$ are distinct, non-trivial, closed ideals of $E(X)$, again in contradiction with the hypothesis. Thus we must have $p = q = r$, so $X = X[p^3]$, getting (iv).

Further suppose $klmE(X) \neq \{0\}$ for all $k, l, m \in \mathbb{P}$. If $\overline{pE(X)} = E(X)$ for all $p \in \mathbb{P}$, it follows from Lemma 1 that $\overline{pX} = \overline{1X} = X$ and $X[p] = X[1] = \{0\}$ for all $p \in \mathbb{P}$, so X is densely divisible and torsion-free, and in this case we are led to (v).

Next assume there exists $p \in \mathbb{P}$ such that $\overline{pE(X)} \neq E(X)$. There are two possibilities: either (1) $\overline{qpE(X)} = \overline{pE(X)}$ for all $q \in \mathbb{P}$, or (2) there is $q \in \mathbb{P}$ such that $\overline{qpE(X)} \neq \overline{pE(X)}$. In the former case, it follows from Lemma 1 that $\overline{q\overline{pX}} = \overline{qpX} = \overline{pX}$ and $X[pq] = X[p]$ for all $q \in \mathbb{P}$, so that $\overline{pX}$ is non-zero, densely divisible and $t(X) = X[p]$, which gives us (vi). In the second case, $\overline{qpE(X)}$ and $\overline{pE(X)}$ are distinct non-trivial closed ideals of $E(X)$. Since, by our assumption, $\overline{pqrE(X)} \neq \{0\}$ for all $p, q, r \in \mathbb{P}$, it follows that $\overline{pqrE(X)} = \overline{pqE(X)}$ for all $r \in \mathbb{P}$, so $\overline{r\overline{pqX}} = \overline{pqrX} = \overline{pqX}$ and $X[pqr] = X[pq]$ for all $r \in \mathbb{P}$, and hence $\overline{pqX}$ is non-zero, densely divisible and $t(X) = X[pq]$, whence (vii). □

Remark 1. We know from [8, (2.3)] that any group X appearing in item (i) of the preceding theorem has a topologically simple ring $E(X)$. It is also clear from [9, (2.2)] and [8, (2.3)] that for any group X appearing in item (ii), the ring $E(X)$ is a topological direct product of two topologically simple rings, and so it has exactly two non-trivial closed ideals. In particular, we see from [10, Theorem 3] that in this case every non-trivial closed ideal of $E(X)$ is strongly topologically maximal.

In the remaining part of this paper we handle the problem stated in Introduction for groups appearing in items (iii) and (iv). Since the groups appearing in items (v), (vi), and (vii) contain non-torsion elements, this furnishes a solution to the considered problem in the case of torsion LCA groups.

4 Groups of finite exponent p^2

Our aim in this section is to describe the groups $X \in \mathcal{L}$ of finite exponent p^2, where $p \in \mathbb{P}$, such that the ring $E(X)$ has no more than two non-trivial closed ideals. First, we note a lemma from [7, (3.8)], which will be frequently used in the sequel.

Lemma 2. *Let $X \in \mathcal{L}$ be a group of finite exponent p^n, where p is a prime and n is a positive integer. If $a \in X$ is an element of order p^n, then $\langle a \rangle$ splits topologically from X. Moreover, the complement of $\langle a \rangle$ can be chosen so as to contain a preassigned open subgroup V of X satisfying $\langle a \rangle \cap V = \{0\}$.*

We continue with five lemmas, that are needed for establishing the desired description.

Lemma 3. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^2. If $\overline{pX} \neq X[p]$, then $E(X)$ has more than two non-trivial closed ideals, which are comparable with respect to set-theoretic inclusion.*

Proof. Assume that $\overline{pX} \neq X[p]$. Clearly, $p1_X \neq 0$ and $p1_X \in ann_{E(X)}\big(X[p]\big)$. It is also clear that $\mathrm{im}(p1_X) \subset \overline{pX}$, so $p1_X \in ann_{E(X)}\big(X/\overline{pX}\big)$, and thus

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \neq \{0\}.$$

Further, since $\overline{pX} \subset X[p]$, we have $ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$. Finally, since $pX \neq \{0\}$, it follows that $\overline{pX} \not\subset \ker(1_X)$, so $1_X \notin ann_{E(X)}\big(\overline{pX}\big)$, and hence $ann_{E(X)}\big(\overline{pX}\big) \neq E(X)$. We shall show that the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \subset ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$$

are strict. Let $\xi : X \to X/X[p]$ and $\eta : X \to X/\overline{pX}$ denote the canonical projections, and fix any $a \in X \backslash X[p]$ and $b \in X[p] \backslash \overline{pX}$. Then $o(a) = p^2$ and $o(\xi(a)) = p = o(\eta(b))$. By Lemma 2, we can write

$$X/X[p] = \langle \xi(a) \rangle \oplus A \qquad \text{and} \qquad X/\overline{pX} = \langle \eta(b) \rangle \oplus B,$$

where A and B are closed subgroups in $X/X[p]$ and $X/\overline{pX}$, respectively. Let $\lambda : \langle \xi(a) \rangle \to X$ and $\mu : \langle \eta(b) \rangle \to X$ be the group homomorphisms given by $\lambda(\xi(a)) = \mu(\eta(b)) = b$. Denoting by φ the canonical projection of $X/X[p]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{pX}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{pX}$), so $ann_{E(X)}\big(X[p]\big)$ properly contains $ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big)$. Similarly, denoting by ψ the canonical projection of $X/\overline{pX}$ onto $\langle \eta(b) \rangle$ with kernel B, we see that $\mu \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$ and $\mu \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p]\big)$ (because $b \in X[p]$ and $(\mu \circ \psi \circ \eta)(b) = b$), so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p]\big)$. Consequently, the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \subset ann_{E(X)}\big(X[p]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$$

are strict. □

Lemma 4. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^2 such that $\overline{pX} = X[p]$, and let C be a non-zero closed ideal of $E(X)$. Further, let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$, and for each $(a, G) \in \mathcal{P}$ let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Then:*

