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Vague Lie Ideals of Lie Algebras

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Abstract. In this paper, we have introduced the notion of vague Lie ideal and have studied their related properties. The cartesian products of vague Lie ideals are discussed. In particular, the Lie homomorphisms between the vague Lie ideals of a Lie algebra and the relationship between the domains and the co-domains of the vague Lie ideals under these Lie homomorphisms are investigated.

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1 Introduction

Lie algebras were first discovered by Sophus Lie (1842–1899) when he attempted to classify certain smooth subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. To study more about Lie algebras see [12]. There are many applications of Lie algebras in many branches of mathematics and physics [9].

The notion of fuzzy sets was first introduced by Zadeh [18]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics [15, 16]. Later many authors applied fuzzy set theory in Lie algebras [2–6, 10, 13, 14, 17].

The notion of vague theory was first introduced by Gau and Buechrer [11] in 1993. Later vague theory of the "group" concept into "vague group" was made by Biswas [7]. This work was the first vagueness of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. Further, in [1] Akram and Shum have studied vague Lie subalgebras over a vague field. Recently, Borumand Saeid applied vague set theory in $BCI/BCK-$algebras in [8]. The theory of vague sets started with the aim of interpreting the real life problems in a better way than the fuzzy sets do.

In this paper, we have introduced the notion of vague Lie ideals of Lie algebras and have studied their related properties. Characterization of vague Lie ideals on Lie homomorphisms is also presented.

2 Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper.

Definition 2.1. A *Lie algebra* is a vector space L over a field F (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $L \times L \longrightarrow L$ denoted by $(x, y) \longrightarrow [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,
(L2) $[x, x] = 0$ for all $x \in L$,
(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

In what follows, we denote L for Lie algebra, unless otherwise specified.

We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is *anticommutative*, i.e., $[x, y] = -[y, x]$. We call a subspace H of L closed under $[\cdot, \cdot]$ a *Lie subalgebra*. A subspace I of L with the property $[I, L] \subseteq I$ is called a *Lie ideal* of L. Obviously, any Lie ideal is a subalgebra.

Definition 2.1 [13]. A fuzzy set $\mu : L \to [0, 1]$ is said to be a *fuzzy Lie ideal* of L if the following conditions are satisfied:

(F1) $((\forall x, y \in L),\ \mu(x + y) \geq \min\{\mu(x), \mu(y)\})$,
(F2) $((\forall x, y \in L$ and $\alpha \in F),\ \mu(\alpha x) \geq \mu(x))$,
(F3) $((\forall x, y \in L),\ \mu([x, y]) \geq \max\{\mu(x), \mu(y)\})$.

Definition 2.2 [3]. Let μ be a fuzzy set on L, i.e., a map $\mu : L \to [0, 1]$. Then, μ is said to be an *anti fuzzy Lie ideal* of L if the following conditions are satisfied:

(AF1) $((\forall x, y \in L),\ \mu(x + y) \leq \max\{\mu(x), \mu(y)\})$,
(AF2) $((\forall x, y \in L$ and $\alpha \in F),\ \mu(\alpha x) \leq \mu(x))$,
(AF3) $((\forall x, y \in L),\ \mu([x, y]) \leq \mu(x))$.

Definition 2.3 [11]. A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(V1) A true membership function $t_A : U \to [0, 1]$, and
(V2) A false membership function $f_A : U \to [0, 1]$,

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and $t_A(u) + f_A(u) \leq 1$.

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

3 Vague Lie Ideals

In this section, we define the notion of vague Lie ideals.

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as:

$$\operatorname{imax}(I_1, I_2) \simeq [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$. Similarly, we define "imin". The concept of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $[0,1]$.

It is obvious that $L = \{I[0,1], \operatorname{isup}, \operatorname{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

Also, if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$, we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 3.1. Let L be a Lie algebra. A vague set A of L is called a *vague Lie subalgebra* of L if the following axioms hold:

(VLI1) $(\forall x, y \in L), (V(x+y) \succeq \operatorname{imin}\{V(x), V(y)\})$,

(VLI2) $(\forall x \in L, a \in F), (V(ax)) \succeq V(x))$.

(VLI3) $(\forall x, y \in L), (V([x,y]) \succeq \operatorname{imin}\{V(x), V(y)\})$.

