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Topologies on $Spec_g(M)$

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Abstract. Let R be a G-graded commutative ring with identity and let M be a graded R-module. We endow $Spec_g(M)$, the collection of all graded prime submodules of M, analogous to that for $Spec(R)$, the spectrum of prime ideals of R, by two topologies: quasi-Zariski topology and Zariski topology. Then we study some properties of these topological spaces.

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1 Introduction

Throughout this paper all rings are commutative with a nonzero identity, and all modules are unitary. Let R be a commutative ring and consider $Spec(R)$, the spectrum of all prime ideals of R. The Zariski topology on $Spec(R)$ is a useful implement in algebraic geometry. For each ideal I of R, the variety of I is the set $V(I) = \{P \in Spec(R) | I \subseteq P\}$. Then the set $\{V(I) | I \trianglerighteq R\}$ satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology on $Spec(R)$ [1].

Let M be an R-module and let N be a submodule of M. We denote the annihilator of M/N by $(N :_R M)$, i.e. $(N :_R M) = \{r \in R | rM \subseteq N\}$. We recall that a proper submodule N of M is called a prime submodule of M if, for every $a \in R$ and $m \in M$, $am \in N$ implies that either $m \in N$ or $a \in (N :_R M)$. The notion of prime submodules was first introduced and studied in [2] and recently it has received a good deal of attention from several authors. We denote the set of all prime submodules of M by $Spec(M)$. In [7], the $Spec(M)$ topologized with the Zariski topology (quasi-Zariski topology by the notions of [6]) in a similar way to that of $Spec(R)$. For any submodule $N \leq M$, denote by $V^*(N)$ the variety of N, which is the set $V^*(N) = \{P \in Spec(M) | N \subseteq P\}$. Then the set $\tau^*(M) = \{V^*(N) | N \leq M\}$ is not closed under finite unions. The R-module M is called a Top-module provided that $\tau^*(M)$ is closed under finite unions, whence $\tau^*(M)$ constitute the closed sets in a Zariski topology on $Spec(M)$.

A grading on a ring and its modules usually aids computations by allowing one to focus on the homogeneous elements, which are presumably simpler or more controllable than random elements. However, for this to work one needs to know that the constructions being studied are graded. One approach to this issue is to redefine the constructions entirely in terms of the category of graded modules and thus avoid any consideration of non-graded modules or non-homogeneous elements; Sharp gives such a treatment of attached primes in [10]. Unfortunately, while such an approach helps to understand the graded modules themselves, it will only help to understand the original construction if the graded version of the concept happens to coincide with the original one. Therefore, notably, the study of graded modules is very important.

For the sake of completeness, we recall some definitions and notations used throughout. Let G be an arbitrary group. A commutative ring R with a non-zero identity is G-graded if it has a direct sum decomposition $R = \bigoplus_{g\in G} R_g$ such that for all $g, h \in G$, $R_gR_h \subseteq R_{gh}$. The G-graded ring R is called a graded integral domain provided that $ab = 0$ implies that either $a = 0$ or $b = 0$ where $a, b \in h(R) := \bigcup_{g\in G} R_g$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_gM_h \subseteq M_{gh}$. For every $g \in G$, an element of R_g or M_g is said to be a homogeneous element. We denote by $h(M)$ the set of all homogeneous elements of M, that is $h(M) = \bigcup_{g\in G} M_g$. Let M be a G-graded R-module. A submodule N of M is called graded (or homogeneous) if $N = \bigoplus_{g\in G}(N \cap M_g)$ or equivalently N is generated by homogeneous elements. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for each $g \in G$. An ideal I of R is called a graded ideal if it is a graded submodule of R and a graded R-module.

Let R be a G-graded ring. A proper graded ideal I of R is said to be a graded prime ideal if whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I , denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$, where $x_{g_i} \in h(M)$ for every $1 \leq i \leq n$. It is clear that a graded module is finitely generated if and only if it is graded finitely generated. For M, consider the subset $T^g(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in h(R)\}$. If R is a graded integral domain, then $T^g(M)$ is a graded submodule of M. M is called graded torsion-free (g-torsion-free for short) if $T^g(M) = 0$, and it is called graded torsion (g-torsion for short) if $T^g(M) = M$. It is clear that if M is torsion-free, then it is g-torsion-free. Moreover, if M is g-torsion, then it is torsion.

