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Transversals in loops. 3. Loop transversals

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Abstract. The investigation of the new notion of a transversal in a loop to its subloop (begun in [10]) is continued in the present article. This notion generalized the well-known notion of a transversal in a group to its subgroup and can be correctly defined only in the case when some specific condition (Condition A) for a loop and its subloop holds. The connections between loop transversals in some loop to its subloop and loop transversals in multiplicative group of this loop to some suitable subgroup are investigated in this work.

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1 Introduction

In group theory, in group representations theory and in quasigroup theory the following notion is well-known – the notion of a left (right) transversal in a group to its subgroup [1, 5, 6, 11].

Definition 1. Let G be a group and H be a subgroup in G. A complete set $T = \{t_i\}_{i\in E}$ of representatives of the left (right) cosets H_i in G to H ($e = t_1 \in H$, $t_i \in H_i$) is called a **left (right) transversal in G to H**.

In the present work we continue to study a variant of natural generalization of the notion of transversal to the class of loops, begun in [10]. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloop (see the Condition A below).

In Section 2 of this article we remember the most important notions and theorems from the first part of this investigation [10].

In Section 3 different structural theorems are proved. They demonstrate a connection between transversals in a loop to its subloop and transversals in a multiplicative group of this loop to its suitable subgroup.

In Section 4 one of the most important particular cases of transversals in a loop to its subloop is investigated – the case of a loop transversal. Some criteria of the existence of a loop transversal in a given loop to its subloop are proved.

Further we shall use the following notations:

$\langle L, \cdot, e\rangle$ is an initial loop with the unit e;

$\langle R, \cdot, e\rangle$ is its proper subloop;

E is a set of indexes ($1 \in E$) of the left (right) cosets R_i in L to R (moreover, $R_1 = R$).

2 Preliminaries

Definition 2. A system $\langle E, \cdot\rangle$ is called [2] a **right (left) quasigroup** if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If the system $\langle E, \cdot\rangle$ is both a right and left quasigroup, then it is called a **quasigroup**. If in a right (left) quasigroup $\langle E, \cdot\rangle$ there exists an element $e \in E$ such that $x \cdot e = e \cdot x = x$ for every $x \in E$, then the system $\langle E, \cdot\rangle$ is called a **right (left) loop** (the element e is called a **unit** or **identity element**). If a system $\langle E, \cdot\rangle$ is both a right and left loop, then it is called a **loop**.

At the beginning let us define a partition of a loop by left (right) cosets to its proper subloop.

Definition 3 (see [12])**.** Let $\langle L, \cdot\rangle$ be a loop and $\langle R, \cdot\rangle$ be its proper subloop. Then a **left coset** in L to R is a set of the form $xR = \{xr \mid r \in R\}$, and a **right coset** is a set of the form $Rx = \{rx \mid r \in R\}$.

In a general case the cosets in a loop to its subloop do not necessarily form a partition of the loop. This leads us to the following definition.

Definition 4 (see [12])**.** A loop L has a **left (right) coset decomposition by its proper subloop** R, if the left (right) cosets form a partition of the loop L, i.e. for some set of indexes E:

1. $\bigcup\limits_{i \in E} (a_i R) = L$;
2. For every $i, j \in E$, $i \neq j$, $\qquad (a_i R) \cap (a_j R) = \varnothing$.

In order to define correctly the notion of a left (right) transversal in a loop to its proper subloop, the following condition must be necessarily fulfilled.

Definition 5 (see [10])**.** (**Left Condition A**) Let R be a subloop of a loop L. For all $a, b \in L$ there exists an element $c \in L$ such that

$$a(bR) = cR. \tag{1}$$

The **right Condition A** is defined analogously.

Let us denote (see [13]) $\forall a, b \in L$: a **left inner mapping**

$$l_{a,b}(x) = (a \cdot b) \setminus (a \cdot (b \cdot x)), \qquad x \in L, \tag{2}$$

where "\" is a left division in the loop $\langle L, \cdot, e\rangle$, and a **right inner mapping**

$$r_{a,b}(x) = ((x \cdot b) \cdot a) / (b \cdot a), \qquad x \in L, \tag{3}$$

where "/" is a right division in the loop $\langle L, \cdot, e\rangle$.