(i) *If C contains elements of order p^2, then $C \supset (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(ii) *If $pC = \{0\}$, then $C \supset (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

Proof. For $(a, G) \in \mathcal{P}$ and $b \in X$, we define $f_{a,G,b} \in E(X)$ by the rule

$$f_{a,G,b}(t) = \begin{cases} b, & \text{if } t = a; \\ 0, & \text{if } t \in G. \end{cases}$$

(i) Pick any $u \in C$ with $o(u) = p^2$. Since $pu \neq 0$, there exists $x \in X$ such that $(pu)(x) \neq 0$, and so $o(u(x)) = p^2$. It then follows from Lemma 2 that there exists a closed subgroup Y of X such that $X = \langle u(x) \rangle \oplus Y$. Now, given any $(a, G) \in \mathcal{P}$, it is straightforward to check that $\varepsilon_{a,G} = f_{u(x),Y,a} \circ u \circ f_{a,G,x}$, so $\varepsilon_{a,G} \in C$.

(ii) Pick any non-zero $u \in C$ and any $x \in X$ such that $u(x) \neq 0$. Since $pu = 0$, we have $pX \subset \ker(u)$, so $X[p] \subset \ker(u)$, and therefore $o(x) = p^2$. In particular, by Lemma 2 we may write $X = \langle x \rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $u(x) \notin U$. Since $X[p] = \overline{pX}$, there exists $z \in X$ satisfying $pz - u(x) \in U$. As $u(x) \notin U$, we cannot have $pz = 0$, and so $o(z) = p^2$. Let π denote the canonical projection of X onto the quotient group X/U. Clearly, $\pi(u(x)) \neq 0$ and $\pi(u(x)) = \pi(pz) = p\pi(z)$, so $o(\pi(z)) = p^2$. Hence we can write $X/U = \langle \pi(z) \rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle \pi(z) \rangle$ with kernel Γ and letting $h : \langle \pi(z) \rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p\varepsilon_{x,Y}$, so $p\varepsilon_{x,Y} \in C$. Finally, given any $(a, G) \in \mathcal{P}$, we have $p\varepsilon_{a,G} = f_{x,Y,a} \circ (p\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$. □

Lemma 5. *Let $X \in \mathcal{L}$ be a group of finite exponent p^n, where p is a prime and n is a positive integer. If the subgroup A of X is a finite direct sum of cyclic groups of order p^n, then A splits topologically from X. Moreover, the complement of A can be chosen so as to contain a preassigned open subgroup V of X with property $A \cap V = \{0\}$.*

Proof. We induct on the number of summands, k, in the decomposition of A as a direct sum $A = A_1 \oplus \ldots \oplus A_k$ of cyclic groups of order p^n. If $k = 1$, the assertion holds trivially since this is just Lemma 2. Assume $k \geq 2$, and assume the result is true for any group of finite exponent p^n in $\mathcal{L}$ and any its subgroup written as a direct sum of $k-1$ cyclic subgroups of order p^n. Given an arbitrary open subgroup V of X satisfying $A \cap V = \{0\}$, it is clear that $V_1 = A_2 \oplus \ldots \oplus A_k \oplus V$ is an open subgroup of X and $A_1 \cap V_1 = \{0\}$. By Lemma 2, we can write $X = A_1 \oplus X_1$ for some subgroup X_1 of X containing V_1. Now, applying the inductive hypothesis to X_1, $A_2 \oplus \ldots \oplus A_k$, and V, we can find a subgroup X_k of X such that $X_k \cap V = \{0\}$ and $X_1 = A_2 \oplus \ldots \oplus A_k \oplus X_k$. Then $X = A_1 \oplus A_2 \oplus \ldots \oplus A_k \oplus X_k$. □

Lemma 6. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^2 satisfying $\overline{pX} = X[p]$. For any compact subset K of X and any neighbourhood U of zero in X, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^2 of K'.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary neighbourhood U of zero in X. Since X is totally disconnected, we can find a compact open subgroup U_0 of X such that $U_0 \subset U$ [5, (7.7)]. Let $K_0 = \langle K \cup U_0 \rangle$. Then K_0 is compact [5, (9.8)], and $U_0 \subset K_0$. In particular, K_0 is topologically isomorphic to a topological direct product of cyclic p-groups of order at most p^2 [5, (25.9)], and so there exist two disjoint sets I_1 and I_2 such that $K_0 \cong \prod_{i \in I_1 \cup I_2} C_i$, where $C_i = \mathbb{Z}(p)$ for $i \in I_1$ and $C_i = \mathbb{Z}(p^2)$ for $i \in I_2$. Fix a topological isomorphism f from K_0 onto