Most of our results are related to the references [6,7] which have been proved for the graded case.

2 Results

Let R be a G-graded R-module and consider $Spec_g(R)$, the spectrum of all graded prime ideals of R. The Zariski topology on $Spec_g(R)$ is defined in a similar way to that of $Spec(R)$. For each graded ideal I of R, the graded variety of I is the set $V_R^g(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then the set $\{V_R^g(I) | I \text{ is a graded ideal of } R\}$ satisfies the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$ (see [9]).

Let R be a G-graded ring and M a graded R-module. We recall from [3] that a proper graded submodule N of M is called graded prime if $rm \in N$, then $m \in N$

or $r \in (N :_R M) = \{r \in R | rM \subseteq N\}$, where $r \in h(R)$, $m \in h(M)$. It is shown in [3, Proposition 2.7] that if N is a graded prime submodule of M, then $P := (N :_R M)$ is a graded prime ideal of R. Let N be a graded submodule of M. Then N is a graded prime submodule of M if and only if $P := (N :_R M)$ is a graded prime ideal of R and M/N is a g-torsion-free R/P-module. Note that some graded R-modules M have no graded prime submodules. We call such graded modules g-primeless. For example, the zero module is clearly g-primeless. A submodule S of M will be called graded semiprime if S is an intersection of graded prime submodules of M. Let $Spec_g(M)$ denote the set of all graded prime submodules of M. Our goal is to endow $Spec_g(M)$ with some topologies. To this end, for each subset $E \subseteq h(M)$, let

$$V_*^g(E) = \{P \in Spec_g(M) | E \subseteq P\}.$$

Let N be a graded submodule of M. The graded radical of N in M, denoted by $Gr_M(N)$ is defined to be the intersection of all graded prime submodules of M containing N [5]. In the other words, $Gr_M(N) = \bigcap_{P \in V_*^g(N)} P$, and it is equal to M if $V_*^g(M) = \emptyset$. It is obvious that $N \subseteq Gr_M(N)$ and that $Gr_M(N) = M$ or $Gr_M(N)$ is a graded semiprime submodule of M.

Assume that N is the graded submodule generated by $E \subseteq h(M)$. Then from $E \subseteq N \subseteq Gr_M(N)$ we clearly have $V_*^g(Gr_M(N)) \subseteq V_*^g(N) \subseteq V_*^g(E)$. On the other hand, N is the smallest graded submodule of M containing E, so that if $P \in V_*^g(E)$, then $P \in V_*^g(N)$. Therefore $V_*^g(E) = V_*^g(N)$. Moreover $Gr_M(N)$ is the intersection of all graded prime submodules of M containing N; so $V_*^g(N) = V_*^g(Gr_M(N))$. Therefore $V_*^g(E) = V_*^g(N) = V_*^g(Gr_M(N))$. Consider the cases when $E = \{0\}$ or $E = M$. Then $V_*^g(0) = Spec_g(M)$ and $V_*^g(M) = \emptyset$. Now let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a family of graded submodules of M. Then $\bigcap_{\lambda \in \Lambda} V_*^g(N_\lambda) = V_*^g(\sum_{\lambda \in \Lambda} N_\lambda)$. Moreover, for every pair N and K of graded submodules of M, we have $V_*^g(N) \cup V_*^g(K) \subseteq V_*^g(N \cap K)$. Summarizing, we have proved:

Proposition 1. *Let M be a graded R-module. Then*

(1) For each subset $E \subseteq h(M)$, $V_*^g(E) = V_*^g(N) = V_*^g(Gr_M(N))$, where N is the graded submodule of M generated by E.

(2) $V_*^g(0) = Spec_g(M)$, and $V_*^g(M) = \emptyset$.

