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Categorial aspects of the semireflexivity

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Abstract. We examine the properties of semireflexive product, the relations between semireflexive subcategories, the right product of two subcategories and the factorization structures. We construct examples of semireflexive subcategories, also some problems are formulated.

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1 Introduction

For the theory of locally convex spaces we refer the reader to the monograph of Schaeffer (see [6]). Semireflexive and reflexive spaces are defined using the dual space. Many scientists have studied different classes of semireflexive spaces (see [1, 2, 5, 6]) by modifying this definition.

Definition 1 (see [6, Section IV 5.4]). *A locally convex space E is called semireflexive if the canonic inclusion $E \longrightarrow (E'_\beta)'$ is a surjective mapping: $E = (E'_\beta)'$.*

Definition 2 (see [6, Section IV 5.5]). *A locally convex space E is called reflexive if the canonic inclusion $E \longrightarrow (E'_\beta)'_\beta$ is a topological isomorphism of the space E on the second dual space with strong topology: $E = (E'_\beta)'_\beta$.*

Proposition 1 (see [6, Section IV 5.5]). *For a locally convex space E the following statements are equivalent:*

(a) the space E is semireflexive;

(b) every functional $\beta(E', E)$-continuous on E' is also continuous in the weak topology $\sigma(E', E)$;

(c) the space E'_τ (the space E' endowed with Mackey topology τ) is tunneled;

(d) every bounded set in E is compact in the weak topology $\sigma(E, E')$;

(e) the space E is quasicomplete in the weak topology $\sigma(E, E')$.

The criterium (e) permits a categorial formulation. It is used in the definition of the semireflexive product and of the semireflexive subcategories (see Definition 7).

We study the properties of semireflexive subcategories, the relations of the semireflexive product with the right product and we construct some examples.

Concerning the factorization structures (bicategory structures) see [4].

In Section 2 we examine the problem of factorization of one reflector functor within the factorization structures. In Section 3 we introduce the notion of $\varkappa$-functor (Definition 5), and Theorem 5 allows to construct examples of such functors. The property $(\mathcal{SR}t)$ generalizes the property $(\mathcal{SR})$ which often takes place in locally convex spaces. These conditions permit us to characterize and to construct examples of semireflexive subcategories in category $\mathcal{C}_2\mathcal{V}$ of locally convex topological Hausdorff vector spaces.

Definition 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is $\mathcal{B}$-hereditary if $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$ imply that $g \in \mathcal{A}$.*

Dual notion: the class is $\mathcal{B}$-cohereditary.

Definition 4. *The class $\mathcal{A}$ of morphisms of the category $\mathcal{C}$ is called right stable if from the fact that $u' \cdot v = v' \cdot u$ is pullback and $u \in \mathcal{A}$ it follows that $u' \in \mathcal{A}$.*

Dual notion: the class of morphisms is left stable.

We denote by $\mathcal{M}_u$ the class of right stable monomorphisms.

In the category $\mathcal{C}_2\mathcal{V}$ the monomorphism $m : X \longrightarrow Y$ belongs to the class $\mathcal{M}_u$ iff any functional defined on X is expanded through m (see [4]).

2 The factorization of the reflector functor

Any factorization structure $(\mathcal{P}, \mathcal{I})$ of the category $\mathcal{C}_2\mathcal{V}$ divides the class $\mathbb{R}$ of the non-zero reflective subcategories into three classes:

a) The class $\mathbb{R}(\mathcal{P})$ of the $\mathcal{P}$-reflective subcategories.

b) The class $\mathbb{R}(\mathcal{I})$ of the $\mathcal{I}$-reflective subcategories.

c) The class $\mathbb{R}(\mathcal{P}, \mathcal{I}) = (\mathbb{R} \backslash (\mathbb{R}(\mathcal{P}) \cup \mathbb{R}(\mathcal{I}))) \cup \{\mathcal{C}_2\mathcal{V}\}$ consisting of the subcategories which are neither $\mathcal{P}$-reflective nor $\mathcal{I}$-reflective (with the exception of the element $\mathcal{C}_2\mathcal{V}$). All these classes have the last element $\mathcal{C}_2\mathcal{V}$.

Theorem 1. *1 (see [7, Theorem 1.3]). The class $\mathbb{R}(\mathcal{P})$ possesses the first element $\overline{\mathcal{S}}$.*

2 (see [7, Theorem 2.2]). Let $(\mathcal{I} \cap \mathcal{E}pi, (\mathcal{I} \cap \mathcal{E}pi)^{\perp})$ be a right factorization structure. Then $\mathbb{R}(\mathcal{I})$ possesses the first element $\overline{\mathcal{A}}$ and

$$\mathbb{R}(\mathcal{I}) = \{\mathcal{R} \in \mathbb{R} \mid \overline{\mathcal{A}} \subset \mathcal{R}\}.$$

We mention that in the category $\mathcal{C}_2\mathcal{V}$ a proper class of the factorization structures has been constructed which possesses the property indicated in the previous theorem.