That is,

$$\begin{aligned}
t_A(x+y) &\geq \min\{t_A(x), t_A(y)\}) \\
1 - f_A(x+y) &\geq \min\{1 - f_A(x), 1 - f_A(y)\} \\
t_A(ax) &\geq t_A(x) \\
1 - f_A(ax) &\geq 1 - f_A(x)) \\
t_A([x,y]) &\geq \min\{t_A(x), t_A(y)\}) \\
1 - f_A([x,y]) &\geq \min\{1 - f_A(x), 1 - f_A(y)\}.
\end{aligned}$$

Definition 3.2. Let L be a Lie algebra. A vague set A of L is called a *vague Lie ideal* of L if the following axioms hold:

It satisfies (VLI1), (VLI2) and (VLI4)$(\forall x, y \in L), (V([x,y]) \succeq \operatorname{imax}\{V(x), V(y)\})$.

That is,

$$\begin{aligned}
t_A([x,y]) &\geq \max\{t_A(x), t_A(y)\}) \\
1 - f_A([x,y]) &\geq \max\{1 - f_A(x), 1 - f_A(y)\}.
\end{aligned}$$

Example 3.3. Let $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\mathbb{R}^3$ with the bracket $[\cdot,\cdot]$ defined as the usual cross product, i.e.,

$$[x,y] = x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

forms a real Lie algebra over the field $\mathbb{R}$.

(1) Let A be the vague set in $\mathbb{R}^3$ defined as follows:

$$A = \begin{cases} [0.8, 0.1] & if \;\; x = y = z = 0, \\ [0.7, 0.2] & if \;\; x \neq 0, y = z = 0, \\ [0.5, 0.3] & \;\; otherwise. \end{cases}$$

By routine calculations, it is clear that A is a vague Lie subalgebra of $\mathbb{R}^3$, but not a vague ideal of $\mathbb{R}^3$, since

$$t_A([(1, 0, 0), (1, 1, 1)]) = t_A(0, -1, 1) = 0.5$$

and

$$\max\{t_A(1, 0, 0), t_A(1, 1, 1)\} = \max\{0.7, 0.5\} = 0.7.$$

Also,

$$1 - f_A([(1, 0, 0), (1, 1, 1)]) = 1 - f_A(0, -1, 1) = 0.7$$

and

$$\max\{1 - f_A(1, 0, 0), 1 - f_A(1, 1, 1)\} = \max\{0.8, 0.7\} = 0.8.$$

(2) Let A be the vague set in $\mathbb{R}^3$ defined as follows:

$$A = \begin{cases} [0.8, 0.1] & \text{if} \;\; (x, y, z) = (0, 0, 0), \\ [0.6, 0.2] & \text{otherwise}. \end{cases}$$

By routine calculations, it is clear that A is a vague Lie ideal of $\mathbb{R}^3$.

Proposition 3.4. *Let A be a vague set of L. Then A is a vague ideal of L if and only if t_A is a fuzzy ideal of L and f_A is an anti fuzzy ideal of L.*

Proof. The proof is obvious. □

For $\alpha, \beta \in [0, 1]$, now we define $(\alpha, \beta) - cut$ and $\alpha - cut$ of a vague set.

Definition 3.5. Let A be a vague set in L with true membership function t_A and the false membership function f_A. The $(\alpha, \beta) - cut$ of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set L given by

$$A_{(\alpha,\beta)} = \{x \in L \,|V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly, $A_{(0,0)} = L$. The $(\alpha, \beta)-$cuts of the vague set A are also called *vague sets* of A.

Definition 3.6. The $\alpha - cut$ of the vague set A is a crisp subset A_α of the set L given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = L$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$. Equivalently, we can define the $\alpha-$cut as

$$A_{(\alpha)} = \{x \in L \,|t_A(x) \geq \alpha\}.$$

Theorem 3.7. *Let A be a vague set of L. Then A is a vague Lie ideal of L if and only if $A_{(\alpha,\beta)}$ is a Lie ideal of L for every $\alpha, \beta \in (0, 1]$.*

Proof. Let A be a vague set of L. Suppose A is a vague Lie ideal of L.

For all $x, y \in A_{(\alpha,\beta)}$ and $\alpha, \beta \in (0, 1]$, then

$$t_A(x), t_A(y) \geq \alpha \text{ and } 1 - f_A(x), 1 - f_A(y) \geq \beta.$$

Then we have

(i)

$$t_A(x + y) \geq t_A(x) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$$

and

$$1 - f_A(x + y) \geq 1 - f_A(x) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta.$$

Thus $x + y \in A_{(\alpha,\beta)}$.

(ii)

$$t_A(ax) \geq t_A(x) \geq \alpha \text{ and } 1 - f_A(ax) \geq 1 - f_A(x) \geq \beta.$$

Thus $ax \in A_{(\alpha,\beta)}$.