Lemma 1 (see [10])**.** *Let the left **Condition A** be fulfilled. Then* $\forall a, b \in L$*:* $l_{a,b}(R) = R$*.*

Lemma 2 (see [10])**.** *Let the right **Condition A** be fulfilled. Then* $\forall a, b \in L$*:* $r_{a,b}(R) = R$*.*

Definition 6 (see [9])**.** Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and the left **Condition A** be fulfilled. Let $\{R_x\}_{x\in E}$ be the set of all left cosets in L to R that form a left coset decomposition of the loop L. A set $T = \{t_x\}_{x\in E} \subset L$ is called a **left transversal** in L to R if T is a complete set of representatives of the left cosets R_x in L to R, i.e. there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.

A right and two-sided transversal in L to R is defined analogously.

On a set E it is possible to define correctly the following operations:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \tag{4}$$

if T is a left transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \tag{5}$$

if T is a right transversal in L to R.

Definition 7. Let T be a left (right) transversal in L to R. If the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ ($\langle E, \overset{(T)}{\circ}, 1\rangle$) is a loop then the transversal T is called a **left (right) loop transversal** in L to R.

Let still $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and the left **Condition A** be fulfilled. Let $T = \{t_x\}_{x\in E}$ be a left transversal in L to R. Define the following map:

$$\begin{aligned} f \quad &: \quad L \times E \to E, \\ f \quad &: \quad (g, x) \to y = \hat{g}(x), \\ \hat{g}(x) \quad &= \quad y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot R) = t_y \cdot R. \end{aligned} \tag{6}$$

By virtue of the left **Condition A** this definition (a *left action* of the loop L on a set E) is correct.

Lemma 3 (see [10])**.** *A map* $\hat{g}$ *is a permutation on a set* E *for every element* $g \in L$*.*

Lemma 4 (see [10]). *For an arbitrary left transversal $T = \{t_x\}_{x\in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$ the following propositions are true:*

1. $\forall r \in R$: $\hat{r}(1) = 1$;

2. $\forall x, y \in E$: $\hat{t}_x(y) = x \overset{(T)}{\cdot} y$, $\hat{t}_x^{-1}(y) = x\backslash y$,

where $\hat{t}_x^{-1}$ is an inverse permutation to a permutation $\hat{t}_x$ in S_E, and "\" is a left division in a left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \quad \hat{t}_x^{-1}(1) = x\backslash 1, \quad \hat{t}_x^{-1}(x) = 1.$$

Lemma 5 (see [10]). *The following conditions are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop transversal in a loop L to its subloop R;*

2. *A set $\hat{T} = \{\hat{t}_x\}_{x\in E}$ is a sharply transitive set of permutations in the group S_E.*

3 Semidirect products of loops and suitable subgroups

Remind the definition (see [8, 13]) of the semidirect product of a left loop $L = \langle E, \cdot, 1\rangle$ with two-sided unit 1 on a suitable permutation group H on the set E ($H \subseteq St_1(S_E)$). Let the following conditions hold:

1. $\forall a, b \in E:\ l_{a,b}^{-1} L_a L_b \in H$;

2. $\forall u \in E$ and $\forall h \in H:\quad \varphi(u,h) L_{h(u)}^{-1} h\, L_u h^{-1} \in H$,

where L_a is the left translation by an element a in $\langle E, \cdot, 1\rangle$ (i.e. $L_{a(x)} = a \cdot x$).