In the case of the factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$ = (the class of universal epimorphisms, the class of exact monomorphisms) = (the class of surgective mappings, the class of topological embeddings) we have the following division of the lattice $\mathbb{R}$ in three complete sublattices:

a) The sublattice $\mathbb{R}(\mathcal{E}_u)$ of the $\mathcal{E}_u$-reflective subcategories. A $\mathcal{E}_u$-reflective subcategory $\mathcal{R}$ is characterized by the fact that the $\mathcal{R}$-replica of every object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Another characteristic is:

$$\mathbb{R}(\mathcal{E}_u) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\},$$

where $\mathcal{S}$ is the subcategory of spaces with the weak topology.

b) The sublattice $\mathbb{R}(\mathcal{M}_p)$ of the $\mathcal{M}_p$-reflective subcategories, that means the class of those reflective subcategories $\mathcal{R}$ for which the $\mathcal{R}$-replica for any object of the category $\mathcal{C}_2\mathcal{V}$ is a topological embedding:

$$\mathbb{R}(\mathcal{M}_p) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\},$$

where Γ_0 is the subcategory of complete spaces.

c) $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p) = (\mathbb{R}\backslash(\mathbb{R}(\mathcal{E}_u) \cup \mathbb{R}(\mathcal{M}_p))) \cup \{\mathcal{C}_2\mathcal{V}\}$.
$\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p)$ is also a complete sublattice with the first element Π and the last element $\mathcal{C}_2\mathcal{V}$.

Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ a non-zero reflective subcategory. For any object X of the category $\mathcal{C}_2\mathcal{V}$ let $l^X : X \to lX$ be the $\mathcal{L}$-replica, and

$$l^X = i^X \cdot p^X, \tag{1}$$

the $(\mathcal{P}, \mathcal{I})$-factorization of respective morphism. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{L})$ the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ formed from all objects isomorphic with the objects bX when $X \in\mid \mathcal{C}_2\mathcal{V} \mid$. The subcategory $\mathcal{B}$ is $\mathcal{P}$-reflective, and $b^X : X \to bX$ is the $\mathcal{B}$-replica of object X.

Let $\mathcal{A}'' = \mathcal{A}''(\mathcal{L})$ be the full subcategory of all objects A with the property:

For any object X of the category $\mathcal{C}_2\mathcal{V}$, every morphism $f : bX \to A$ is extended via the morphism i^X: $\quad f = g \cdot i^X$ for some morphism g.

The subcategory $\mathcal{A}''$ is closed under products and $\mathcal{M}_f$-subobjects. So it is reflective, and $i^X : bX \longrightarrow lX$ is the $\mathcal{A}''$-replica of the object bX.

Let $l : \mathcal{C}_2\mathcal{V} \to \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \to \mathcal{B}$ and $a'' : \mathcal{C}_2\mathcal{V} \to \mathcal{A}''$ be the respective reflector functors. Then

$$l = a'' \cdot b. \tag{2}$$

Starting from this remarks we will denote:

by $G(\mathcal{L})$ the class of all $\mathcal{I}$-reflective subcategories $\mathcal{A}$ of the category $\mathcal{C}_2\mathcal{V}$ for which the reflector functor $a : \mathcal{C}_2\mathcal{V} \to \mathcal{A}$ verifies the relation $l = a \cdot b$;

by $\overline{G}(\mathcal{L})$ the class of all reflective subcategories $\mathcal{A}$ for which $l = a \cdot b$.

It is possible that $G(\mathcal{L})$ be the empty class. Also we mention that $G(\mathcal{L}) = \overline{G}(\mathcal{L}) \cap \mathbb{R}(\mathcal{I})$.

Theorem 2 (see [7, Theorem 3.2])**.** *Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$ so that $(\mathcal{I} \cap \mathcal{E}pi, (\mathcal{I} \cap \mathcal{E}pi)^{\perp})$ is a right factorization structure. Then for every element $\mathcal{L} \in \mathbb{R}$ we have:*

1. $\mathcal{A}''(\mathcal{L}) \in G(\mathcal{L})$.

2. The subcategory $\mathcal{A}' = \mathcal{A}'(\mathcal{L}) = \cap\{\mathcal{R} \mid \mathcal{R} \in G(\mathcal{L})\}$ belongs to class $\mathbb{R}(\mathcal{I})$.