$\prod_{i\in I_1\cup I_2} C_i$. Given an arbitrary subset J of $I_1 \cup I_2$, we denote by C'_J the subgroup of all $(c_i)_{i\in I_1\cup I_2} \in \prod_{i\in I_1\cup I_2} C_i$ satisfying $c_i = 0$ for all $i \notin J$. Since U_0 is open in K_0, there exist finite subsets $J_1 \subset I_1$ and $J_2 \subset I_2$ such that $f(U_0) \supset C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)}$. We then have

$$\prod_{i\in I_1\cup I_2} C_i = \Big(\bigoplus_{i\in J_1\cup J_2} C'_i\Big) \oplus C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)},$$

so

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2} f^{-1}(C'_i)\Big) \oplus f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)}),$$

where C'_i stands for $C'_{\{i\}}$. Set $U' = f^{-1}(C_{(I_1\setminus J_1)\cup(I_2\setminus J_2)})$ and, for $i \in J_1 \cup J_2$, let a_i be a generator of $f^{-1}(C'_i)$. Then U' is an open subgroup of X contained in U_0 and

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2} \langle a_i\rangle\Big) \oplus U'.$$

We also have $o(a_i) = p$ if $i \in J_1$, and $o(a_i) = p^2$ if $i \in J_2$. In the following, we shall construct a compact subgroup $K' \supset K_0$ which admits a decomposition similar to that of K_0, by replacing the elements a_i with $i \in J_1$ by elements of order p^2. If $J_1 = \varnothing$, we set $K' = K_0$. Suppose $J_1 \neq \varnothing$, and pick an arbitrary $j \in J_1$. Since $X[p] = \overline{pX}$, there exists $b_j \in X$ such that $a_j - pb_j \in U'$. As $a_j \notin U'$, we cannot have $pb_j = 0$, so $o(b_j) = p^2$. We claim that

$$\langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Indeed, given any $x \in \langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg)$, we can write

$$x = lpb_j = \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2} l_i a_i\Big) + y'$$

for some non-negative integers l, l_i and some $y' \in U'$. Since $y' + l(a_j - pb_j) \in U'$, it follows that

$$\begin{aligned} la_j &= \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2} l_i a_i\Big) + y' + l(a_j - pb_j) \\ &\in \langle a_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) \;=\; \{0\}, \end{aligned}$$

so p divides l, and hence $x = 0$. This proves our claim that

$$\langle pb_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2}\langle a_i\rangle\Big)\oplus U'\Big) = \{0\}.$$

We replace K_0 by

$$K_1 = \langle b_j\rangle \oplus \Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2}\langle a_i\rangle\Big)\oplus U'.$$

Now, if $J_1 \setminus \{j\} \neq \varnothing$, we can apply the preceding procedure to K_1, and so, after a finite number of steps, we shall arrive at a compact subgroup K' of X having the following form:

$$K' = \Big(\bigoplus_{i\in J_1}\langle b_i\rangle\Big)\oplus\Big(\bigoplus_{i\in J_2}\langle a_i\rangle\Big)\oplus U',$$

where $o(b_i) = p^2$ for all $i \in J_1$. Since $a_i \in K'$ for all $i \in J_1$, we also have $K \cup U' \subset K_0 \subset K'$, so K' and U' are those required. □

Lemma 7. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^2 satisfying $\overline{pX} = X[p]$, and let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X = \langle a\rangle \oplus G$. Then the ideal $(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$, where $\varepsilon_{a,G} \in E(X)$ denotes the canonical projection of X onto $\langle a\rangle$ with kernel G, is dense in $E(X)$.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary open neighbourhood U of zero in X. It suffices to show that

$$(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P}) \cap [1_X + \Omega(K, U)] \neq \varnothing.$$

By Lemma 6, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^2 of K'. Further, by Lemma 5 there is a subgroup G of X such that $U' \subset G$ and $X = \bigoplus_{i\in I}\langle a_i\rangle \oplus G$. Then $(a_j, \bigoplus_{i\in I\setminus\{j\}}\langle a_i\rangle \oplus G) \in \mathcal{P}$ for all $j \in I$, and

$$\sum_{j\in I}\varepsilon_{(a_j,\bigoplus_{i\in I\setminus\{j\}}\langle a_i\rangle\oplus G)} - 1_X \in \Omega(K', U') \subset \Omega(K, U).$$

□

We now combine the preceding lemmas to obtain the main result of this section.

Theorem 2. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^2. The following statements are equivalent:*

(i) *$E(X)$ has only one non-trivial closed ideal.*

(ii) *Every non-trivial closed ideal of* $E(X)$ *is strongly topologically maximal.*

(iii) *Every non-trivial closed ideal of* $E(X)$ *is topologically maximal.*

(iv) $X[p] = \overline{pX}$.

Proof. Obviously, (i) implies (ii), and (ii) implies (iii). The fact that (iii) implies (iv) follows from Lemma 3.

Assume (iv), and let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^2 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Further, for $(a, G) \in \mathcal{P}$, let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Now, pick an arbitrary non-zero closed ideal C of $E(X)$. We distinguish cases when C contains elements of order p^2 and when $pC = \{0\}$.