(3) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of graded submodules of M, then $\bigcap_{\lambda \in \Lambda} V_*^g(N_\lambda) = V_*^g(\sum_{\lambda \in \Lambda} N_\lambda)$.

(4) For every pair N and K of graded submodules of M, we have $V_*^g(N) \cup V_*^g(K) \subseteq V_*^g(N \cap K)$.

Therefore if we set

$$\zeta_*^g(M) = \{V_*^g(N) | \text{N is a graded submodule of M}\}$$

then $\zeta_*^g(M)$ contains the empty set and $Spec_g(M)$, and $\tau_*^g(M)$ is closed under arbitrary intersections, but it is not necessarily closed under finite unions.

Definition 1. Let M be a graded R-module.

(1) We shall say that M is a $g-Top$ module if $\zeta^g_*(M)$ is closed under finite unions, i.e. for any graded submodules N and L of M there exists a graded submodule K of M such that $V^g_*(N) \cup V^g_*(L) = V^g_*(K)$.

(2) A graded prime submodule N of M will be called graded extraordinary, or g-extraordinary for short, if whenever K and L are graded semiprime submodules of M with $K \cap L \subseteq N$ then $K \subseteq N$ or $L \subseteq N$.

Assume that M is a $g - Top$ module. In this case $\zeta^g_*(M)$ satisfies the axioms for the closed sets of a unique topology τ^g_* on $Spec_g(M)$. Then the topology $\tau^g_*(M)$ on $Spec_g(M)$ is called the quasi-Zariski topology. Note that we are not excluding the trivial case where $Spec_g(M)$ is empty; i.e. g-primeless graded R-modules are $g-Top$ modules. Also any graded prime ideal of the graded ring R is an extraordinary graded prime submodule of the graded R-module R.

Theorem 1. *Let M be a graded R-module. Then, the following statements are equivalent:*

(i) M is a $g - Top$ module.

(ii) Every graded prime submodule of M is g-extraordinary.

(iii) $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$ for any graded semiprime submodules N and L of M.

Proof. The result is clear when $Spec_g(M) = \emptyset$. So assume that $Spec_g(M) \neq \emptyset$.

$(i) \Rightarrow (ii)$ Let M be a $g - Top$ module. Assume that P is a graded prime submodule of M and that N, L are graded semiprime submodules of M with $N \cap L \subseteq P$. By assumption, there exists a graded submodule K of M with $V^g_*(N) \cup V^g_*(L) = V^g_*(K)$. Since N is a graded semiprime submodule, $N = \bigcap_{i\in I} P_i$ in which $\{P_i\}_{i\in I}$ is a collection of graded prime submodules of M. For every $i \in I$, we have

$$P_i \in V^g_*(N) \subseteq V^g_*(K) \Rightarrow K \subseteq P_i \Rightarrow K \subseteq \textstyle\bigcap_{i\in I} P_i = N.$$

Similarly, $K \subseteq L$. So $K \subseteq N \cap L$. Now we have

$$V^g_*(N) \cup V^g_*(L) \subseteq V^g_*(N \cap L) \subseteq V^g_*(K) = V^g_*(N) \cup V^g_*(L).$$

Consequently, $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$. Now from $N \cap L \subseteq P$ we have $P \in V^g_*(N \cap L) = V^g_*(N) \cup V^g_*(L)$. Hence either $P \in V^g_*(N)$ or $P \in V^g_*(L)$, that is either $N \subseteq P$ or $L \subseteq P$. So P is g-extraordinary.

$(ii) \Rightarrow (iii)$ Suppose that every graded prime submodule of M is g-extraordinary. Assume that N and L are two graded semiprime submodules of M. Clearly $V^g_*(N) \cup V^g_*(L) \subseteq V^g_*(N \cap L)$. For the other containment, choose $P \in V^g_*(N \cap L)$. Then $N \cap L \subseteq P$. By assumption, P is g-extraordinary. So $N \subseteq P$ or $L \subseteq P$, that is either $P \in V^g_*(N)$ or $P \in V^g_*(L)$. Therefore $V^g_*(N \cap L) \subseteq V^g_*(N) \cup V^g_*(L)$, and so $V^g_*(N) \cup V^g_*(L) = V^g_*(N \cap L)$.