3. $G(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{A}' \subset \mathcal{R} \subset \mathcal{A}''\}$.

For the class $\overline{G}(\mathcal{L})$ things are easier.

Theorem 3 (see [5, Theorem 2.7]). *For any factorization structure* $(\mathcal{P}, \mathcal{I})$ *we have:*

$$\overline{G}(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{L} \subset \mathcal{R} \subset \mathcal{A}''\}.$$

3 $\varkappa$-Functors

Definition 5. *A functor* $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ *is called a* $\varkappa$*-functor if*

$$t(E, u) = (E, t(u)), \quad u \leq t(u)$$

for every object (E, u).

Any non-zero coreflector functor $t : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ in composition with the embedding functor $i : \mathcal{K} \longrightarrow \mathcal{C}_2\mathcal{V}$ is a $\varkappa$-functor.

Let $\mathcal{R}$ and $\mathcal{K}$ be two non-zero subcategories of the category $\mathcal{C}_2\mathcal{V}$, where the $\mathcal{R}$ is reflective and the $\mathcal{K}$ is coreflective. For every object X of the category $\mathcal{C}_2\mathcal{V}$ let $r^X : X \longrightarrow rX$ and $k^{rX} : krX \longrightarrow rX$ be the $\mathcal{R}$-replica and the $\mathcal{K}$-coreplica of the respective objects. On these two morphisms we construct a pullback:

$$r^X \cdot t^X = k^{rX} \cdot u^X. \tag{1}$$

Theorem 4 (see [5, Theorem 3.4]). *The correspondence* $X \longrightarrow tX$ *defines a* $\varkappa$*-functor in the category* $\mathcal{C}_2\mathcal{V}$.

Definition 6. *The functor* t *defined in the previous section is called the* $\varkappa$*-functor generated by the reflective subcategory* $\mathcal{R}$ *and the coreflective one* $\mathcal{K}$.

Remark 1. We mention that a $\varkappa$-functor is not always a coreflector functor, since a $\varkappa$-functor is not necessarily idempotent.

Let $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ be a $\varkappa$-functor. For subcategory $\mathcal{R}$ we define the following condition:

($\mathcal{SR}t$) *Let* $(E, u) \in\mid \mathcal{R} \mid$. *Then, for every locally convex topology* v *on the vector space* E

$$u \leq v \leq t(u),$$

the space (E, v) *belongs to subcategory* $\mathcal{R}$.

Remark 2. 1. Let $\mathcal{M}$ be the coreflective subcategory of the spaces with Mackey topology, $m\colon \mathcal{C}_2\mathcal{V} \to \mathcal{M}$ be the coreflector functor. We denote the ($\mathcal{SR}m$) condition simply by ($\mathcal{SR}$).

2. Categorically, the condition ($\mathcal{SR}t$) can be formulated as follows:

($\mathcal{SR}t$) If $X \in\mid \mathcal{R} \mid$ and $f\colon Y \to X$ is a monomorphism such that $t^X = f \cdot g$ for some morphism g, then $Y \in |\mathcal{R}|$.

Since t^X is a bijective mapping, we deduce that so is f. In the given equality f and t^X are bijective mappings, so it follows that g is also a bijective mapping.

We will examine the property $(\mathcal{SR})$ for any elements of the classes $\mathbb{R}(\mathcal{E}_u), \mathbb{R}(\mathcal{M}_p)$ and $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p)$.

Theorem 5 (see [5, Theorem 3.8])**.** *1. Every element of the class $\mathbb{R}(\mathcal{M}_p)$ possesses the property $(\mathcal{SR})$.*

2. Let an element $\mathcal{L}$ of the class $\mathbb{R}(\mathcal{E}_u)$ possess the property $(\mathcal{SR})$. Then $\mathcal{L} = \mathcal{C}_2\mathcal{V}$.

4 Semireflexive product of two subcategories

Definition 7. *1. Let $\mathcal{R}$ be a reflective subcategory, and $\mathcal{A}$ be a subcategory of the category $\mathcal{C}$. The object X of the category C is called $(\mathcal{R}, \mathcal{A})$-semireflexive if its $\mathcal{R}$-replica belongs to the subcategory $\mathcal{A}$.*

2. The full subcategory of all $(\mathcal{R}, \mathcal{A})$-semireflexive objects is called the semireflexive product of the subcategories $\mathcal{R}$ and $\mathcal{A}$, and is denoted by $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$.