First, suppose C contains elements of order p^2. Then $(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P}) \subset C$ by Lemma 4, and hence $C = E(X)$ by Lemma 7.

Next suppose that $pC = \{0\}$. Then $\overline{(p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})} \subset C$ by Lemma 4. In order to establish the reverse inclusion, pick any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 6, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset u^{-1}(U)$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^2 of K'. It then follows from Lemma 5 that $X = \bigoplus_{i \in I} \langle a_i \rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p]$ and $X[p] = \overline{pX}$, for every $i \in I$ there exists $b_i \in X$ such that $pb_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. Then clearly $pv - u \in \Omega(K', U')$. Finally, by Lemma 7, there exists $w \in (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. Since U' is a subgroup in X, we have $p(w - v) \in \Omega(K', U')$, and hence

$$pw - u = p(w - v) + (pv - u) \in \Omega(K', U') \subset \Omega(K, U).$$

As $pw \in (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$, we conclude that $u \in \overline{(p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})}$. □

Remark 2. Lemma 3 and Theorem 2 give an answer to the considered question in the case of LCA groups of finite exponent p^2, where $p \in \mathbb{P}$.

5 Groups of finite exponent p^3

In this section, we determine the groups $X \in \mathcal{L}$ of finite exponent p^3, where $p \in \mathbb{P}$, such that the ring $E(X)$ has at most two non-trivial closed ideals. In preparation to this we establish four lemmas, which are similar to Lemmas 3, 4, 6, and 7 of the preceding section.

Lemma 8. *Let* $p \in \mathbb{P}$, *and let* $X \in \mathcal{L}$ *be a group of finite exponent* p^3. *If* $E(X)$ *has no more than two non-trivial closed ideals, then* $\overline{pX} = X[p^2]$ *and* $\overline{p^2X} = X[p]$.

Proof. Suppose first that $\overline{pX} \neq X[p^2]$. To get a contradiction, it is enough to indicate three distinct, non-trivial, closed ideals of $E(X)$. Clearly, $p^2 1_X \neq 0$

and $p^2 1_X \in ann_{E(X)}\big(X[p^2]\big)$. It is also clear that $\mathrm{im}(p^2 1_X) = p^2 X \subset \overline{pX}$, so $p^2 1_X \in ann_{E(X)}\big(X/\overline{pX}\big)$, and thus

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) \neq \{0\}.$$

Further, since $\overline{pX} \subset X[p^2]$, we have $ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(\overline{pX}\big)$. Now, given any $u \in ann_{E(X)}\big(\overline{pX}\big)$, we have $pu(X) = u(pX) = \{0\}$, so $\mathrm{im}(u) \subset X[p] \subset X[p^2]$, and hence $u \in ann_{E(X)}\big(X/X[p^2]\big)$. As $u \in ann_{E(X)}\big(\overline{pX}\big)$ was arbitrary, it follows that $ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big)$. Finally, since $p^2 X \neq \{0\}$, it follows that $\mathrm{im}(1_X) \not\subset X[p^2]$, so $1_X \notin ann_{E(X)}\big(X/X[p^2]\big)$, and hence $ann_{E(X)}\big(X/X[p^2]\big) \neq E(X)$. We shall show that at least two of the inclusions

$$\begin{aligned} ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) &\subset ann_{E(X)}\big(X[p^2]\big) \\ &\subset ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big) \end{aligned}$$

are strict. Let $\xi : X \to X/X[p^2]$ and $\eta : X \to X/\overline{pX}$ be the canonical projections, and fix any $a \in X \setminus X[p^2]$ and $b \in X[p^2] \setminus \overline{pX}$. Then $o(a) = p^3$ and $o(\xi(a)) = p = o(\eta(b))$. By Lemma 2, we can write

$$X/X[p^2] = \langle \xi(a) \rangle \oplus A, \quad X/\overline{pX} = \langle \eta(b) \rangle \oplus B, \quad \text{and} \quad X = \langle a \rangle \oplus Y,$$

where A, B, and Y are closed subgroups in $X/X[p^2]$, $X/\overline{pX}$, and X, respectively. In the following, we distinguish the cases when $o(b) = p$ and when $o(b) = p^2$.

First assume that $o(b) = p$. Let $\lambda : \langle \xi(a) \rangle \to X$ and $\mu : \langle \eta(b) \rangle \to X$ be the group homomorphisms given by the rule $\lambda(\xi(a)) = \mu(\eta(b)) = b$. Denoting by φ the canonical projection of $X/X[p^2]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p^2]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{pX}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{pX}$), so $ann_{E(X)}\big(X[p^2]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big)$. Similarly, denoting by ψ the canonical projection of $X/\overline{pX}$ onto $\langle \eta(b) \rangle$ with kernel B, we see that $\mu \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$, and $\mu \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p^2]\big)$ (because $b \in X[p^2]$ and $(\mu \circ \psi \circ \eta)(b) = b$), so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$ as well.