Then on the set
$$E \times H = \{(u,h) | u \in E, h \in H\}$$
it is possible to define correctly the operation

$$(u, h_1) * (v, h_2) \overset{def}{=} (u \cdot h_1(v),\, l_{u,h_1(v)} \varphi(v, h_1) h_1 h_2).$$

The system $G = \langle E \times H, *, (1, id)\rangle$ is a group, which is called the **semidirect product** $G = L \leftthreetimes H$ of the left loop L on the group H. This product satisfies the following properties:

1. The map $\widehat{(u,h)}: E \to E$:
$$(\widehat{u,h}))\,(x) \overset{def}{=} u \cdot h(x)$$
is an **action**, i.e.

   (a) It is a permutation on E;

   (b) If $(\widehat{u,h_1})(x) = (\widehat{v,h_2})(x)\quad \forall x \in E$, then $u = v$ and $h_1 = h_2$;

(c) If $\widehat{(u,h)}(x) = (x) \quad \forall x \in E$, then $(u,h) \equiv (1, id)$.

2. $\forall x \in E$ it is true that

$$((u,h_1)\widehat{*(v},h_2))(x) = \widehat{(u,h_1)}(\widehat{(v,h_2)}(x)) = L_u h_1 L_v h_2(x).$$

3. $(u,h)^{-1} = (h^{-1}(u\backslash 1), L^{-1}_{h^{-1}(u\backslash 1)} h^{-1} L_u^{-1})$,

   and, in particular $(u, id)^{-1} = (u\backslash 1, L^{-1}_{u\backslash 1} L_u^{-1})$.

4. The system $\hat{H} = \langle H^*, *, (1, id)\rangle$ (where $H^* = \{(1,h) |\, h \in H\}$) is a subgroup in G, isomorphic to the group H.

5. The set $\hat{T} = \{(u, id) |\, u \in E\}$ is a left transversal in G to $\hat{H}$, and the operation $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ coincides with the operation $\langle E, \cdot, 1\rangle$.

We remind the definitions of the **left multiplicative group** of a left loop L:

$$LM(L) \overset{def}{=} \langle L_x | x \in L, L_x(u) = x \cdot u \rangle\,,$$

and the **left inner permutation group** of a left loop L:

$$LI(L) \overset{def}{=} \langle l_{a,b} | a, b \in L\rangle\,.$$

It was shown in [8] that:

$$LI(L) = St_1(LM(L)) \subset LM(L), \qquad LM(L) = L \leftthreetimes LI(L).$$

Lemma 6. *Let $L = \langle E, \cdot, 1\rangle$ be a loop, $R = \langle E_1, \cdot, 1\rangle$ be its subloop, and the left* ***Condition A*** *be fulfilled for them. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R.*

Assume

$$G = LM(L) = L \leftthreetimes LI(L), \qquad H = LI(L).$$

Then:

1. *The set $K = \{(r,h) |\, r \in R,\ h \in H\}$ is a subgroup in G, and $H \subseteq K \subset G$;*

2. *The set $T_0^* = \{(t_x, id) |\ t_x \in T_0,\ x \in E_0\}$ is a left transversal in G to K, and*

$$\left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle \equiv \left\langle E_0, \overset{(T_0^*)}{\cdot}, 1 \right\rangle.$$

Proof. **1**. Let the conditions of the lemma hold. According to properties of semidirect product we have

$$H = \{(1,h) | h \in H = LI(L)\} \subset \{(u,h) | u \in L, h \in H\} = G.$$

Since $R \subseteq L$, then

$$\begin{aligned} \forall a,b \in R: \quad l_{a,b} \in LI(L) = H, \\ \forall u \in R \ \forall h \in H: \quad \varphi(u,h) \in \{\varphi(u,h) |\, u \in R,\, h \in H\} \subseteq \\ \subseteq \{\varphi(u,h) |\, u \in L,\, h \in H\} \subseteq LI(L) = H. \end{aligned}$$

Then it is possible to define correctly a semidirect product on the set

$$K = R \times H = \{(r,h) |\, r \in R, h \in H\} \subseteq G.$$

It is obvious that $H \subseteq K$.

