A Generalization of Hardy-Hilbert’s Inequality for Non-homogeneous Kernel

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Abstract. This paper deals with a generalization of Hardy-Hilbert’s inequality for non-homogeneous kernel by considering sequences \((s_n, t_n)\), the functions \(\phi_p, \phi_q\) and parameter \(\lambda\). This inequality generalizes both Hardy-Hilbert’s inequality and Mulholland’s inequality, which includes most of the recent results of this type. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

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1 Introduction

If \(a_n, b_n \geq 0\) satisfy \(\sum_{n=1}^{\infty} a_n^2 < \infty\) and \(\sum_{n=1}^{\infty} b_n^2 < \infty\), then the well known Hilbert’s inequality (see [1]) is given by

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{1/2}
\]

and an equivalent form is given by

\[
\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2,
\]

where the constant factors \(\pi\) and \(\pi^2\) are the best possible. In 1925, Hardy [2] gave some extensions of (1) and (2) by introducing the \((p, q)\)-parameters as: if \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\), \(a_n, b_n \geq 0\) satisfy \(\sum_{n=1}^{\infty} a_n^p < \infty\) and \(\sum_{n=1}^{\infty} b_n^q < \infty\), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}
\]

and an equivalent form is given by

\[
\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p,
\]
where the constant factors \(\frac{\pi}{\sin(\pi/p)}\) and \([\frac{\pi}{\sin(\pi/p)}]^p\) are the best possible. Inequality (3) is called Hardy-Hilbert’s inequality and is important in analysis and its applications (cf. Mintrinovic et al. [4]). Recently many generalizations and refinements of these inequalities were also obtained. Some of them are given in [5–15].

If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0\) satisfy \(0 < \sum_{n=2}^{\infty} \frac{1}{n} a_n^p < \infty\) and \(0 < \sum_{n=2}^{\infty} \frac{1}{n} b_n^q < \infty\), then the Mulholland’s inequality (cf. [1,3]) is given by

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=2}^{\infty} \frac{1}{n} a_n^p \right)^{1/p} \left( \sum_{n=2}^{\infty} \frac{1}{n} b_n^q \right)^{1/q};
\]

where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible. Replacing \(a_m\) with \(ma_m\) and \(b_n\) with \(nb_n\) we have the following inequality:

If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0\) satisfy \(0 < \sum_{n=2}^{\infty} n^{p-1} a_n^p < \infty\) and \(0 < \sum_{n=2}^{\infty} n^{q-1} b_n^q < \infty\), then the inequality

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_mb_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=2}^{\infty} n^{p-1} a_n^p \right)^{1/p} \left( \sum_{n=2}^{\infty} n^{q-1} b_n^q \right)^{1/q};
\]

holds, where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible. The inequality (6) is also referred to as Mulholland’s inequality. Some generalizations of these inequalities are given in [16,17].

Most of the recent generalizations of inequalities (1) and (3) (cf. [5–15]) estimate the upper bounds of the double sum of the form \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K(m,n) a_m b_n\), where the kernel \(K(m,n)\) is homogeneous in \(m\) and \(n\). In this paper, we give a generalization of Hardy-Hilbert’s inequality for non-homogeneous kernel \(K(m,n) = (s_m + t_n)^{-1}\) by considering the sequences \((s_n), (t_n)\), the functions \(\phi_p, \phi_q\) and parameter \(\lambda\). This inequality generalizes both Hardy-Hilbert’s inequality and Mulholland’s inequality, from which all the inequalities given in [5–17] are obtained as particular cases. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

2 Some Lemmas

We first set the following notations. Suppose \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\) and \(\phi_r (r = p, q)\) is a function of \(r\) such that \(0 < \phi_r < \lambda (r = p, q)\). Let \(m_0, n_0 \in \mathbb{N}\) and \(s(x), t(x)\) are differentiable strictly increasing functions in \((m_0 - 1, \infty)\) and \((n_0 - 1, \infty)\), respectively, such that \(s(m_0 - 1) = t(n_0 - 1) = 0\) and \(s(\infty) = t(\infty) = \infty\), \(s'(x)/(s(x))^{1-\sigma}\) and \(t'(x)/(t(x))^{1-\sigma}\) are decreasing in \((m_0 - 1, \infty)\) and \((n_0 - 1, \infty)\), respectively. We write \(s(m) = s_m, s'(m) = s'_m, t(n) = t_n\) and \(t'(n) = t'_n\).

