BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(67), 2011, Pages 84–90
ISSN 1024–7696

Inclusion Radii for the Zeros of Special Polynomials

Matthias Dehmer

Abstract. To locate the zeros of complex-valued polynomials is a classical problem in algebra and function theory. For this, numerous inclusion radii have been established to estimate the moduli of the zeros of an underlying polynomial. In this note, we particularly state bounds for polynomials whose coefficients satisfy special conditions.

Mathematics subject classification: 30C15.
Keywords and phrases: Complex polynomials, zeros, inequalities.

1 Introduction

The analytic theory of polynomials [5] investigates properties of polynomials representing analytic functions. In particular the location of zeros of complex and real-valued polynomials has been extensively investigated [1–5]. To tackle this problem, we determine disks in the complex plane

$$K(z_0, r) := \{z \in \mathbb{C} \mid |z - z_0| \leq r\},\ z_0 \in \mathbb{C},\ r \in \mathbb{R}_+,$$

containing all zeros of a complex valued polynomial

$$f(z) = \sum_{i=0}^{n} a_i z^i,\ a_i \in \mathbb{C}, a_n \neq 0.$$

r is called inclusion radius. Clearly, $r = r(a_0, a_1, \ldots, a_n)$.

In this paper, we examine the location of zeros of special complex-valued polynomials of the form

$$f(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1 - 1}(z))(c_{n_2} z^{n_2} + g_{n_2 - 1}(z)). \tag{1}$$

That means, we infer bounds for the moduli of their zeros given by an inclusion radius. It turns out that these bounds are more practicable for this class of polynomials rather than applying existing zero bounds for general polynomials, see [1–5].

2 Results

In [1], Dehmer proved the following theorem.

Theorem 2.1. *Let $f(z)$ be a complex polynomial, such that $f(z)$ is reducible in $\mathbb{C}[z]$, namely*

$$f(z) := f_{n_1}(z)g_{n_2}(z) = (b_{n_1}z^{n_1} + f_{n_1-1}(z))(c_{n_2}z^{n_2} + g_{n_2-1}(z))$$

where

$$|b_{n_1}| > |b_i|, 0 \leq i \leq n_1 - 1, |c_{n_2}| > |c_i|, 0 \leq i \leq n_2 - 1. \tag{2}$$

If $n_1 + n_2 > 1$, then all zeros of the polynomial $f(z)$ lie in the closed disk $K(0, \delta)$, where $\delta > 1$ is the positive root of the equation

$$z^{n_1+n_2+2} - 4z^{n_1+n_2+1} + 2z^{n_1+n_2} + z^{n_2+1} + z^{n_1+1} - 1 = 0. \tag{3}$$

It holds $1 < \delta < 2 + \sqrt{2}$.

In the following, we prove some related theorems for this class of polynomials (see Equation (1)). An improvement of Theorem 2.1 is

Theorem 2.2. *Let*

$$f(z) := f_{n_1}(z)g_{n_2}(z) = (b_{n_1}z^{n_1} + f_{n_1-1}(z))(c_{n_2}z^{n_2} + g_{n_2-1}(z)),$$

where

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \textit{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|}, \tag{4}$$

and

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1. \tag{5}$$

All zeros of the polynomial $f(z)$ lie in the closed disk

$$K\left(0, \max\left[\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1-1)^2+4}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2-1)^2+4}}{2}\right]\right).$$

Proof. We start the proof by obtaining the estimation

$$\begin{aligned}|f_{n_1}(z)| &= |b_{n_1}z^{n_1} + f_{n_1-1}(z)| = \\ &= |b_{n_1}z^{n_1} + b_{n_1-1}z^{n_1-1} + \cdots + b_1 z + b_0| \geq \\ &\geq |b_{n_1}||z|^{n_1} - \left[|b_{n_1-1}||z|^{n_1-1} + \cdots + |b_1||z| + |b_0|\right].\end{aligned}$$

Using the relations $|b_{n_1}| > |b_i|, 0 \leq i \leq n_1 - 1$ (see Inequalities (5)), Equation (4) and $|z| > 1$, we further obtain

$$|f_{n_1}(z)| \geq |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \left[|z|^{n_1-2} + \cdots + |z| + 1 \right] \right] \tag{6}$$
$$= |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \frac{|z|^{n_1-1} - 1}{|z| - 1} \right]$$
$$> |b_{n_1}| \left[|z|^{n_1} - \phi_1 |z|^{n_1-1} - \frac{|z|^{n_1-1}}{|z| - 1} \right]$$
$$= \frac{|b_{n_1}||z|^{n_1-1}}{|z| - 1} \left[|z|^2 - |z|(1 + \phi_1) + (\phi_1 - 1) \right].$$