Besides for any two elements (r_1, h_1) and (r_2, h_2) from K we have:

$$(r_1, h_1) * (r_2, h_2) = (r_1 \overset{(R)}{\cdot} h_1(r_2), l_{r_1, h_1(r_2)} \varphi(r_2, h_1) h_1 h_2).$$

In order that the group K be a subgroup in G it is necessary and sufficient that the following condition be fulfilled:

$$\forall r_1, r_2 \in R \quad \forall h \in H: \quad (r_1 \overset{(R)}{\cdot} h(r_2)) \in R.$$

But it is equivalent to the following: $\forall h \in H: \quad h(R) \subseteq R$, i.e. $\forall a, b \in L: l_{a,b}(R) \subseteq R$. According to Lemma 1 the last conditions are equivalent to the left Condition A for loops L and R.

2. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R. Then we consider the set

$$T_0^* = \{(t_x, id) |\ t_x \in T_0,\ x \in E_0\}.$$

For an arbitrary $x \in E_0$ we consider the set:

$$\begin{aligned} (t_x, id) * K &= \{(t_x, id) * (r,h) | r \in R, h \in H\} = \\ &= \{(t_x \overset{(L)}{\cdot} r, l_{t_x, r} h) | r \in R, h \in H\}. \end{aligned} \tag{7}$$

Let us show that this set is a left coset in G to K. Since the set

$$\{t_x \overset{(L)}{\cdot} r |\ r \in R\} = t_x \overset{(L)}{\cdot} R$$

is a left coset in L to R, if $x_1 \neq x_2$ then by (7) we have:

$$((t_{x_1}, id) * K) \cap ((t_{x_2}, id) * K) = \varnothing.$$

Further, let g_0 be an arbitrary element from G; by virtue of the representation $G = L \times H$ we have that $g_0 = (u_0, h_0)$, where $u_0 \in L,\ h_0 \in H$. Since $T_0 = \{t_x\}_{x \in E_0}$ is the left transversal in L to R, then $u_0 = t_{x_0} \overset{(L)}{\cdot} r_0$, where $t_{x_0} \in T_0$, $r_0 \in R$. Therefore supposing $h_1 = l^{-1}_{t_{x_0}, r_0} h_0 \in H$, we obtain

$$(t_{x_0}, id)_{\in T_0^*} * (r_0, h_1)_{\in K} = (t_{x_0} \overset{(L)}{\cdot} r_0, l_{t_{x_0}, r_0} h_1) = (u_0, h_0) = g_0.$$

So, sets of the form $(t_x, id) * K$, $x \in E_0$ are left cosets in G to K. Therefore the set

$$T_0^* = \{(t_x, id) |\, t_x \in T_0,\, x \in E_0\}$$

is a left transversal in G to K. The corresponding transversal operation is $\left\langle E_0, \overset{(T_0^*)}{\cdot}, 1 \right\rangle$, for which we have:

$$\begin{gathered}
x \overset{(T_0^*)}{\cdot} y = z \quad \Leftrightarrow \quad (t_x, id) * (t_y, id) = (t_z, id) * (r, h), \quad (r, h \in K), \\
(t_x \overset{(L)}{\cdot} t_y, l_{t_x, t_y}) = (t_z \overset{(L)}{\cdot} r, l_{t_z, r} h), \\
t_x \overset{(L)}{\cdot} t_y = t_z \overset{(L)}{\cdot} r; \quad r \in R, \\
x \overset{(T_0)}{\cdot} y = z,
\end{gathered}$$

i.e.

$$x \overset{(T_0^*)}{\cdot} y = x \overset{(T_0)}{\cdot} y, \quad \forall x, y \in E_0,$$

as required. □

Let us prove one additional lemma.

Lemma 7. *Let $T_0 = \{t_x\}_{x \in E_0}$ be a left transversal in L to R. Then $\forall t_u, t_x \in T_0$ and $\forall r \in R$ it is true that:*

$$(t_u \cdot r) \cdot t_x = t_u \cdot (r \cdot l^{-1}_{t_u, r}(t_x)),$$

where $l_{a,b} \in LI(L)$.