(iii)

$$t_A([x, y]) \geq \max\{t_A(x), t_A(y)\} \geq \alpha,$$

and

$$1 - f_A([x, y]) \geq \max\{1 - f_A(x), 1 - f_A(y)\} \geq \beta,$$

which implies $[x, y] \in A_{(\alpha,\beta)}$. Thus $A_{(\alpha,\beta)}$ is a Lie ideal of L.

Conversely, assume that $A_{(\alpha,\beta)} \neq \emptyset$ is a Lie ideal of L for every $\alpha, \beta \in (0, 1]$. Assume that

$$V(x + y) \prec \text{imin}\{V(x), V(y)\}$$

for some $x, y \in L$. Taking

$$\alpha_1 = \frac{1}{2}\{t_A(x + y) + \min\{t_A(x), t_A(y)\}\}$$

and

$$\beta_2 = \frac{1}{2}\{1 - f_A(x + y) + \min\{1 - f_A(x), 1 - f_A(y)\}\}$$

for some $x, y \in L$, we have

$$t_A(x + y) < \alpha_1 < \min\{t_A(x), t_A(y)\}$$

and

$$1 - f_A(x+y) < \beta_2 < \min\{1 - f_A(x), 1 - f_A(y)\}.$$

So, we have $x+y \notin A_{(\alpha_1,\beta_2)}$, for all $x, y \in A_{(\alpha_1,\beta_2)}$. This is a contradiction. Thus

$$V(x+y) \succeq \mathrm{imin}\{V(x), V(y)\}.$$

Similarly, we can prove (VLI2), (VLI3) and (VLI4). Hence A is a vague ideal of L. This completes the proof. □

Theorem 3.8. *If $\{A_i | i \in I\}$ is an arbitrary family of vague Lie ideals of L then $\bigcap A_i$ is a vague Lie ideals of L, where $\bigcap A_i(x) = inf\{A_i(x) | i \in I)\}$, for all $x \in L$.*

Proof. The proof is trivial. □

However, the union of two vague Lie ideals cannot be a vague ideal. Let A and B be two vague Lie ideals of L. Define

$$(A \cup B)(x) = \max\{A(x), B(x)\}, \text{ for all } x \in L.$$

The following example shows that $A \cup B$, cannot be a vague Lie ideal of L.

Example 3.9. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space V over a field F. Then, it is not difficult to see that, by putting: $[e_1, e_2] = e_5$, $[e_1, e_3] = e_6$, $[e_1, e_4] = e_7$, $[e_1; e_5] = -e_8$, $[e_2, e_3] = e_8$, $[e_2; e_4] = e_6$, $[e_2, e_6] = -e_7$, $[e_3, e_4] = -e_5$, $[e_3, e_5] = -e_7$, $[e_4; e_6] = -e_8$, $[e_i; e_j] = -[e_j, e_i]$ and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over the field F. Define the vague sets A and B for all $x \in V$ as follows:

$$A = \begin{cases} [0.8, 0.1] & if \;\; x = 0, e_8, \\ [0.6, 0.2] & if \;\; x = e_7, \\ [0.2, 0.6] & otherwise. \end{cases}$$

and

$$B = \begin{cases} [0.8, 0.1] & if \;\; x = 0, e_7, \\ [0.5, 0.3] & if \;\; x = e_8, \\ [0.2, 0.6] & otherwise. \end{cases}$$

Then A and B are vague Lie ideal of V, since by Theorem 3.7, the vague-cut sets, $A_{(0.8,0.1)} = \langle e_8 \rangle$, $B_{(0.8,0.1)} = \langle e_7 \rangle$ and $A_{(0.6,0.2)} = B_{(0.5,0.3)} = \langle e_7, e_8 \rangle$ are vague Lie ideals of V, but

$$(t_A \cup t_B)(e_7 + e_8) = \max\{t_A(e_7 + e_8), t_B(e_7 + e_8)\} \geq$$

$$\geq \max\{\min\{t_A(e_7), t_A(e_8)\}, \min\{t_B(e_7), t_B(e_8)\}\} = \max\{0.6, 0.5\} = 0.6.$$

and

$$(1 - f_A \cup f_B)(e_7 + e_8) = \max\{1 - f_A(e_7 + e_8), 1 - f_B(e_7 + e_8)\} \geq$$

$$\geq \max\{\min\{1 - f_A(e_7), 1 - f_A(e_8)\}, \min\{1 - f_B(e_7), 1 - f_B(e_8)\}\} =$$

$$= \max\{0.8, 0.7\} = 0.8.$$

On the other hand

$$\min\{(t_A \cup t_B)(e_7), (t_A \cup t_B)(e_8)\} =$$
$$= \min\{\max\{t_A(e_7), t_B(e_7)\}, \max\{t_A(e_8), t_B(e_8)\}\} =$$
$$= \min\{0.8, 0.8\} = 0.8.$$

and

$$\min\{1 - (f_A \cup f_B)(e_7), 1 - (f_A \cup f_B)(e_8)\} =$$
$$= \min\{\max\{1 - f_A(e_7), 1 - f_B(e_7)\}, \max\{1 - f_A(e_8), 1 - f_B(e_8)\}\} =$$
$$= \min\{0.9, 0.9\} = 0.9.$$