We need the formula of the \(\beta\)-function as (cf. Wang et al. [18]):

\[
B(p, q) = \int_0^\infty \frac{1}{(1 + u)^{p+q}} u^{p-1} du = B(q, p) \quad (p, q > 0),
\]
Lemma 1. Define the weight functions $\omega_{\lambda}(s, t, p, m)$ and $\omega_{\lambda}(t, s, q, n)$ as

$$\omega_{\lambda}(s, t, p, m) = \sum_{n=n_0}^{\infty} \frac{1}{(s_m + t_n)^{\lambda}} \frac{s_m}{(s_m)^{1-\phi_q}}, \ (m \geq m_0); \quad (8)$$

$$\omega_{\lambda}(t, s, q, n) = \sum_{m=m_0}^{\infty} \frac{1}{(s_m + t_n)^{\lambda}} \frac{t_n}{(s_m)^{1-\phi_p}}, \ (n \geq n_0). \quad (9)$$

Then

$$\omega_{\lambda}(s, t, p, m) < B(\phi_p, \lambda - \phi_p)(s_m)^{\phi_p - \lambda}, \ (m \geq m_0); \quad (10)$$

$$\omega_{\lambda}(t, s, q, n) < B(\phi_q, \lambda - \phi_q)(t_n)^{\phi_q - \lambda}, \ (n \geq n_0). \quad (11)$$

Proof. Since $\lambda > 0$, $s(x)$, $t(x)$ are differentiable, strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in $((m_0 - 1, \infty)$ and $(n_0 - 1, \infty)$, respectively.

So

$$\omega_{\lambda}(s, t, p, m) < \sum_{n=n_0}^{\infty} \int_{n-1}^{n} \frac{1}{(s_m + t(y))^\lambda} \frac{t'(y)}{(t(y))^{1-\phi_p}} dy$$

$$= \int_{n_0-1}^{\infty} \frac{(t(y))^{\phi_q - 1}t'(y)}{(s_m + t(y))^\lambda} dy$$

$$= (s_m)^{\phi_q - \lambda} \int_{0}^{\infty} \frac{1}{(1 + u)^\lambda} u^{\phi_q - 1} du \quad \text{setting} \ u = \frac{t(y)}{s_m}. \quad (1)$$

Then by (7), we get (10). Similarly, (11) can be proved. The lemma is proved. \square

Lemma 2. If $\phi_p + \phi_q = \lambda$ and $0 < \varepsilon < q \phi_p$, then

$$\sum_{1} := \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{s_m}{(s_m + t_n)^{\lambda}} \frac{t_n}{(t_n)^{1-\phi_p}} \frac{s_m}{(s_m)^{1-\phi_q + \frac{\varepsilon}{q}} \frac{t_n}{(t_n)^{1-\phi_p + \frac{\varepsilon}{q}}} \geq \frac{1}{\varepsilon (s_m)^{\frac{\varepsilon}{q}}} B(\frac{\phi_p - \varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}) - O(1). \quad (12)$$

Proof. Since $\lambda > 0$, $s(x)$, $t(x)$ are differentiable strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in $((m_0 - 1, \infty)$ and $(n_0 - 1, \infty)$, respectively,
Then by (7), (12) is valid. The lemma is proved.

Theorem 1. If we have

Proof. By Hölder’s inequality with weight (cf. Kuang [19]), we have

Then by (7), (12) is valid. The lemma is proved.