Clearly, applying this procedure to $f_{n_2}(z)$ also yields

$$|f_{n_2}(z)| > \frac{|c_{n_2}||z|^{n_2-1}}{|z| - 1} \left[|z|^2 - |z|(1 + \phi_2) + (\phi_2 - 1) \right].$$

By defining

$$H_{1,2}(z) := z^2 - z(1 + \phi_{1,2}) + (\phi_{1,2} - 1),$$

we get

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > \frac{|b_{n_1}||z|^{n_1-1}}{|z| - 1} \cdot \frac{|c_{n_2}||z|^{n_2-1}}{|z| - 1} H_1(|z|) \cdot H_2(|z|),$$

and

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0 \quad \text{if} \quad H_1(|z|) \cdot H_2(|z|) > 0.$$

Solving the last inequality requires to determine the zeros of $H_{1,2}(z)$. The zeros of $H_1(z)$ and $H_2(z)$ are

$$\frac{1 + \phi_1}{2} \pm \frac{\sqrt{(\phi_1 - 1)^2 + 4}}{2},$$

and

$$\frac{1 + \phi_2}{2} \pm \frac{\sqrt{(\phi_2 - 1)^2 + 4}}{2},$$

respectively. We easily see that

$$\alpha_1 := \frac{1 + \phi_1}{2} + \frac{\sqrt{(\phi_1 - 1)^2 + 4}}{2} > 1,$$

and

$$\alpha_2 := \frac{1 + \phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + 4}}{2} > 1.$$

This finally implies

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0,$$

if

$$|z| > \max\left(\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1-1)^2+4}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2-1)^2+4}}{2}\right),$$

and all zeros of $f(z)$ lie in $|z| \leq \max(\alpha_1, \alpha_2)$. □

Remark 1. The bound given by Theorem 2.2 is an improvement of the upper bound of Equation (3) given in Theorem 2.1 since

$$\frac{1+\phi_{1,2}}{2} + \frac{\sqrt{(\phi_{1,2}-1)^2+4}}{2} < 2+\sqrt{2}$$

if $\phi_{1,2} < 3$. But this is fulfilled by assumption, see Inequalities (5).

Assuming the special conditions for the polynomial's coefficients also leads to a bound whose value does not depend on any coefficients.

Theorem 2.3. *Let*

$$f(z) := f_{n_1}(z)g_{n_2}(z) = (b_{n_1}z^{n_1} + f_{n_1-1}(z))(c_{n_2}z^{n_2} + g_{n_2-1}(z)),$$

where

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \textit{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|},$$

and

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1. \tag{7}$$

All zeros of the polynomial $f(z)$ lie in the closed disk $K(0,2)$.

Proof. Using the Inequalities (7) and $|z| > 1$, we obtain

$$\begin{aligned}
|f_{n_1}(z)| &\geq |b_{n_1}|\left[|z|^{n_1} - \left[\frac{|b_{n_1-1}|}{|b_{n_1}|}|z|^{n_1-1} + \cdots + \frac{|b_1|}{|b_{n_1}|}|z| + \frac{|b_0|}{|b_{n_1}|}\right]\right] \\
&= |b_{n_1}|\left[|z|^{n_1} - \frac{|z|^{n_1}-1}{|z|-1}\right] > |b_{n_1}|\left[|z|^{n_1} - \frac{|z|^{n_1}}{|z|-1}\right] \\
&= \frac{|b_{n_1}||z|^{n_1}}{|z|-1}\left[|z|-2\right].
\end{aligned}$$

Analogously, we also conclude ($|z| > 1$)

$$|f_{n_2}(z)| > \frac{|c_{n_2}||z|^{n_1}}{|z|-1}\left[|z|-2\right].$$

Finally,

$$|f_{n_1}(z) \cdot f_{n_2}(z)| > 0 \quad \text{if} \quad |z| > 2,$$

and, hence, all zeros of $f(z)$ lie in $|z| \leq 2$. □

A more general statement is

Theorem 2.4. *Let*

$$f(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1 - 1}(z))(c_{n_2} z^{n_2} + g_{n_2 - 1}(z)).$$

Define

$$\phi_1 := \frac{|b_{n_1-1}|}{|b_{n_1}|} \quad \text{and} \quad \phi_2 := \frac{|c_{n_2-1}|}{|c_{n_2}|},$$

$$M_1 := \max_{0 \leq i \leq n_1 - 2} \left| \frac{b_i}{b_{n_1}} \right| \quad \text{and} \quad M_2 := \max_{0 \leq i \leq n_2 - 2} \left| \frac{c_i}{c_{n_2}} \right|.$$