Proof. Really, by virtue of the definition of $l_{a,b}$,

$$l^{(z)}_{a,b} = (a \cdot b) \backslash (a \cdot (b \cdot z)).$$

Then

$$(t_u \cdot r) \backslash (t_u \cdot (r \cdot l^{-1}_{t_u, r}(r_x))) = l_{t_u, r} l^{-1}_{t_u, r}(t_x) = t_x,$$

i.e.

$$(t_u \cdot r) \cdot t_x = t_u \cdot (r \cdot l^{-1}_{t_u, r}(t_x)),$$

as required. □

Let us consider the permutation representations of loop L by left cosets to a subloop R and group G by left cosets to a subgroup K.

Lemma 8. *Let $\hat{L}$ be the permutation representation of a loop L by left cosets to a subloop R, i.e. $\forall g \in L$:*

$$\hat{g}(x) = y \Leftrightarrow g \overset{(L)}{\cdot} (t_x \overset{(L)}{\cdot} R) = t_y \overset{(L)}{\cdot} R,$$

where $T_0 = \{t_x\}_{x \in E_0}$ *is a left transversal in* L *to* R*. Then in the group* G *to its subgroup* K *(see Lemma 6) there exists such a left transversal* $T_0^* = \{t_x^*\}_{x \in E_0}$ *that for a suitable permutation representation* $\breve{G}$ *of the group* G *by left cosets to its subgroup* K *the following is true:*

$$\forall g \in L \;\; \exists \;\; g' \in G \quad \textit{such} \quad \textit{that} \quad \hat{g}(x) = \breve{g}'(x) \quad \forall x \in E_0.$$

Proof. Let the conditions of the lemma hold. According to Lemma 6, we can consider the following left transversal

$$T_0^* = \{(t_x, id) |\, t_x \in T_0\}.$$

We have in the loop L: if $g = t_u \cdot r$ (where $t_u \in T_0$, $r \in R$), then

$$\begin{gathered} \hat{g}(x) = y\,, \\ g \cdot (t_x \cdot R) = t_y \cdot R\,, \\ g \cdot t_x = t_y \cdot r'; \quad r' \in R; \\ (t_u \cdot r) \cdot t_x = t_y \cdot r'. \end{gathered}$$

By virtue of Lemma 7 we obtain:

$$t_u \cdot (r \cdot l^{-1}_{t_u,r}(t_x)) = t_y \cdot r'. \tag{8}$$

Now pass to the group G. As an element g' we take

$$g' = (t_u, k') = (t_u, id) \cdot (r, l^{-1}_{t_u,r}),$$

where $k' \in K$, $k' = (r, l^{-1}_{t_u,r})$. Then we have:

$$\breve{g}'(x) = z \quad \Leftrightarrow \quad g' t_x^* K = t_z^* K \quad \Leftrightarrow \quad g' t_x^* = t_z^* k', \quad k' \in K. \tag{9}$$

And so

$$\begin{gathered} (t_u, id) * (r, l^{-1}_{t_u,r}) * (t_x, id) = \\ (t_u, id) * (r \cdot l^{-1}_{t_u,r}(t_x),\ l_{r, l^{-1}_{t_u,r}(t_*)} \varphi(t_x, l^{-1}_{t_u,r}) l^{-1}_{t_u,r}) = \\ (t_u \cdot (r \cdot l^{-1}_{t_u,r}(t_x)),\ l_{t_u, r \cdot l^{-1}_{t_u,r}(t_x)} l_{r, l^{-1}_{t_u,r}(t_x)} \varphi(t_x, l^{-1}_{t_u,r}) l^{-1}_{t_u,r}) = \\ \overset{(8)}{=} (t_y \cdot r', \underbrace{l_{t_u, r \cdot l^{-1}_{t_u,r}(t_x)} l_{r, l^{-1}_{t_u,r}(t_x)} \varphi(t_x, l^{-1}_{t_u,r}) l^{-1}_{t_u,r}}_{h'}) = (t_y, id) * \underbrace{(r', h'')}_{\in K}, \end{gathered}$$

where $h',\ h'' \in LI(L)$.