Thus we have

$$(t_A \cup t_B)(e_7 + e_8) = 0.6 \not\geq 0.8 = \min\{(t_A \cup t_B)(e_7), (t_A \cup t_B)(e_8)\}$$

and

$$1 - (f_A \cup f_B)(e_7 + e_8) = 0.8 \not\geq 0.9 = \min\{1 - (f_A \cup f_B)(e_7), 1 - (f_A \cup f_B)(e_8)\}.$$

Therefore, $(A \bigcup B)$ is not a vague Lie ideal.

Definition 3.10. Let A and B be two vague Lie ideals of L. We define the *sup − min product* $[AB]$ of A and B by

$$[t_A t_B](x) = \begin{cases} \sup\limits_{x=[yz]} \min\{t_A(y), t_B(z)\}, \\ \\ 0, \qquad x \neq yz \end{cases}$$

and

$$[1 - f_A f_B](x) = \begin{cases} \sup\limits_{x=[yz]} \min\{1 - f_A(y), 1 - f_B(z)\}, \\ \\ 0, \qquad x \neq yz. \end{cases}$$

Let A and B be vague Lie ideals of the Lie algebra L. Then $[AB]$ may not be a vague Lie ideal of L as this can be seen in the following counter-example:

Example 3.11. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space over a field F. Then, it is not difficult to see that, by putting: $[e_1, e_2] = e_5$, $[e_1, e_3] = e_6$, $[e_1, e_4] = e_7$, $[e_1; e_5] = -e_8$, $[e_2, e_3] = e_8$, $[e_2; e_4] = e_6$, $[e_2, e_6] = -e_7$, $[e_3, e_4] = -e_5$, $[e_3, e_5] = -e_7$, $[e_4; e_6] = -e_8$, $[e_i; e_j] = -[e_j, e_i]$ and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over the field F. The following vague sets

$$A = \begin{cases} [0.7, 0.1] & if\ \ x = 0, e_1, e_5, e_6, e_7, e_8, \\ \\ [0.2, 0.6] & otherwise. \end{cases}$$

and

$$B = \begin{cases} [0.7, 0.1] & if \;\; x = 0, \\ \\ [0.5, 0.2] & if \;\; x = e_2, e_5, e_6, e_7, e_8, \\ \\ [0.2, 0.6] & otherwise. \end{cases}$$

Thus A and B are vague Lie ideals of L because the cut Lie ideals of L $A_{(0.7,0.1)} =< e_1, e_5, e_6, e_7, e_8 >$ and $B_{(0.5,0.2)} =< e_2, e_5, e_6, e_7, e_8 >$ are vague-cut Lie ideals of L. But $[AB]$ is not a vague Lie ideal because the following condition does not hold:

$$[V_A V_B](e_7 + e_8) \succeq \mathrm{imin}\{[V_{AB}](e_7), [V_{AB}](e_8)\},$$

$$t_A t_B(e_7) = \sup \begin{cases} \min\{t_A(e_1), t_B(e_4)\} = \min\{0.7, 0.2\} = 0.2, e_7 = [e_1, e_4], \\ \min\{t_A(e_2), t_B(e_6)\} = \min\{0.2, 0.5\} = 0.2, e_7 = -[e_2, e_6], \\ \min\{t_A(e_3), t_B(e_5)\} = \min\{0.2, 0.5\} = 0.2, e_7 = -[e_3, e_5], \\ \min\{t_A(e_4), t_B(e_1)\} = \min\{0.2, 0.2\} = 0.2, e_7 = -[e_4, e_1], \\ \min\{t_A(e_6), t_B(e_2)\} = \min\{0.7, 0.5\} = 0.5, e_7 = [e_6, e_2], \\ \min\{t_A(e_5), t_B(e_3)\} = \min\{0.7, 0.2\} = 0.2, e_7 = [e_5, e_3] \end{cases}$$