3. The subcategory $\mathcal{L} \in \mathbb{R}(\mathcal{E}_u, \mathcal{M}_p), \mathcal{L} \neq \mathcal{C}_2\mathcal{V}$ of the category $\mathcal{C}_2\mathcal{V}$ is called semireflexive if these exists a reflective subcategory $\mathcal{R} \in \mathbb{R}(\mathcal{E}_u)$ and a reflective subcategory $\Gamma \in \mathbb{R}(\mathcal{M}_p)$ of the category $\mathcal{C}_2\mathcal{V}$ so that $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.

Remark 3. The respective condition from the definition of the semireflexive subcategories has been imposed to exclude the trivial cases. So every subcategory $\mathcal{L}$ of the category $\mathcal{C}$ can be presented as $\mathcal{L} = \mathcal{C} \times_{sr} \mathcal{L}$.

Let $(\mathcal{P}, \mathcal{I})$ be a factorization structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ be a non-zero reflective subcategory. The $(\mathcal{P}, \mathcal{I})$-factorization of the reflector functor $l\colon \mathcal{C}_2\mathcal{V} \to \mathcal{L}$ generates the $\mathcal{P}$-reflective subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and the lattice $\overline{G}(\mathcal{L})$. Let $\Gamma \in \overline{G}(\mathcal{L})$. We examine the following conditions:

A. *$\mathcal{L} = \mathcal{B} \times_{sr} \Gamma$, where $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and $\Gamma \in \overline{G}(\mathcal{L})$.*

B. *There exists a pair of reflective subcategories $\mathcal{R} \in \mathbb{R}(\mathcal{P})$ and $\Gamma \in \mathbb{R}(\mathcal{I})$ of the category $\mathcal{C}_2\mathcal{V}$ so that $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.*

C. *The subcategory $\mathcal{L}$ is closed under $(\mathcal{P} \cap \mathcal{M}_u)$-subobjects.*

D. *The subcategory $\mathcal{L}$ verifies the condition $(\mathcal{SR})$ that means the subcategory $\mathcal{L}$ is closed under $(\mathcal{E}_u \cap \mathcal{M}_u)$-subobjects.*

E. *The subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$ verifies the condition $(\mathcal{SR}t)$ for $\varkappa$-functor $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$ generated by the reflective subcategory $\Gamma \in \overline{G}(\mathcal{L})$ and the coreflective subcategory $\mathcal{M}$ of the spaces with Mackey topology.*

Lemma 1. *1. In the previous conditions we have $\mathcal{L} \subset \mathcal{B} \times_{sr} \Gamma$, where $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and $\Gamma \in \overline{G}(\mathcal{L})$.*

2. For the objects of the subcategories $\mathcal{L}(\mathcal{L} \subset \mathcal{B}(\mathcal{L}))$ the condition $(\mathcal{SR}t)$ coincides with the condition $(\mathcal{SR})$.

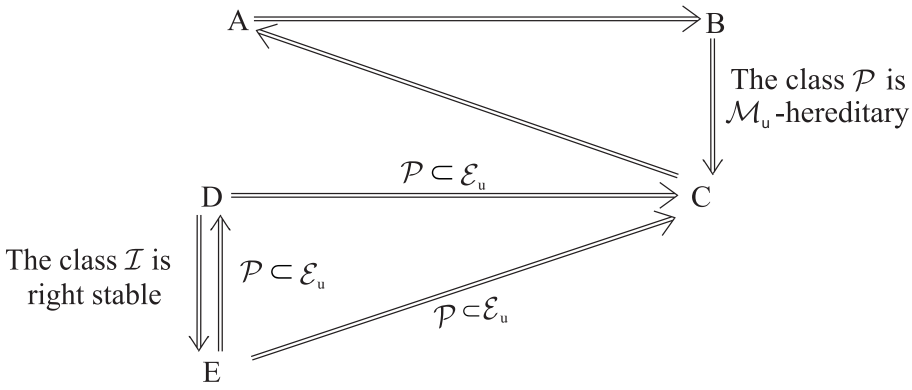
Theorem 6 (see [5, Theorem 4.5])**.** *The following implications are true:*

1. $C \Longrightarrow A \Longrightarrow B$.

2. Let $\mathcal{P}$ be an $\mathcal{M}_u$-hereditary class. Then $B \Longrightarrow C$.

3. Let $\mathcal{P} \subset \mathcal{E}_u$. Then $E \Longrightarrow D \Longrightarrow C$.

4. Let $\mathcal{I}$ be a right stable class. Then $D \Longrightarrow E$.



Theorem 7 (see [5, Theorem 4.6]). *In the case when $(\mathcal{P}, \mathcal{I}) = (\mathcal{E}_u, \mathcal{M}_p)$ the conditions $A - E$ are equivalent.*

5 Examples, conclusions, problems

Let $q\Gamma_0$ be a subcategory of the quasicomplete spaces, $s\mathcal{R}$ be a subcategory of semireflexive spaces [6], $\mathcal{S}$ be a subcategory of the spaces with weak topology, $\mathcal{N}$ be a subcategory of nuclear spaces. Then

$$\mathcal{R} \times_{sr} (q\Gamma_0) = s\mathcal{R},$$

for any reflective subcategory $\mathcal{R}$ with the property $\mathcal{S} \subset \mathcal{R} \subset \mathcal{N}$.