Next we consider the case when $o(b) = p^2$. Let $\mu' : \langle \eta(b) \rangle \to X$ denote the group homomorphism given by the rule $\mu'(\eta(b)) = pb$. Then $\mu' \circ \psi \circ \eta \in ann_{E(X)}\big(\overline{pX}\big)$ and $\mu' \circ \psi \circ \eta \notin ann_{E(X)}\big(X[p^2]\big)$, so $ann_{E(X)}\big(\overline{pX}\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$. Further, let $v \in E(X)$ be defined by $v(a) = b$ and $v(y) = 0$ for all $y \in Y$. Since $v(pa) = pb \neq 0$, we conclude that $v \notin ann_{E(X)}\big(\overline{pX}\big)$. On the other hand, since $p^2 v(a) = p^2 b = 0$, it is clear that $\mathrm{im}(v) \subset X[p^2]$, so $v \in ann_{E(X)}\big(X/X[p^2]\big)$, and hence $ann_{E(X)}\big(X/X[p^2]\big)$ properly contains $ann_{E(X)}\big(\overline{pX}\big)$.

We have shown that at least two of the inclusions

$$\begin{aligned} ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{pX}\big) &\subset ann_{E(X)}\big(X[p^2]\big) \\ &\subset ann_{E(X)}\big(\overline{pX}\big) \subset ann_{E(X)}\big(X/X[p^2]\big) \end{aligned}$$

are strict, a contradiction. Consequently, we must have $\overline{pX} = X[p^2]$.

Now suppose that $\overline{p^2X} \neq X[p]$. As we already mentioned, $p^2 1_X \neq 0$ and $p^2 1_X \in ann_{E(X)}\big(X[p^2]\big)$. Since $\operatorname{im}(p^2 1_X) \subset \overline{p^2X}$, we also have $p^2 1_X \in ann_{E(X)}\big(X/\overline{p^2X}\big)$, so

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \neq \{0\}.$$

Further, since $X[p] \subset X[p^2]$, we have $ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$. Finally, since $X[p] \neq \{0\}$, it follows that $X[p] \not\subset \ker(1_X)$, so $1_X \notin ann_{E(X)}\big(X[p]\big)$, and hence $ann_{E(X)}\big(X[p]\big) \neq E(X)$. We shall show that the inclusions

$$ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \subset ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$$

are strict. Let $\xi : X \to X/X[p^2]$ denote the canonical projection, and fix any $a \in X \setminus X[p^2]$ and $b \in X[p] \setminus \overline{p^2X}$. Then $o(a) = p^3$, so $o(\xi(a)) = p = o(b)$. By Lemma 2, we can write

$$X/X[p^2] = \langle \xi(a) \rangle \oplus A,$$

where A is a closed subgroup of $X/X[p^2]$. Let $\lambda : \langle \xi(a) \rangle \to X$ be the group homomorphism given by $\lambda(\xi(a)) = b$. Denoting by φ the canonical projection of $X/X[p^2]$ onto $\langle \xi(a) \rangle$ with kernel A, we see that $\lambda \circ \varphi \circ \xi \in ann_{E(X)}\big(X[p^2]\big)$, and $\lambda \circ \varphi \circ \xi \notin ann_{E(X)}\big(X/\overline{p^2X}\big)$ (because $(\lambda \circ \varphi \circ \xi)(a) = b \notin \overline{p^2X}$), so $ann_{E(X)}\big(X[p^2]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big)$. Finally, since $p1_X \notin ann_{E(X)}\big(X[p^2]\big)$ and $p1_X \in ann_{E(X)}\big(X[p]\big)$, $ann_{E(X)}\big(X[p]\big)$ properly contains $ann_{E(X)}\big(X[p^2]\big)$. Consequently, the inclusions

$$ann_{E(X)}\big(X[p]\big) \cap ann_{E(X)}\big(X/\overline{p^2X}\big) \subset ann_{E(X)}\big(X[p^2]\big) \subset ann_{E(X)}\big(X[p]\big)$$

are strict. As this contradicts our hypothesis, we must have $\overline{p^2X} = X[p]$. □

Lemma 9. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^3 such that $\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$, and let C be a non-zero closed ideal of $E(X)$. Further, let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$, and for each $(a, G) \in \mathcal{P}$ let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Then:*

(i) *If C contains elements of order p^3, then $C \supset (\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(ii) *If $p^2C = \{0\}$ and $pC \neq \{0\}$, then $C \supset (p\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

(iii) *If $pC = \{0\}$, then $C \supset (p^2\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$.*

Proof. As in the proof of Lemma 4, for any $(a, G) \in \mathcal{P}$ and $b \in X$, we let $f_{a,G,b} \in E(X)$ be defined by the rule

$$f_{a,G,b}(t) = \begin{cases} b, & \text{if } t = a; \\ 0, & \text{if } t \in G. \end{cases}$$

(i) Pick any $u \in C$ with $o(u) = p^3$. Since $p^2u \neq 0$, there exists $x \in X$ such that $(p^2u)(x) \neq 0$, and so $o(u(x)) = p^3$. By Lemma 2, there exists a closed subgroup Y of X such that $X = \langle u(x)\rangle \oplus Y$. Given any $(a, G) \in \mathcal{P}$, we then have $\varepsilon_{a,G} = f_{u(x),Y,a} \circ u \circ f_{a,G,x}$, so $\varepsilon_{a,G} \in C$.