$(iii) \Rightarrow (i)$ Let N, L be two graded submodules of M. We can assume that $V^g_*(N)$ and $V^g_*(L)$ are both nonempty, for otherwise $V^g_*(N) \cup V^g_*(L) = V^g_*(N)$ or $V^g_*(N) \cup V^g_*(L) = V^g_*(L)$. We know that $Gr_M(N)$ and $Gr_M(L)$ are both graded semiprime submodules of M. Setting $K = Gr_M(N) \cap Gr_M(L)$ we have:

$V_*^g(N) \cup V_*^g(L) = V_*^g(Gr_M(N)) \cup V_*^g(Gr_M(L)) = V_*^g(Gr_M(N) \cap Gr_M(L)) = V_*^g(K)$ by (iii). Hence M is a $g-Top$ module. $\square$

Proposition 2. *Let M be a graded R-module with the property that for every graded prime submodule N of M, $(K :_R M) \subseteq (N :_R M)$ implies that $K \subseteq N$ for each graded semiprime submodule K of M. Then M is a $g-Top$ module.*

Proof. Let N be a graded prime submodule of M and assume that $S_1 \cap S_2 \subseteq N$, where S_1, S_2 are graded semiprime submodules of M. It follows from $(S_1 :_R M) \cap (S_2 :_R M) = (S_1 \cap S_2 :_R M) \subseteq (N :_R M)$ that either $(S_1 :_R M) \subseteq (N :_R M)$ or $(S_2 :_R M) \subseteq (N :_R M)$ since $(N :_R M)$ is a graded prime ideal of R. Now by assumption we have $S_1 \subseteq N$ or $S_2 \subseteq N$, that is N is g-extraordinary. Hence M is a $g-Top$ module by Theorem 1. $\square$

Theorem 2. *Let M be a $g-Top$ R-module.*

(1) If K is a graded submodule of M, then M/K is a $g-Top$ R-module.

(2) The graded R_P-module M_P is a $g-Top$ module for every graded prime ideal P of R.

(3) If $Gr_M(N) = N$ for every graded submodule N of M, then M is a graded distributive module.

Proof. There will be nothing to prove if M has no graded prime submodules. So assume that $Spec_g(M) \neq \emptyset$.

(1) By [3, Lemma 2.8], the graded prime submodules of M/K are just the submodules N/K where N is a graded prime submodule of M with $K \subseteq N$. Consequently, any graded semiprime submodule of M/K is of the form S/K in which S is a graded semiprime submodule of M with $K \subseteq S$. Assume that S_1/K and S_2/K are two graded semiprime submodules of M/K. Then, by Theorem 1, $V_*^g(S_1) \cup V_*^g(S_2) = V_*^g(S_1 \cap S_2)$ since M is a $g-Top$ module. Thus $V_*^g(S_1/K) \cup V_*^g(S_2/K) = V_*^g(S_1/K \cap S_2/K)$. It follows from Theorem 1 that M/K is a $g-Top$ module.

(2) By Theorem 1, it is enough to show that every graded prime submodule of M_P is g-extraordinary. Let N be a graded prime submodule of M_P, and let $S_1 \cap S_2 \subseteq N$ for some graded semiprime submodules S_1, S_2 of M_P. Clearly, $N \cap M$ is a proper graded submodule of M. Assume that $r \in h(R)$ and $m \in h(M)$ are such that $rm \in N \cap M$. Then, $r/1 \in h(R_P)$ and $m/1 \in h(M_P)$ with $(r/1)(m/1) = (rm)/1 \in N$. It follows that either $(r/1)M_P \subseteq N$ or $m/1 \in N$ since N is graded prime. Therefore, either $r \in (N \cap M :_R M)$ or $m \in N \cap M$. This implies that $N \cap M$ is a graded prime submodule of M. Hence N is g-extraordinary by Theorem 1. As another consequence, $S_1 \cap M$ and $S_2 \cap M$ are graded semiprime submodules of M with $(S_1 \cap M) \cap (S_2 \cap M) \subseteq N \cap M$. Therefore, $S_1 \cap M \subseteq N \cap M$ or $S_2 \cap M \subseteq N \cap M$. It follows that either $S_1 = (S_1 \cap M)R_P \subseteq (N \cap M)R_P$ or $S_2 = (S_2 \cap M)R_P \subseteq (N \cap M)R_P$. Hence N is a g-extraordinary submodule of M_P.