All zeros of the polynomial $f(z)$ lie in the closed disk

$$K\left(0, \max\left[\frac{1+\phi_1}{2} + \frac{\sqrt{(\phi_1 - 1)^2 + 4M_1}}{2}, \frac{1+\phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + 4M_2}}{2}\right]\right).$$

Proof. Similar to Inequality (6) and by assuming $|z| > 1$, we infer

$$|f_{n_1}(z)| \geq |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \left[|z|^{n_1-2} + \cdots + |z| + 1 \right] \right]$$

$$= |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \frac{|z|^{n_1-1} - 1}{|z| - 1} \right]$$

$$> |b_{n_1}| \left[|z|^{n_1} - \frac{|b_{n_1-1}|}{|b_{n_1}|} |z|^{n_1-1} - M_1 \frac{|z|^{n_1-1}}{|z| - 1} \right]$$

$$= \frac{|b_{n_1}|}{|z| - 1} \left[|z|^{n_1+1} - |z|^{n_1} \left(1 + \frac{|b_{n_1-1}|}{|b_{n_1}|} \right) + |z|^{n_1-1} \left(\frac{|b_{n_1-1}|}{|b_{n_1}|} - M_1 \right) \right]$$

$$= \frac{|b_{n_1}||z|^{n_1-1}}{|z| - 1} \left[|z|^2 - |z| \left(1 + \phi_1 \right) + \left(\phi_1 - M_1 \right) \right],$$

and

$$|f_{n_1}(z)| \geq \frac{|c_{n_2}||z|^{n_1-1}}{|z| - 1} \left[|z|^2 - |z| \left(1 + \phi_2 \right) + \left(\phi_2 - M_2 \right) \right].$$

The rest of the proof is analogous to the proof steps of Theorem (2.2). □

3 Numerical Results

In this section, we evaluate the obtained bounds by using the following polynomials:

$$f_1(z) := (100z^3 - z^2 + iz + 50) \cdot (4z^4 + z^3 + 3z - 1),$$

$$f_2(z) := \left(2z^3 - z^2 + \frac{z}{2} + \frac{1}{10}\right) \cdot \left(\frac{z^3}{2} + \frac{z^2}{3} - \frac{z}{5} + \frac{1}{3}\right).$$

We start by evaluating the statements for $f_1(z)$ and first determine its zeros:

$$\begin{aligned} z_1 &\doteq -1.3039, \\ z_2 &\doteq -0.7903 + 0.0041i, \\ z_3 &\doteq \ 0.1285 - 0.7933i, \\ z_4 &\doteq \ 0.1285 + 0.7933i, \\ z_5 &\doteq \ 0.2968, \\ z_6 &\doteq 0.3965 + 0.6852i, \\ z_7 &\doteq 0.4038 - 0.6894i. \end{aligned}$$

$\max(|z_1|, |z_2|, \dots, |z_7|) = 1.3039$. Then, we yield $K(0, 3.3734)$ (Theorem 2.1), $K(0, 1.8827)$ (Theorem 2.2), $K(0, 2)$ (Theorem 2.3) and $K(0, 1.75)$ (Theorem 2.4). We see that Theorem 2.2 – Theorem 2.4 clearly outperform Theorem 2.1. For polynomials for which the conditions of the Equations (2) are satisfied, the bound given by Theorem 2.4 is always an improvement of Theorem 2.2 as $M_1, M_2 < 1$.

For $f_2(z)$, we get

$$\begin{aligned} z_1 &\doteq -1.3380, \\ z_2 &\doteq -0.1454, \\ z_3 &\doteq \ 0.3227 - 0.4896i, \\ z_4 &\doteq \ 0.3227 + 0.4896i, \\ z_5 &\doteq \ 0.3356 - 0.6209i, \\ z_6 &\doteq \ 0.3356 + 0.6209i. \end{aligned}$$

$\max(|z_1|, |z_2|, \dots, |z_6|) = 1.3380$. This leads to the disks $K(0, 3.3499)$ (Theorem 2.1), $K(0, 1.847127)$ (Theorem 2.2), $K(0, 2)$ (Theorem 2.3) and $K(0, 1.6666)$ (Theorem 2.4). By inspecting the bound values for this polynomial, we see that we get the same situation as in the case of $f_1(z)$.

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Matthias Dehmer
UMIT, Institute for Bioinformatics
and Translational Research
Eduard Wallnoefer Zentrum 1
A-6060, Hall in Tyrol, Austria
E-mail: *matthias.dehmer@umit.at*

Received July 23, 2011