(ii) Pick any $u \in C$ with $o(u) = p^2$. Since $pu \neq 0$, there exists $x \in X$ such that $(pu)(x) \neq 0$, so $px \notin \ker(u)$ and $o(u(x)) = p^2$. On the other hand, since $p^2u = 0$, we have $p^2X \subset \ker(u)$, so $X[p] \subset \ker(u)$. It follows that $px \notin X[p]$, so $p^2x \neq 0$, and hence $o(x) = p^3$. In particular, we can write $X = \langle x\rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $pu(x) \notin U$. Since $u(x) \in X[p^2] = \overline{pX}$, there exists $z \in X$ such that $pz - u(x) \in U$. As $p(pz - u(x)) \in U$ and $pu(x) \notin U$, we cannot have $p^2z = 0$, so $o(z) = p^3$. Let π denote the canonical projection of X onto X/U. Clearly, $\pi(pu(x)) \neq 0$ and $\pi(pu(x)) = \pi(p^2z) = p^2\pi(z)$, so $o(\pi(z)) = p^3$. Hence we can write $X/U = \langle\pi(z)\rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle\pi(z)\rangle$ with kernel Γ and letting $h : \langle\pi(z)\rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p\varepsilon_{x,Y}$, so $p\varepsilon_{x,Y} \in C$. Now, given any $(a, G) \in \mathcal{P}$, we have $p\varepsilon_{a,G} = f_{x,Y,a} \circ (p\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$.

(iii) Pick any non-zero $u \in C$ and any $x \in X$ such that $u(x) \neq 0$. Since $pu = 0$, we have $pX \subset \ker(u)$, so $X[p^2] \subset \ker(u)$, and therefore $o(x) = p^3$. In particular, $X = \langle x\rangle \oplus Y$ for some closed subgroup Y of X. Now, fix an arbitrary open subgroup U of X such that $u(x) \notin U$. Since $u(x) \in X[p] = \overline{p^2X}$, there exists $z \in X$ such that $p^2z - u(x) \in U$. As $u(x) \notin U$, we cannot have $p^2z = 0$, so $o(z) = p^3$. Let $\pi : X \to X/U$ be the canonical projection. Clearly, $\pi(u(x)) \neq 0$ and $\pi(u(x)) = \pi(p^2z) = p^2\pi(z)$, so $o(\pi(z)) = p^3$. Hence we can write $X/U = \langle\pi(z)\rangle \oplus \Gamma$ for some subgroup Γ of X/U [3, Lemma 15.1]. Denoting by φ the canonical projection of X/U onto $\langle\pi(z)\rangle$ with kernel Γ and letting $h : \langle\pi(z)\rangle \to X$ be the group homomorphism defined by $h(\pi(z)) = x$, it is clear that $h \circ \varphi \circ \pi \in E(X)$ and $(h \circ \varphi \circ \pi) \circ u \circ \varepsilon_{x,Y} = p^2\varepsilon_{x,Y}$, so $p^2\varepsilon_{x,Y} \in C$. Consequently, for any $(a, G) \in \mathcal{P}$, we have $p^2\varepsilon_{a,G} = f_{x,Y,a} \circ (p^2\varepsilon_{x,Y}) \circ f_{a,G,x} \in C$. $\square$

Lemma 10. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^3 satisfying $\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$. For any compact subset K of X and any neighbourhood U of zero in X, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary neighbourhood U of zero in X. Since X is totally disconnected, we can find a compact open subgroup U_0 of X such that $U_0 \subset U$ [5, (7.7)]. Let $K_0 = \langle K \cup U_0\rangle$. Then K_0 is compact [5, (9.8)], and $U_0 \subset K_0$. In particular, K_0 is topologically isomorphic to a topological direct product of cyclic p-groups of order at most p^3 [5, (25.9)], and so there exist three disjoint sets I_1, I_2, and I_3 such that $K_0 \cong \prod_{i\in I_1\cup I_2\cup I_3} C_i$, where $C_i = \mathbb{Z}(p)$ for $i \in I_1$, $C_i = \mathbb{Z}(p^2)$ for $i \in I_2$, and $C_i = \mathbb{Z}(p^3)$ for $i \in I_3$. Fix a topological isomorphism f from K_0 onto $\prod_{i\in I_1\cup I_2\cup I_3} C_i$. Given an arbitrary subset J of $I_1 \cup I_2 \cup I_3$, we denote by C'_J the subgroup of all $(c_i)_{i\in I_1\cup I_2\cup I_3} \in \prod_{i\in I_1\cup I_2\cup I_3} C_i$ satisfying $c_i = 0$ for all $i \notin J$. Since U_0 is open in K_0, there exist finite subsets $J_1 \subset I_1$, $J_2 \subset I_2$, and $J_3 \subset I_3$

such that $f(U_0) \supset C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)}$. We then have

$$\prod_{i\in I_1\cup I_2\cup I_3} C_i = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} C'_i\Big) \oplus C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)},$$

so

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} f^{-1}(C'_i)\Big) \oplus f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)}),$$

where C'_i stands for $C'_{\{i\}}$. Set $U' = f^{-1}(C'_{(I_1\setminus J_1)\cup(I_2\setminus J_2)\cup(I_3\setminus J_3)})$ and, for $i \in J_1 \cup J_2 \cup J_3$, let a_i be a generator of $f^{-1}(C'_i)$. Then U' is an open subgroup of X contained in U_0 and

$$K_0 = \Big(\bigoplus_{i\in J_1\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'.$$