(3) For every graded submodules N, K and L of M we have:

$$
\begin{aligned}
(K+L)\cap N &= Gr_M((K+L)\cap N)\\
&= \bigcap\{P|P\in V_*^g((K+L)\cap N)\}\\
&= \bigcap\{P|P\in V_*^g(K+L)\cup V_*^g(N)\}\\
&= \bigcap\{P|P\in (V_*^g(K)\cap V_*^g(L))\cup V_*^g(N)\}\\
&= \bigcap\{P|P\in (V_*^g(K)\cup V_*^g(N))\cap(V_*^g(L)\cup V_*^g(N))\}\\
&= \bigcap\{P|P\in (V_*^g(K\cap N))\cap(V_*^g(L\cap N))\}\\
&= \bigcap\{P|P\in V_*^g((K\cap N)+(L\cap N))\}\\
&= Gr_M((K\cap N)+(L\cap N)) = (K\cap N)+(L\cap N).
\end{aligned}
$$

Thus M is graded distributive. □

Let M be a $g-Top$ module and let $X = Spec_g(M)$. We know that any closed subset of X is of the form $V_*^g(N)$ for some graded prime submodule N of M. But now the question arises as to what open subsets of X look like. To say that any open subset of X is of the form $X - V_*^g(N)$ for some graded prime submodule N of M, though true, is not very helpful. For every subset S of $h(M)$, define

$$X_S = X - V_*^g(S)$$

In particular, if $S = \{f\}$, we denote X_S be X_f.

Proposition 3. *The set $\{X_f|f\in h(M)\}$ is a basis for the quasi-Zariski topology on X.*

Proof. Let U be a non-void open subset in X. Then $U = X - V_*^g(N)$ for some graded submodule N of M. Assume that N is generated by some subset $E\subseteq h(M)$. Then we have

$$U = X - V_*^g(N) = X - V_*^g(E) = X - V_*^g(\bigcup_{f\in E}\{f\}) = X - \bigcap_{f\in E} V_*^g(f) = \bigcup_{f\in E}(X - V_*^g(f)) = \bigcup_{f\in E} X_f$$

Therefore the set $\{X_f|f\in h(M)\}$ is a basis for X. □

Let R be a G-graded ring. A graded R-module M is said to be a graded multiplication module if for each graded submodule N of M, $N = IM$ for some graded ideal I of R [4]. One can easily show that if N is a graded submodule of a graded multiplication module M, then $N = (N:_R M)M$. A graded multiplication module need not be multiplication. We first recall some results concerning graded prime submodules and graded multiplication modules.

Theorem 3 (see [8]). *Let M be a graded multiplication R-module, and N a proper graded submodule of M. Then, the following statements are equivalent:*

(1) N is a graded prime submodule;

(2) $(N :_R M)$ is a graded prime ideal of R;

(3) $N = PM$ for some graded prime ideal P of R with $Ann(M) \subseteq P$.

Suppose that M is a graded multiplication R-module, $N = IM$ and $K = JM$ are graded submodules of M, where I and J are graded ideals of R. The product of N and K, denoted by NK, is defined by $NK = (IJ)M$. It is proved in [8, Theorem 4] that this product is independent of the choice of I and J. For each pair m, m' of elements of $h(M)$, we define $mm' = (IJ)M$, where $Rm = IM$ and $Rm' = JM$.

Theorem 4. *Let N be a proper graded submodule of the graded multiplication R-module M. Then, the following statements are equivalent.*

(1) N is a graded prime submodule;

(2) $AB \subseteq N$ implies that $A \subseteq N$ or $B \subseteq N$ for each graded submodules A and B of M;

(3) $m.m' \subseteq N$ implies that $m \in N$ or $m' \in N$ for every $m, m' \in h(M)$.