3 Main Result

Theorem 1. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \) \( 0 < \phi_p < \lambda (r = p, q) \) and \( a_n, b_n \geq 0 \) satisfy

0 < \( \sum_{m=n_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s_m)^{p-1}} a_m^p < \infty \) and 0 < \( \sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t_n)^{q-1}} b_n^q < \infty, \)

then

\[
\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} a_m b_n (s_m+t_n)^{\lambda} < H_\lambda(\phi_p, \phi_q) \left\{ \sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s_m)^{p-1}} a_m^p \right\}^{\frac{1}{p}} \\
\times \left\{ \sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t_n)^{q-1}} b_n^q \right\}^{\frac{1}{q}}
\]

(13)

where \( H_\lambda(\phi_p, \phi_q) = B_{\phi_p}^{\frac{1}{p}}(\phi_p, \lambda - \phi_p) B_{\phi_q}^{\frac{1}{q}}(\phi_q, \lambda - \phi_q). \)

Proof. By Hölder’s inequality with weight (cf. Kuang [19]), we have

\[
\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} a_m b_n (s_m+t_n)^{\lambda} \\
= \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{1}{(s_m+t_n)^{\lambda}} \left\{ \frac{(t_n)^{(\phi_q-1)/p}(t_n')^{1/p}}{(s_m)^{(\phi_q-1)/q}(s_m')^{1/q} a_m} \right\} \left\{ \frac{(s_m)^{(\phi_p-1)/q}(s_m')^{1/q}}{(t_n)^{(\phi_p-1)/p}(t_n')^{1/p} b_n} \right\}
\leq \left\{ \sum_{m=m_0}^{\infty} \frac{t_n'}{(s_m+t_n)^{\lambda}} \left( t_n \right)^{1-\phi_p} \right\} \left( s_m \right)^{(p-1)(1-\phi_q)} a_m \left\{ \frac{1}{p} \right\}^{\frac{1}{p}} \\
\times \left\{ \sum_{n=n_0}^{\infty} \frac{s_m'}{(s_m+t_n)^{\lambda}} \left( s_m \right)^{1-\phi_q} \right\} \left( t_n' \right)^{(q-1)(1-\phi_p)} b_n \left\{ \frac{1}{q} \right\}^{\frac{1}{q}}.
\]
Then by (8) and (9), we have

$$
\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m + t_n)^\lambda} \leq \left\{ \sum_{m=m_0}^{\infty} \omega_\lambda(s, t, p, m) \frac{(s_m)^{(p-1)(1-\phi)}}{(s'_m)^{p-1}} a_m^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=n_0}^{\infty} \omega_\lambda(t, s, q, n) \frac{(t_n)^{(q-1)(1-\phi)}}{(t'_n)^{q-1}} b_n^q \right\}^{\frac{1}{q}}
$$

and in view of (10) and (11), it follows that (13) is valid. The theorem is proved. □

**Theorem 2.** If $p > 1$, \( \frac{1}{p} + \frac{1}{q} = 1 \), $0 < \phi_r < \lambda$ ($r = p, q$) and $a_n \geq 0$ satisfy

$$
0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi - p(1-\phi)}(s'_m)^{p-1}}{(s_m)^p} a_m^p < \infty,
$$

then we obtain an equivalent inequality of (13) as follows:

$$
\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_q + (p-1)(1-\phi)}} \left[ \sum_{m=m_0}^{\infty} \frac{a_m}{(s_m + t_n)^\lambda} \right]^p < \left\{ H_\lambda(\phi_p, \phi_q) \right\}^p \sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi - p(1-\phi)}(s'_m)^{p-1}}{(s_m)^p} a_m^p.
$$

**Proof.** Setting $b_n = \frac{t'_n}{(t_n)^{1-\phi_q + (p-1)(1-\phi)}} \left[ \sum_{m=m_0}^{\infty} \frac{a_m}{(s_m + t_n)^\lambda} \right]^{p-1}$ and using (13) we obtain

$$
0 < \sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q - \lambda + (q-1)(1-\phi)}}{(t'_n)^{q-1}} b_n^q
$$

$$
= \sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_q + (p-1)(1-\phi)}} \left[ \sum_{m=m_0}^{\infty} \frac{a_m}{(s_m + t_n)^\lambda} \right]^p
$$