We also have $o(a_i) = p$ if $i \in J_1$, $o(a_i) = p^2$ if $i \in J_2$, and $o(a_i) = p^3$ if $i \in J_3$. In the following, we shall construct a compact subgroup $K' \supset K_0$ which admits a decomposition similar to that of K_0, by replacing the elements a_i with $i \in J_1 \cup J_2$ by elements of order p^3. If $J_1 \cup J_2 = \varnothing$, we set $K' = K_0$. Suppose $J_1 \cup J_2 \neq \varnothing$, and fix an arbitrary $j \in J_1 \cup J_2$. We distinguish the cases when $j \in J_1$ and when $j \in J_2$. In the former case we use the equality $X[p] = \overline{p^2X}$ to find an element $b_j \in X$ such that $a_j - p^2b_j \in U'$. As $a_j \notin U'$, we cannot have $p^2b_j = 0$, so $o(b_j) = p^3$. We claim that

$$\langle p^2b_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Indeed, given any $x \in \langle p^2b_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg)$, we can write

$$x = lp^2b_j = \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} l_ia_i\Big) + y'$$

for some non-negative integers l, l_i and some $y' \in U'$. Since $y' + l(a_j - p^2b_j) \in U'$, it follows that

$$\begin{aligned} la_j &= \Big(\sum_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} l_ia_i\Big) + y' + l(a_j - p^2b_j) \\ &\in \langle a_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}, \end{aligned}$$

so p divides l, and hence $x = 0$. This proves our claim that

$$\langle p^2b_j\rangle \cap \bigg(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3} \langle a_i\rangle\Big) \oplus U'\bigg) = \{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big)=\{0\}.$$

In this case, we replace K_0 by

$$K_1=\langle b_j\rangle \oplus \Big(\bigoplus_{i\in(J_1\setminus\{j\})\cup J_2\cup J_3}\langle a_i\rangle\Big)\oplus U'.$$

Next we consider the second case when $j\in J_2$. We use the equality $X[p^2]=\overline{pX}$ to find an element $b_j\in X$ such that $a_j-pb_j\in U'$. Since then $pa_j-p^2b_j\in U'$ and $pa_j\notin U'$, we cannot have $p^2b_j=0$, so $o(b_j)=p^3$. We claim that

$$\langle pb_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big)=\{0\}.$$

Indeed, given any $x\in\langle pb_j\rangle\cap\Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big)$, we can write

$$x=lpb_j=\Big(\sum_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}l_ia_i\Big)+y'$$

for some non-negative integers l,l_i and $y'\in U'$. Since $y'+l(a_j-pb_j)\in U'$, it follows that

$$\begin{aligned} la_j &= \sum_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}l_ia_i+y'+l(a_j-pb_j)\\ &\in \langle a_j\rangle\cap\Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big) \;=\; \{0\},\end{aligned}$$

so p^2 divides l, and hence $x=0$. This proves our claim that

$$\langle pb_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big)=\{0\}.$$

Clearly, we then also have

$$\langle b_j\rangle \cap \Big(\Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'\Big)=\{0\}.$$

Consequently, in this case we can enlarge K_0 by considering

$$K_1=\langle b_j\rangle \oplus \Big(\bigoplus_{i\in J_1\cup(J_2\setminus\{j\})\cup J_3}\langle a_i\rangle\Big)\oplus U'.$$

Now, if $(J_1 \cup J_2) \setminus \{j\} \neq \varnothing$, we can apply the preceding procedure to K_1, and so, after a finite number of steps, we shall arrive at a compact subgroup K' of X having the following form:

$$K' = \Big(\bigoplus_{i \in J_1 \cup J_2} \langle b_i \rangle \Big) \oplus \Big(\bigoplus_{i \in J_3} \langle a_i \rangle \Big) \oplus U',$$

where $o(b_i) = p^3$ for all $i \in J_1 \cup J_2$. Since $a_i \in K'$ for all $i \in J_1 \cup J_2$, we also have $K \cup U' \subset K_0 \subset K'$, so K' and U' are those required. □

Lemma 11. *Let $p \in \mathbb{P}$, let $X \in \mathcal{L}$ be a group of finite exponent p^3 satisfying $\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$, and let $\mathcal{P}$ be the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Then the ideal $(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P})$, where $\varepsilon_{a,G} \in E(X)$ denotes the canonical projection of X onto $\langle a \rangle$ with kernel G, is dense in $E(X)$.*

Proof. Pick an arbitrary compact subset K of X and an arbitrary open neighbourhood U of zero in X. It suffices to show that

$$(\varepsilon_{a,G} \mid (a, G) \in \mathcal{P}) \cap [1_X + \Omega(K, U)] \neq \varnothing.$$

By Lemma 10, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i \in I} \langle a_i \rangle \oplus U'$ for some finite family $(a_i)_{i \in I}$ of elements of order p^3 of K'. Further, by Lemma 5 there is a subgroup G of X such that $U' \subset G$ and $X = \bigoplus_{i \in I} \langle a_i \rangle \oplus G$. Then $(a_j, \bigoplus_{i \in I \setminus \{j\}} \langle a_i \rangle \oplus G) \in \mathcal{P}$ for all $j \in I$, and

$$\sum_{j \in I} \varepsilon_{(a_j, \bigoplus_{i \in I \setminus \{j\}} \langle a_i \rangle \oplus G)} - 1_X \in \Omega(K', U') \subset \Omega(K, U).$$

□

With this preparation, we can now state the main result of this section.