Proof. It is a direct consequence of [8, Theorem 4 and Corollary 2]. □

Theorem 5. *Every graded multiplication module is a $g - Top$ module.*

Proof. Let M be a graded multiplication R-module. Assume that N and L are two graded semiprime submodules of M. Clearly $V_*^g(N) \cup V_*^g(L) \subseteq V_*^g(NL)$. For the converse containment, pick $P \in V_*^g(NL)$. Then from $NL \subseteq P$ we get either $N \subseteq P$ or $L \subseteq P$ by Theorem 4. Therefore $P \in V_*^g(N) \cup V_*^g(L)$, that is $V_*^g(NL) \subseteq V_*^g(N) \cup V_*^g(L)$. Consequently $V_*^g(N) \cup V_*^g(L) = V_*^g(NL)$. It follows from Theorem 1 that M is a $g - Top$ module. □

Corollary 1. *Let M be a graded multiplication R-module. Then $V_*^g(N) \cup V_*^g(L) = V_*^g(NL) = V_*^g(N \cap L)$ for each pair N and L of graded submodules of M.*

We end this paper by endowing $Spec_g(M)$ by another topology, called the Zariski topology on M. Let M be a graded module over the G-graded ring R. For every graded submodule N of M, set

$$V^g(N) = \{P \in Spec_g(M) | (P :_R M) \supseteq (N :_R M)\}$$

and

$$\zeta^g(M) = \{V^g(N) | N \text{ is a graded submodule of } M\}.$$

Then

Proposition 4. *(1) $V^g(0) = Spec_g(M)$, and $V^g(M) = \emptyset$.*

(2) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a family of graded submodules of M, then $\bigcap_{\lambda \in \Lambda} V^g(N_\lambda) = V^g(\sum_{\lambda \in \Lambda} N_\lambda)$.

(3) For each pair N and K of graded submodules of M, we have $V^g(N) \cup V^g(K) = V^g(N \cap K)$.

Therefore for any graded R-module M there always exists a topology τ^g on $Spec_g(M)$ in which $\zeta^g(M)$ is the family of all closed sets. τ^g is called the Zariski topology on $Spec_g(M)$. Now consider the set

$$\zeta^g_{**}(M) = \{V^g_*(IM) | I \text{ is a graded ideal of } R\}.$$

In contrast with $\zeta`^g_*(M)$, $\zeta^g_{**}(M)$ is always closed under finite unions, and so it always induces a topology τ^g_{**} on $Spec_g(M)$. It is easy to verify that, for every $g - Top$ module, the topology τ^g_{**} is coarser than the topology τ^g_*.

Lemma 1. *Let R be a G-graded ring and let M be a graded R-module. For every graded prime ideal p of R, denote by $Spec^p_g(M)$, the set $\{P \in Spec_g(M) | (P :_R M) = p\}$. Then, for every graded submodules N and L of M, the following statements are satisfied.*

(1) If $(N :_R M) = (L :_R M)$, then $V^g(N) = V^g(L)$.

(2) Let both N and L be graded prime. Then $(N :_R M) = (L :_R M)$ if and only if $V^g(N) = V^g(L)$.

(3) $V^g(N) = \bigcup_{p \in V^g_R(N :_R M)} Spec^p_g(M)$.

Theorem 6. *Let R be a G-graded ring and let M be a graded R-module.*

(1) The Zariski topology τ^g on $Spec_g(M)$ and the topology τ^g_{**} are identical.

(2) If M is $g - Top$ module, then the quasi-Zariski topology τ^g_* on $Spec_g(M)$ is finer than the Zariski topology τ^g.

Proof. It is easy to show that $V^g(N) = V^g((N :_R M)M) = V^g_*((N :_R M)M)$ and $V^g(IM) = V^g_*(IM)$ for every graded submodule N of M and every graded ideal I of R. Therefore $\zeta^g(M) = \zeta^g_{**}(M) \subseteq \zeta^g_*(M)$. So the result follows. $\square$

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