$$
= \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m + t_n)^\lambda}
$$

$$
\leq H_\lambda(\phi_p, \phi_q) \left\{ \sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi - p(1-\phi)}(s'_m)^{p-1}}{(s_m)^p} a_m^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q - \lambda + (q-1)(1-\phi)}}{(t'_n)^{q-1}} b_n^q \right\}^{\frac{1}{q}}.
$$
Hence

\[ 0 < \left\{ \sum_{n=m_0}^\infty \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t_n')^{q-1}b_n^q} \right\}^{\frac{1}{p}} \]

\[ = \left\{ \sum_{n=m_0}^\infty \frac{t_n'}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[ \sum_{m=m_0}^\infty \frac{a_m}{(s_m+t_n)^\lambda} \right]^{p} \right\}^{\frac{1}{p}} \frac{1}{b_n^q} \]

\[ \leq H_{\lambda}(\phi_p, \phi_q) \left\{ \sum_{m=m_0}^\infty \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s_m')^{p-1}a_m} \right\}^{\frac{1}{p}} < \infty. \]

By using (13) it follows that (15) takes the form of strict inequality; so does (16). Hence we get (14).

On the other hand, if (14) holds, then by Hölder’s inequality, we have

\[ \sum_{m=m_0}^\infty \sum_{n=n_0}^\infty \frac{a_mb_n}{(s_m+t_n)^\lambda} \]

\[ = \sum_{n=n_0}^\infty \left[ \frac{(t_n')^{1/p}}{(t_n)^{(1-\phi_p+(p-1)(\phi_q-\lambda))}/p} \sum_{m=m_0}^\infty \frac{a_m}{(s_m+t_n)^\lambda} \right] \left[ \frac{(t_n)^{(1-\phi_q+(p-1)(\phi_q-\lambda))/p}}{(t_n')^{1/p}} b_n \right] \]

\[ \leq \left\{ \sum_{n=n_0}^\infty \frac{t_n'}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[ \sum_{m=m_0}^\infty \frac{a_m}{(s_m+t_n)^\lambda} \right]^{p} \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^\infty \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t_n')^{q-1}b_n^q} \right\}^{\frac{1}{q}}. \]

Hence by (14), (13) yields. Thus it follows that (13) and (14) are equivalent. The theorem is proved.

**Theorem 3.** If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \phi_p > 0 (r = p, q), \phi_p + \phi_q = \lambda, a_n, b_n \geq 0 \) satisfy

\[ 0 < \sum_{m=m_0}^\infty \frac{(s_m)^{\phi_q(r-1)-1}}{(s_m')^{p-1}a_m} < \infty \text{ and } 0 < \sum_{n=n_0}^\infty \frac{(t_n)^{\phi_q(r-1)-1}}{(t_n')^{q-1}b_n^q} < \infty, \]

\[ 0 < \sum_{m=m_0}^\infty \sum_{n=n_0}^\infty \frac{a_mb_n}{(s_m+t_n)^\lambda} < B(\phi_p, \phi_q) \left\{ \sum_{m=m_0}^\infty \frac{(s_m)^{\phi_q(r-1)-1}}{(s_m')^{p-1}a_m} \right\}^{\frac{1}{r}} \left\{ \sum_{n=n_0}^\infty \frac{(t_n)^{\phi_q(r-1)-1}}{(t_n')^{q-1}b_n^q} \right\}^{\frac{1}{q}}, \]

where the constant factor \( B(\phi_p, \phi_q) \) is the best possible.