For the subcategory $\mathcal{S}c$ of the Schwartz spaces and the subcategory Γ_0 of the complete spaces we have

$$\mathcal{S}c \times_{sr} \Gamma_0 = \mathcal{K} \times_d (\mathcal{S}c \cap \Gamma_0) = i\mathcal{R},$$

where $i\mathcal{R}$ is the subcategory of semireflexive inductive spaces (see [1, Theorem 1.5]), and $\mathcal{K}$ is the coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ which forms with the subcategory $\mathcal{S}c$ a pair of conjugate subcategories [4].

The subcategory Π of complete spaces with weak topology is semireflexive. For the case $(\mathcal{P}, \mathcal{I}) = (\mathcal{E}_u, \mathcal{M}_p)$ we have $\mathcal{B}(\Pi) = \mathcal{S}$, the subcategory of spaces with weak topology, $\mathcal{A}'(\Pi) = \Gamma_0$, and $\mathcal{A}''(\Pi)$ contains all normed spaces. From this, it follows that $G(\Pi)$ is a proper class.

The condition D from Theorem 6 indicates the fact that the property of any subcategory to be semireflexive does not depend on the factorization structure $(\mathcal{P}, \mathcal{I})$.

Definition 8. *The subcategory $\mathcal{A}$ of the category $\mathcal{C}_2\mathcal{V}$ is called closed under extensions if $f \colon A \to B \in \mathcal{E}pi \cap \mathcal{M}_p$ and $A \in \mid \mathcal{A} \mid$ implies also that $B \in \mid \mathcal{A} \mid$.*

Problem 1. Let $\mathcal{R}$ be a reflective subcategory closed under extensions, and $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. When the right product $\mathcal{K} \times_d \mathcal{R}$ of the subcategories $\mathcal{K}$ and $\mathcal{R}$ is closed under extensions?

Let $\mathcal{B} = \mathcal{K} \times_d \mathcal{R}$ and assume that $\mathcal{B}$ is a reflective subcategory (see [3, Theorem 2.5] and [2, Theorem 5.3]), and moreover $\mathcal{B}$ is closed under extensions. In this case for every $\Gamma \in \mathbb{R}(\mathcal{M}_p)$ we have

$$\mathcal{B} \cap \Gamma = \mathcal{B} \times_{sr} \Gamma_1, \quad \Gamma_1 \in \widetilde{G}(\mathcal{B} \cap \Gamma).$$

Based on Theorem 2.12 [5] the subcategory $\mathcal{B}$ verifies the condition $(\mathcal{SR}t)$, where $t\colon \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ is the coreflector functor.

Problem 2. Is it true that $\mathcal{B} \cap \Gamma$ is a semireflexive subcategory?

Often, semireflexive subcategories can be presented as the right product of some subcategories [2, Theorem 5.4].

Problem 3. Is it true that every semireflexive subcategory is the right product of two subcategories?

References

[1] Berezansky I. A. *Inductive reflexive locally convex spaces*, DAN SSSR, 1968, **182**, No. 1, p. 20–22 (in Russian).

[2] Botnaru D., Cerbu O. *Semireflexive product of two subcategories*. Proceedings of the 6^{th}. Congress of Romanian Mathematicians, 2007, **I**, 5–19.

[3] Botnaru D., Turcanu A. *Les produits de gauche et de droite de deux souscategories*. Acta et Com., 2003, **III**, Chisinau, 57–73.

[4] Botnaru D., Turcanu A. *On Giraux subcategories in locally convex spaces*. ROMAI Journal, 2005, **I**, No. 1, 7–30.

[5] Cerbu O., Botnaru D., *Some property of semireflexivity*. Non-cummutative Structures in Mathematics and Physics, July 22–26, 2008, Brussel, p. 71-84.

[6] Schaeffer H. H. *Topological vector spaces*. The Macmillan Company, New York, Collier-Macmillan Limited, London, 1966.

[7] Turcanu A. *The factorization of reflective functors*. Bull. Inst. Politechn., Iasi, 2007, T. LII, (**LVII**), fasc. 5, 377–391.

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