Since $(r', h'') \in K$, then from (9) we obtain

$$t_z^* k' = g' t_x^* = \underbrace{(t_y, id)}_{t_y^*} * \underbrace{(r', h'')}_{\in K}.$$

Since $T_0^* = \{t_x^*\}_{x \in E_0}$ is a left transversal in G to K then

$$t_z^* \equiv t_y^* \Leftrightarrow t_z = t_y; \quad \Leftrightarrow \quad z = y,$$

i.e. $\breve{g}'(x) = \hat{g}(x)$, as required. $\square$

4 Loop transversal in loop by its subloop

Let again L be a loop, R be its subloop, and **Condition A** be fulfilled for them. Define under what conditions a left transversal $T_0 = \{t_x\}_{x\in E_0}$ will be a left loop transversal in a loop L by its subloop R.

First prove one preliminary lemma.

Lemma 9. *Let L be a loop, R be its subloop and **Condition A** be fulfilled for them. Then*

1. $\forall a, b, c \in L$:
$$c\backslash(a\cdot(b\cdot R)) = (c\backslash(a\cdot b))\cdot R; \tag{10}$$

2. $\forall a, b, c \in L$:
$$a\cdot(b\cdot(c\backslash R)) = (a\cdot(b\cdot(c\backslash 1)))\cdot R.$$

3. $\forall h \in LI(L)$:
$$h(a\cdot R) = h(a)\cdot R, \quad \forall a \in L. \tag{11}$$

Proof. **1**. $\forall a, b, c \in L$ by virtue of **Condition** A we have:

$$c\cdot[(c\backslash(a\cdot b))\cdot R] = (c\cdot(c\backslash(a\cdot b))\cdot R = (a\cdot b)\cdot R = a\cdot(b\cdot R),$$

i.e.

$$c\backslash(a\cdot(b\cdot R) = (c\backslash(a\cdot b)\cdot R.$$

2. Using **1** we have for $a\cdot b = 1$:

$$c\backslash R = c\backslash(1\cdot R) = c\backslash((a\cdot b)\cdot R) = (c\backslash(a\cdot b))\cdot R = (c\backslash 1)\cdot R. \tag{12}$$

Then by virtue of **Condition** A and (12) we have:

$$a\cdot(b\cdot(c\backslash R)) = a\cdot(b\cdot((c\backslash 1)\cdot R)) = a\cdot((b\cdot(c\backslash 1))\cdot R) =$$

$$= (a\cdot(b\cdot(c\backslash 1)))\cdot R.$$

3. For arbitrary $l_{a,b} \in LI(L)$ using **1** and **Condition A** we have: $\forall c \in L$

$$l_{a,b}(c\cdot R) = (a\cdot b)\backslash(a\cdot(b\cdot(c\cdot R))) = (a\cdot b)\backslash((a\cdot(b\cdot c))\cdot R) =$$

$$= ((a\cdot b)\backslash(a\cdot(b\cdot c)))\cdot R = l_{a,b}(c)\cdot R.$$

Besides $\forall a, b \in L$ we have: $\forall c \in L$

$$l^{-1}_{a,b}(c\cdot R) = b\backslash(a\backslash((a\cdot b)\cdot(c\cdot R))) =$$

$$= b\backslash(a\backslash(((a\cdot b)\cdot c)\cdot R)) = b\backslash((a\backslash((a\cdot b)\cdot c))\cdot R) =$$
$$= (b\backslash((a\cdot b)\cdot c)))\cdot R = l^{-1}_{a,b}(c)\cdot R.$$

Since any $h \in LI(L)$ may be represented in the form

$$h = l_{a_1,b_1}^{\pm 1} \cdot \ldots \cdot l_{a_k,b_k}^{\pm 1},$$

then $\forall h \in LI(L)$ we have: $\forall a \in L$

$$h(a \cdot R) = l_{a_1,b_1}^{\pm 1} \cdot \ldots \cdot l_{a_k,b_k}^{\pm 1}(a \cdot R) = l_{a_1,b_1}^{\pm 1} \cdot \ldots \cdot l_{a_k,b_k}^{\pm 1}(a) \cdot R = h(a) \cdot R.$$