and

$1 - f_A f_B(e_7) =$

$$= \sup \begin{cases} \min\{1 - f_A(e_1), 1 - f_B(e_4)\} = \min\{0.9, 0.4\} = 0.4, e_7 = [e_1, e_4], \\ \min\{1 - f_A(e_2), 1 - f_B(e_6)\} = \min\{0.4, 0.8\} = 0.4, e_7 = -[e_2, e_6], \\ \min\{1 - f_A(e_3), 1 - f_B(e_5)\} = \min\{0.4, 0.8\} = 0.4, e_7 = -[e_3, e_5], \\ \min\{1 - f_A(e_4), 1 - f_B(e_1)\} = \min\{0.9, 0.4\} = 0.4, e_7 = -[e_4, e_1], \\ \min\{1 - f_A(e_6), 1 - f_B(e_2)\} = \min\{0.9, 0.8\} = 0.8, e_7 = [e_6, e_2], \\ \min\{1 - f_A(e_5), 1 - f_B(e_3)\} = \min\{0.9, 0.4\} = 0.4, e_7 = [e_5, e_3]. \end{cases}$$

Thus $t_A t_B(e_7) = 0.5$ and $1 - f_A f_B(e_7) = 0.8$.

Similarly, we can get $t_A t_B(e_8) = 0.5$ and $1 - f_A f_B(e_8) = 0.8$.

On the other hand, we have

$$t_A t_B(e_7 + e_8) = sup\{i - vi\}$$

and

$$1 - f_A f_B(e_7 + e_8) = sup\{i - vi\}$$

(i) if $e_7 + e_8 = [e_1(e_4 - e_5)]$, then

$$\min\{t_A(e_1), t_B(e_4 - e_5)\} = \min\{t_A(e_1), t_B(e_4), t_B(e_5)\} = 0.2,$$

since $t_B(e_4) = 0.2$, and if $e_7 + e_8 = [(e_5 - e_4)e_1]$, then

$$\min\{t_A(e_5 - e_4), t_B(e_1)\} = \min\{t_A(e_5), t_A(e_4), t_B(e_1)\} = 0.2,$$

since $t_A(e_4) = 0.2$.

Similarly,
(ii) the cases $e_7 + e_8 = [e_2(e_3 - e_6)]$; then the value is 0.2,
(iii) the cases $e_7 + e_8 = [e_3(-e_2 - e_5)]$; then the value is 0.2,
(iv) the cases $e_7 + e_8 = [e_4(-e_1 - e_6)]$; then the value is 0.2,
(v) the cases $e_7 + e_8 = [e_5(-e_3 - e_1)]$; then the value is 0.2,
(vi) the cases $e_7 + e_8 = [e_6(-e_2 - e_4)]$; then the value is 0.2.
Thus,

$$t_At_B(e_7 + e_8) = \min\{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\} = 0.2.$$

Hence, we have proved that

$$t_At_B(e_7 + e_8) = 0.2 \ngeq 0.5 = \min\{t_At_B(e_7), t_At_B(e_8)\}.$$

On the other hand, we can prove
$1 - f_Af_B(e_7 + e_8) = 0.4 \ngeq 0.8 = \min\{1 - f_Af_B(e_7), 1 - f_Af_B(e_8)\}$.
Now we redefine the product of two vague Lie ideals A and B of L to an extended form.

Definition 3.12. Let A and B be two vague sets of L. Then, we define the *sup − min product* $[\![AB]\!]$ of A and B, as follows, for all $x, y, z \in L$:

$$[\![t_At_B]\!](x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_iy_i]} \left\{\min\limits_{i\in N}\{\min(t_A(x_i), t_B(y_i))\}\right\}, \\ 0, \qquad x \neq \sum\limits_{i=1}^{n}[x_iy_i] \end{cases}$$

and

$$[\![1 - f_Af_B]\!](x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_iy_i]} \left\{\min\limits_{i\in N}\{\min(1 - f_A(x_i), 1 - f_B(y_i))\}\right\}, \\ 0, \qquad x \neq \sum\limits_{i=1}^{n}[x_iy_i]. \end{cases}$$

From the definitions of $[AB]$ and $[\![AB]\!]$, we can easily see that $[AB] \subseteq [\![AB]\!]$ and $[AB] \neq [\![AB]\!]$.

The following theorem proves $[\![AB]\!]$ is a vague Lie ideal of L if A and B are vague Lie ideals of L.