Theorem 3. *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}$ be a group of finite exponent p^3. The following statements are equivalent:*

(i) *$E(X)$ has exactly two non-trivial closed ideals.*

(ii) *$\overline{pX} = X[p^2]$ and $\overline{p^2X} = X[p]$.*

Moreover, in case these conditions hold, the corresponding ideals are comparable with respect to set-theoretic inclusion.

Proof. The fact that (i) implies (ii) follows from Lemma 8. Assume (ii), and let $\mathcal{P}$ denote the set of all ordered pairs (a, G), where a is an element of order p^3 of X and G is a closed subgroup of X satisfying $X = \langle a \rangle \oplus G$. Further, for $(a, G) \in \mathcal{P}$, let $\varepsilon_{a,G} \in E(X)$ denote the canonical projection of X onto $\langle a \rangle$ with kernel G. Now, fix a non-zero closed ideal C of $E(X)$. We can have three possibilities for C.

First, suppose C contains elements of order p^3. Then $(\varepsilon_{a,G} \mid (a,G) \in \mathcal{P}) \subset C$ by Lemma 9, and hence $C = E(X)$ by Lemma 11.

Next suppose $p^2C = \{0\}$ and $pC \neq \{0\}$. Then $\overline{(p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})} \subset C$ by Lemma 9. To show the opposite inclusion, we pick any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 10, we can find two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'. It then follows from Lemma 5 that $X = \bigoplus_{i\in I}\langle a_i\rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p^2]$ and $X[p^2] = \overline{pX}$, for every $i \in I$ there exists $b_i \in X$ such that $pb_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. Then clearly $pv - u \in \Omega(K', U')$. Finally, by Lemma 11, there exists $w \in (\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. Since U' is a subgroup in X, we have $p(w - v) \in \Omega(K', U')$, and hence

$$pw - u = p(w - v) + (pv - u) \in \Omega(K', U') \subset \Omega(K, U).$$

As $pw \in (p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$, it follows that $u \in \overline{(p\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})}$.

Lastly, suppose $pC = \{0\}$. Then $\overline{(p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})} \subset C$ by Lemma 9. In order to establish the reverse inclusion, fix any $u \in C$, and let K be a compact subset of X and U an open neighbourhood of zero in X. By Lemma 10, there exist two compact open subgroups K', U' of X such that $K \cup U' \subset K'$, $U' \subset U$, and $K' = \bigoplus_{i\in I}\langle a_i\rangle \oplus U'$ for some finite family $(a_i)_{i\in I}$ of elements of order p^3 of K'. Consequently, $X = \bigoplus_{i\in I}\langle a_i\rangle \oplus Y$ for some closed subgroup Y of X containing U'. Since $\mathrm{im}(u) \subset X[p]$ and $X[p] = \overline{p^2X}$, for every $i \in I$ there exists $b_i \in X$ such that $p^2b_i - u(a_i) \in U'$. Define $v \in E(X)$ by setting $v(a_i) = b_i$ for all $i \in I$ and $v(y) = 0$ for all $y \in Y$. It is then clear that $p^2v - u \in \Omega(K', U')$. By Lemma 11, there exists $w \in (\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$ such that $w - v \in \Omega(K', U')$. As U' is a subgroup in X, we have $p^2(w - v) \in \Omega(K', U')$, and hence

$$p^2w - u = p^2(w - v) + (p^2v - u) \in \Omega(K', U') \subset \Omega(K, U).$$

Since $p^2w \in (p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})$, it follows that $u \in \overline{(p^2\varepsilon_{a,G} \mid (a,G) \in \mathcal{P})}$. □

Remark 3. Lemma 8 and Theorem 3 give an answer to our question in the case of LCA groups of finite exponent p^3, where $p \in \mathbb{P}$.

References

[1] Armacost D. L. *The structure of locally compact abelian groups,* Pure and Applied Mathematics Series, Vol. 68 (Marcel Dekker, ed.), New York, 1981.

[2] Braconnier J. *Sur les groupes topologiques localement compact,* J. Math. Pures Apl., 1948, **27**, No. 9, 1–85.

[3] Fuchs L. *Infinite abelian groups,* Vol. 1, Academic Press, New York and London, 1970.

[4] Engelking R. *General topology,* Warszawa, 1977.

[5] Hewitt E., Ross K. *Abstract Harmonic Analysis,* Vol. 1, Moscow, Nauka, 1975.

[6] Perticani F. *Commutative rings in which every proper ideal is maximal,* Fund. Math., 1971, **LXXI**, 193–198.

[7] Popa V. *Units, idempotents and nilpotents of an endomorphism ring. II,* Bul. Acad. Şt. Rep. Moldova, Matematica, 1997, No. 1(23), 93–105.

[8] Popa V. *LCA groups with topologically simple ring of continuous endomorphisms,* Bul. Acad. Şt. Rep. Moldova, Matematica, 1998, No. 2(27), 39–49.

[9] Popa V. *On LCA groups with compact rings of continuous endomorphisms,* Bul. Acad. Şt. Rep. Moldova, Matematica, 2000, No. 1(32), 17–32.

[10] Popa V. *Topological rings with at most two nontrvial closed ideals,* Bul. Acad. Şt. Rep. Moldova, Matematica, 2010, No. 3(64), 77–93.

Valeriu Popa
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova

E-mail: *vpopa@math.md*

Received September 19, 2011