□

Lemma 10. *Let L be an arbitrary loop, R be its subloop, and* ***Condition A*** *be fulfilled for them. Then the following conditions for an arbitrary left transversal $T_0 = \{t_x\}_{x \in E_0}$ in L to R are equivalent:*

1. *T_0 is a left transversal in L to R;*
2. *$\forall u \in L$ and $\forall h \in LI(L)$ the set $T_{u,h}\{u \cdot h(t_x \cdot h^{-1}(u \backslash 1))\}_{x \in E_0}$ is a left transversal in L to R;*
3. *$\forall v \in E_0$ the set $T_v\{t_v \cdot (t_x(t_v \backslash 1))\}_{x \in E_0}$ is a left transversal in L to R;*
4. *$\forall u \in L$ the set $T_u^*\{(u \backslash (t_x \cdot u))\}_{x \in E_0}$ is a left transversal in L to R;*
5. *$\forall v \in E_0$ the set $T_v^*\{t_v \backslash (t_x \cdot t_v)\}_{x \in E_0}$ is a left transversal in L to R.*

Proof. Let conditions of the lemma hold. Using the results of the previous section we have the following sequence of equivalent statements (according to Lemma 6):

– a left transversal $T_0 = \{t_x\}_{x \in E_0}$ in L to R is a left loop by a transversal in L to R

$$\Leftrightarrow \quad \text{the operation } \left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle \text{ is a loop} \quad \Leftrightarrow$$

– the left transversal $T_0^* = \{\underbrace{(t_x, id)}_{t_x^*}\}_{x \in E_0}$ in a group G to its subgroup K is a loop transversal (where $G = L \rightthreetimes LI(L)$, $K = R \rightthreetimes LI(L)$), and $\left\langle E_0, \overset{(T_0^*)}{\cdot}, 1 \right\rangle$ is a loop, coincides with the loop $\left\langle E_0, \overset{(T_0)}{\cdot}, 1 \right\rangle$.

The last statement is equivalent to every of the following statements (see [1,6,11]):

1. $\forall g \in G$ the set $gT_0^*g^{-1}$ is a left transversal in G to K;
2. $\forall x \in E_0$ the set $t_x^*T_0^*t_x^{*-1}$ is a left transversal in G to K;
3. $\forall g \in G$ the set $g^{-1}T_0^*g$ is a left transversal in G to K;
4. $\forall x \in E_0$ the set $t_x^{*-1}T_0^*t_x^*$ is a left transversal in G to K.

Further we have: if $g \in G$, $g = (u, h)$, where $u \in L$, $h \in H = LI(L)$, therefore $\forall x \in E_0$:

$$((u,h) * (t_x, id) * (u,h)^{-1}(z) = (L_u h) * L_{t_x} * (L_u h)^{-1}(z) =$$
$$= L_u h L_{t_x} h^{-1} L_u^{-1}(z). \tag{13}$$

The set $gT_0^* g^{-1}$ is a left transversal in G to K if and only if

1) $\bigcup\limits_{x \in E_0} (g t_x^* g^{-1}) * K = G$;

2) $\forall x_1 \neq x_2$ from E_0:

$$(g t_{x_1}^* g^{-1}) * K \cap (g t_{x_2}^* g^{-1}) * K = \emptyset. \tag{14}$$

So $\forall v \in L$ and $h \in H$ we have

$$(v,h) * K = \bigcup_{r \in R,\ h_1 \in H} ((v,h) * (r, h_1)) =$$
$$= \bigcup_{r \in R,\ h_1 \in H} (v \cdot h(r), l_{v,h(v)} \varphi(r,h) h h_1) =$$
$$= (v \cdot h(R), H) = (L_v h(R), H).$$

Then the conditions (14) (using (13)) are equivalent to the following:

1) $\bigcup\limits_{x \in E_0} (L_u h L_{t_x} h^{-1} L_u^{-1}(R)) = L$;

2) $\forall x_1, x_2 \in E_0$, $x_1 \neq x_2$:

$$(L_u h L_{t_{x_1}}(R)) \cap (L_u h L_{t_{x_2}} h^{-1} L_u^{-1}(R)) = \emptyset. \tag{15}$$

By virtue of item **2** from Lemma 9 we obtain that the conditions (15) are equivalent to the following:

1) $\bigcup\limits_{x \in E_0} [(u \cdot h(t_x \cdot h^{-1}(u \backslash 1))) \cdot R] = L$;

2) $\forall x_1, x_2 \in E_0$, $x_1 \neq x_2$:

$$[(u \cdot h(t_{x_1} \cdot h^{-1}(u \backslash 1))) \cdot R] \cap [(u \cdot h(t_{x_2} \cdot h^{-1}(u \backslash 1))) \cdot R] = \emptyset. \tag{16}$$

The conditions (16) are equivalent to that the set $T_{u,h}\{u \cdot h(t_x \cdot h^{-1}(u \backslash 1))\}$ is a left transversal in L by R. Remembering that the reasoning was carried out $\forall g \in G$, i.e. $\forall u \in L$ and $\forall h \in H = LI(L)$, we obtain item 2 of the present lemma.

The items **3**, **4** and **5** are proved similarly to the previous reasoning, using the corresponding statements and Lemma 9. □

Corollary 1. *Let L be a loop, R be its subloop, and **Condition A** be fulfilled for them. Let $T_0 = \{t_x\}_{x \in E_0}$ be a left loop transversal in L to R. Then $\forall u \in L$ the set $T_u\{u \cdot (t_x \cdot (u \backslash 1))\}_{x \in E_0}$ is a left transversal in L to R.*

Proof. The proof easily follows from Lemma 10, **2**, when $h = id$. □

References

[1] Baer R. *Nets and groups.* Trans. Amer. Math. Soc., 1939, **46**, 110–141.

[2] Belousov V. D. Foundations of quasigroup and loop theory, Moscow, Nauka, 1967 (in Russian).

[3] Bonetti F., Lunardon G., Strambach K. *Cappi di permutazioni.* Rend. Math., 1979, **12**, No. 3-4, 383–395.

[4] Foguel T., Kappe L. C. *On loops covered by subloops.* Expositiones Matematicae, 2005, **23**, 255–270.

[5] Johnson K. W. *S-rings over loops, right mapping groups and transversals in permutation groups.* Math. Proc. Camb. Phil. Soc., 1981, **89**, 433–443.

[6] Kuznetsov E. A. *Transversals in groups.1.Elementary properties.* Quasigroups and related systems, 1994, **1**, No. 1, 22–42.

[7] Kuznetsov E. A. *About some algebraic systems related with projective planes.* Quasigroups and related systems, 1995, **2**, No. 1, 6–33.

[8] Kuznetsov E. A. *Transversals in groups.Semidirect product of a transversal operation and subgroup.* Quasigroups and related systems, 2001, **8**, 37–44.

[9] Kuznetsov E. A. *Transversals in loops.* Abstracts of International Conference "Loops-03", Prague, August 10–17, 2003, 18–20.

[10] Kuznetsov E. A. *Transversals in loops. 1. Elementary properties.* Quasigroups and related systems, 2010, **18**, No. 1, 43–58.

[11] Niemenmaa M., Kepka T. *On multiplication groups of loops.* J. of Algebra, 1990, **135**, 112–122.

[12] Pflugfelder H. *Quasigroups and loops: Introduction.* Sigma Series in Pure Math., 7, Helderman Verlag, New York, 1972.

[13] Sabinin L. V., Mikheev O. I. *Quasigroups and differential geometry, Chapter XII in the book "Quasigroups and loops: Theory and Applications"*, Helderman-Verlag, Berlin, 1990, 357–430.

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