Theorem 3.13. *Let A and B be any two vague Lie ideals of L. Then $[\![AB]\!]$ is also a vague Lie ideal of L.*

Proof. It is easy to prove $[\![AB]\!]$ is a vague Lie subalgebra of L.
(iv) Suppose $x, y \in L$. Let if possible,

$$[\![V_AV_B]\!]([x, y]) \prec \text{imax}\{[\![V_AV_B]\!](x), [\![V_AV_B]\!](y)\}.$$

Then we have

$$[\![V_A V_B]\!]([x,y]) \prec [\![V_A V_B]\!](x) \text{ or } [\![V_A V_B]\!]([x,y]) \prec [\![V_A V_B]\!](y).$$

Choose a number $t < s \in [0,1]$ such that

$$[\![t_A t_B]\!]([x,y]) < t < [\![t_A t_B]\!](x), [\![t_A t_B]\!]([x,y]) < t < [\![t_A t_B]\!](y).$$

and

$$\begin{aligned}
&[\![1 - f_A f_B]\!]([x,y]) < s < [\![1 - f_A f_B]\!](x),\\
&[\![1 - f_A f_B]\!]([x,y]) < s < [\![1 - f_A f_B]\!](y).
\end{aligned}$$

There exist $x_i, y_i \in L$ such that $x = \sum_{i=1}^{n} [x_i y_i]$.
For all i, j we have,

$$t_A(x_i) > t, t_B(y_i) > t$$

and

$$1 - f_A(x_i) > s, 1 - f_B(y_i) > s.$$

Since $[x,y] = \left[\sum_{i=1}^{n} [x_i, y_i], y\right]$, we have

$$\begin{aligned}
[\![t_A t_B]\!]([x,y]) &= [\![t_A t_B]\!]\left(\left[\sum_{i=1}^{n} [x_i, y_i], y\right]\right)\\
&= [\![t_A t_B]\!]\left(\sum_{i=1}^{n} [[x_i, y_i], y]\right)\\
&\geq [\![t_A t_B]\!]([[x_i, y_i], y]), \text{for all } i\\
&= [\![t_A t_B]\!]([[x_i, y], y_i] - [[y_i, y], x_i])\\
&\geq [\![t_A t_B]\!]([[x_i, y], y_i])\\
&\geq \max\{t_A [x_i, y], t_B(y_i)\}\\
&\geq \max\{\max\{t_A(x_i), t_A(y)\}, t_B(y_i)\}\\
&> t.
\end{aligned}$$

and

$$\begin{aligned}
[\![1 - f_A f_B]\!]([x,y]) &= [\![1 - f_A f_B]\!]\left(\left[\sum_{i=1}^{n} [x_i, y_i], y\right]\right)\\
&= [\![1 - f_A f_B]\!]\left(\sum_{i=1}^{n} [[x_i, y_i], y]\right)\\
&\geq [\![1 - f_A f_B]\!]([[x_i, y_i], y]), \text{for all } i\\
&= [\![1 - f_A f_B]\!]([[x_i, y], y_i] - [[y_i, y], x_i])\\
&\geq [\![1 - f_A f_B]\!]([[x_i, y], y_i])
\end{aligned}$$

$$\begin{aligned} &\geq \max\left\{1 - f_A\left[x_i, y\right], 1 - f_B(y_i)\right\} \\ &\geq \max\left\{\max\{1 - f_A(x_i), 1 - f_A(y)\}, 1 - f_B(y_i)\right\} \\ &> s. \end{aligned}$$

Thus, we have

$$[\![1 - f_A f_B]\!]\,([x, y]) > t \text{ and } [\![1 - f_A f_B]\!]\,([x, y]) > s,$$

which is a contradiction. Thus $[\![AB]\!]$ satisfies (VLI4). Hence $[\![AB]\!]$ is a vague Lie ideal of L. □

The following theorem characterized congruence relation on L.

Theorem 3.14. *Let A be a vague Lie ideal of L. Define a binary relation $\sim$ on L by $x \sim y$ if and only if $t_A(x - y) = t_A(0), 1 - f_A(x - y) = 1 - f_A(0)$ for all $x, y \in L$. Then $\sim$ is a congruence relation on L.*

Proof. To prove $\sim$ is an equivalent relation, it is enough to show the transitivity of $\sim$ because the reflectivity and symmetricity of $\sim$ hold trivially. Let $x, y, z \in L$. If $x \sim y$ and $y \sim z$, then $t_A(x - y) = t_A(0)$, $t_A(y - z) = t_A(0)$ and $1 - f_A(x - y) = 1 - f_A(0)$, $1 - f_A(y - z) = 1 - f_A(0)$. Hence it follows that

$$t_A(x - z) = t_A(x - y + y - z) \geq \min\{t_A(x - y), t_A(y - z)\} = t_A(0)$$

and

$$1 - f_A(x - z) = 1 - f_A(x - y + y - z) \geq \min\{1 - f_A(x - y), 1 - f_A(y - z)\} = 1 - f_A(0).$$

Consequently $x \sim z$. We now verify that " $\sim$ " is a congruence relation on L. For this purpose, we let $x \sim y$ and $y \sim z$. Then

$$t_A(x - y) = t_A(0), t_A(y - z) = t_A(0)$$

and

$$1 - f_A(x - y) = 1 - f_A(0), 1 - f_A(y - z) = 1 - f_A(0).$$

Now, for $x_1, x_2, y_1, y_2 \in L$, we have

(i)

$$\begin{aligned} t_A((x_1 + x_2) - (y_1 + y_2)) &= t_A((x_1 - y_1) + (x_2 - y_2)) \\ &\geq \min\{t_A(x_1 - y_1), t_A(x_2 - y_2) = t_A(0) \end{aligned}$$

and

$$\begin{aligned} 1 - f_A((x_1 + x_2) - (y_1 + y_2)) &= 1 - f_A((x_1 - y_1) + (x_2 - y_2)) \\ &\geq \min\{1 - f_A(x_1 - y_1), 1 - f_A(x_2 - y_2)\} \\ &= 1 - f_A(0), \end{aligned}$$

(ii)

$$t_A((ax_1 - ay_1)) = t_A(a(x_1 - y_1)) \geq t_A(x_1, y_1) = t_A(0)$$

and

$$1 - f_A((ax_1 - ay_1)) = 1 - f_A(a(x_1 - y_1)) \geq 1 - f_A(x_1, y_1) = 1 - f_A(0),$$

(iii)

$$\begin{aligned} t_A([x_1, x_2] - [y_1, y_2]) &= t_A([(x_1 - y_1), (x_2 - y_2)]) \\ &\geq \max\{t_A(x_1 - y_1), t_A(x_2 - y_2)\} = t_A(0) \end{aligned}$$

and

$$\begin{aligned} 1 - f_A([x_1, x_2] - [y_1, y_2]) &= 1 - f_A([(x_1 - y_1), (x_2 - y_2)]) \\ &\geq \max\{1 - f_A(x_1 - y_1), 1 - f_A(x_2 - y_2)\} \\ &= 1 - f_A(0). \end{aligned}$$

That is, $x_1 + x_2 \sim y_1 + y_2$, $ax_1 \sim ay_1$ and $[x_1, x_2] \sim [y_1, y_2]$. Thus, " $\sim$ " is indeed a congruence relation on L. $\square$

4 Characterization of vague Lie ideals on Lie Homomorphisms

Definition 4.1. Let L and L' be two Lie algebras over a field F. Then a linear transformation $f : A \to B$ is called a *Lie homomorphism* if $g[x, y] = [g(x), g(y)]$ holds, for all $x, y \in L$.

Let $g : L \longrightarrow L'$ be a Lie homomorphism. For any vague set A in L', we define the *preimage* of A under g, denoted by $g^{-1}(A)$, is a vague set in L defined by

$$g^{-1}(t_A) = t_{A_{g^{-1}}}(x) = t_A(g(x))$$

and

$$1 - g^{-1}(f_A) = 1 - f_{A_{g^{-1}}}(x) = 1 - f_A(g(x)), \forall x \in L.$$

For any vague set A in G, we define the *image* of A under a linear transformation g, denoted by $g(A)$, is a vague set in G' defined by

$$g\left(t_A\right)(y) = \begin{cases} \sup\limits_{x \in g^{-1}(y)} \quad t_A(x) & if \quad g^{-1}(y) \neq \phi, \\ 0 & otherwise. \end{cases}$$

and

$$g\left(f_A\right)(y) = \begin{cases} \inf\limits_{x \in g^{-1}(y)} \quad f_A(x) & if \quad g^{-1}(y) \neq \phi, \\ 0 & otherwise. \end{cases}$$

for all $x \in L$ and $y \in L'$.

Theorem 4.2. *Let g be a surjective Lie homomorphism from L into L'.*

(i) If A and B are two vague Lie ideals of L, then

$$g(A + B) = g(A) + g(B).$$

(ii) If $\{A_i | i \in I\}$ is a set of $g-$invariant vague Lie ideal of L, then

$$g\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} g\left(A_i\right).$$

(iii) If A and B are two vague Lie ideals of L, then

$$g([\![V_A V_B]\!]) \simeq [\![g(V_A)g(V_B)]\!].$$

Proof. The proofs of (i) and (ii) are trivial. To prove (iii), let $x \in L$. Suppose $g([\![V_A V_B]\!])(x) \prec [\![g(V_A)g(V_B)]\!](x)$. Now, we can choose a number $t < s \in [0,1]$ such that

$$g([\![t_A t_B]\!])(x) < t < [\![g(t_A)g(t_B)]\!](x)$$

and

$$g([\![1 - f_A f_B]\!])(x) < s < [\![g(1-f_A)g(1-f_B)]\!](x).$$

Then, there exist $y_i, z_i \in L'$ such that $x = \sum\limits_{i=1}^{n} [y_i, z_i]$ with $g(t_A) > t$, $g(t_B) > t$ and $g(1-f_A) > s$, $g(1-f_B) > s$. Since g is surjective, there exists a $y \in L$ such that $g(y) = x$ and $y = \sum\limits_{i=1}^{n} [a_i, b_i]$, for some $a_i \in g^{-1}(y_i)$ and $b_i \in g^{-1}(z_i)$ with $g(a_i) = y_i$ and $g(a_i) = y_i$, $t_A(a_i) > t$, $t_B(b_i) > t$ and $1 - f_A(a_i) > s$, $1 - f_B(b_i) > s$. Since

$$\begin{aligned} g\left(\sum_{i=1}^{n} [a_i, b_i]\right) &= \sum_{i=1}^{n} g\left([a_i, b_i]\right) = \sum_{i=1}^{n} [g(a_i), g(b_i)] \\ &= \sum_{i=1}^{n} [y_i, z_i] = x, \end{aligned}$$

we have $g([\![t_A t_B]\!])(x) > t$ and $g([\![1 - f_A f_B]\!])(x) > s$. This is a contradiction.

Similarly, for the case $g([\![V_A V_B]\!])(x) \succ [\![g(V_A)g(V_B)]\!](x)$ we get the contradiction. Hence, $g([\![V_A V_B]\!])(x) \simeq [\![g(V_A)g(V_B)]\!](x)$. □

Definition 4.3. Let A and B be two vague Lie ideals of L. Then A is said to be of the *same type* as B if there exists $g \in Aut(L)$ such that $A = B \circ g$, i.e., $V_A(x) \simeq V_B(g(x))$, for all $x \in L$.

Theorem 4.4. *Let A and B be two vague Lie ideals of L. Then A is a vague Lie ideal having the same type as B if and only if A is isomorphic to B.*

Proof. We only need to prove the necessity because the sufficiency part is trivial. Let A be a vague Lie ideal having the same type as B. Then there exists $g \in Aut(L)$ such that $V_A(x) \simeq V_B(g(x))$, for all $x \in L$. Let $\phi : A(L) \longrightarrow B(L)$ be a mapping defined by $\phi(A(x)) = B(g(x))$, for all $x \in L$, that is $\phi(V_A(x)) \simeq V_B(g(x))$, for all $x \in L$. Then it is clear that f is surjective. For all $x, y \in L$, if $\phi(t_A(x)) =$

$\phi(t_A(y))$, then $t_B(g(x)) = t_B(g(y))$ and hence $t_A(x) = t_A(y)$. Similarly, we can prove $\phi(1-f_A(x)) = \phi(1-f_A(y))$, for all $x \in L$ implies $1-f_B(g(x)) = 1-f_B(g(y))$. Thus ϕ is one-to-one. Now we need to prove ϕ is a homomorphism. Let all $x, y \in L$, we have

$$\begin{aligned}\phi(t_A(x+y)) &= t_B(g(x+y)) = t_B(g(x)+g(y)) = t_B(g(x)) + t_B(g(y)) \\ &= \phi(t_A(x)) + \phi(t_A(y))\end{aligned}$$

and

$$\begin{aligned}\phi(1-f_A(x+y)) &= 1-f_B(g(x+y)) = 1-f_B(g(x)+g(y)) \\ &= 1-f_B(g(x)) + 1-f_B(g(y)) \\ &= \phi(1-f_A(x)) + \phi(1-f_A(y)).\end{aligned}$$

Let all $x \in L$ and $a \in F$, we have

$$\phi(t_A(ax)) = t_B(g(ax)) = t_B(ag(x)) = at_B(g(x)) = a\phi(t_A(x)).$$

and

$$\begin{aligned}\phi(1-f_A(ax)) &= 1-f_B(g(ax)) = 1-f_B(ag(x)) = a(1-f_B(g(x))) \\ &= a\phi(1-f_A(x)).\end{aligned}$$

Let all $x, y \in L$, we have

$$\begin{aligned}\phi(t_A([x,y])) &= t_B(g([x,y])) = t_B([g(x),g(y)]) = [t_B(g(x)), t_B(g(y))] \\ &= [\phi(t_A(x)), \phi(t_A(y)]\end{aligned}$$

and

$$\begin{aligned}\phi(1-f_A([x,y])) &= 1-f_B(g([x,y])) = 1-f_B([g(x),g(y)]) \\ &= [1-f_B(g(x)), 1-f_B(g(y))] \\ &= [\phi(1-f_A(x)), \phi(1-f_A(y))].\end{aligned}$$

This completes the proof. □

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