**Proof.** Since \( \phi_p + \phi_q = \lambda \), then by Theorem 1, (17) is valid. For \( 0 < \varepsilon < q\phi_p \), we take

\[ \tilde{a}_m = (s_m)^{-1+\phi_q-\varepsilon/p} s_m' \quad (m \geq m_0), \]

\[ \tilde{b}_n = (t_n)^{-1+\phi_p-\varepsilon/q} t_n' \quad (n \geq n_0). \]
Since \( \frac{s'(x)}{(s(x))^{1+\varepsilon}} = \frac{s'(x)}{(s(x))^{1-\sum_{n=m_0}^{\infty} (s_m)^{1-r}a_m}} \) is decreasing in \((m_0-1, \infty)\), we have

\[
\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} \frac{1}{a_m} \leq \sum_{m=m_0+1}^{\infty} \frac{s'_m}{(s_m)^{1+\varepsilon}} \leq \frac{s'_m}{(s_m)^{1+\varepsilon}} + \int_{m_0}^{\infty} s'(x)(x)^{1+\varepsilon}dx \tag{18}
\]

Similarly,

\[
\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_q)-1}}{(t'_n)^{q-1}} \frac{1}{b_n} \leq \frac{1}{(m_0)} \leq \frac{1}{(m_0)^{1+\varepsilon}} \tag{19}
\]

If the constant factor \(B(\phi_p, \phi_q)\) in (17) is not the best possible, then there exists a positive constant \(K < B(\phi_p, \phi_q)\) such that (17) is still valid if we replace \(B(\phi_p, \phi_q)\) by \(K\). In particular by (12), (18) and (19), we have

\[
\frac{1}{(s_m)^{1+\varepsilon}} \leq \frac{1}{(t_n)^{1+\varepsilon}} \leq \frac{B(\phi_p, \phi_q)}{K}
\]

and taking \(\varepsilon \to 0^+\), we get \(B(\phi_p, \phi_q) \leq K\). This contradiction leads to the conclusion that the constant factor \(B(\phi_p, \phi_q)\) in (17) is the best possible. The theorem is proved.

**Corollary 1.** If \(p > 1\), \(\frac{1}{p} + \frac{1}{q} = 1\) and \(a_m \geq 0\) satisfy \(0 < \sum_{m=m_0}^{\infty} (s'_m)^{1-r}a_m < \infty (r = p, q)\), then

\[
\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{m=m_0}^{\infty} (s'_m)^{1-p}a_m \right)^{\frac{1}{p}} \left( \sum_{m=m_0}^{\infty} (s'_m)^{1-q}a_m \right)^{\frac{1}{q}} \tag{20}
\]

where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible.

**Proof.** Taking \(a_n = b_n, s_n = t_n, \lambda = 1, \phi_r = \frac{1}{r} (r = p, q)\) in (17), we get (20). The corollary is proved.
Corollary 2. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( a_m \geq 0 \) satisfy \( 0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{p-1}}{(s'_m)^p} a_m^p < \infty \) \( (r = p, q) \), then

\[
\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \pi \left( \sum_{m=m_0}^{\infty} \frac{(s_m)^{p-1}}{(s'_m)^p} a_m^p \right) \frac{1}{\beta} \left( \sum_{m=m_0}^{\infty} \frac{(s_m)^{q-1}}{(s'_m)^q} a_m^q \right) \frac{1}{\gamma} \tag{21}
\]

where the constant factor \( \pi \) is the best possible.

Proof. Taking \( a_n = b_n, s_n = t_n, \lambda = 1, \phi_r = \frac{1}{2} (r = p, q) \) in (17), we get (21). The corollary is proved. \( \square \)

Theorem 4. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \phi_r > 0 (r = p, q), \phi_p + \phi_q = \lambda, a_n \geq 0 \) satisfy \( 0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{(p-1)\phi_q)}{(s'_m)^p} a_m^p < \infty \), then we obtain an equivalent inequality of (17) as follows:

\[
\sum_{n=n_0}^{\infty} \frac{t'_n}{(s'_m)^{1-p\phi_p}} \left[ \sum_{m=m_0}^{\infty} \frac{a_m}{s_m + t_n} \right]^{p} \frac{B(\phi_p, \phi_q)}{B(\phi_q, \phi_p)} \sum_{m=m_0}^{\infty} \frac{(s_m)^{(p-1)\phi_q)}{(s'_m)^p} a_m^p \tag{22}
\]

where the constant factor \( \frac{B(\phi_p, \phi_q)}{B(\phi_q, \phi_p)} \) is the best possible.

Proof. Since \( \phi_p + \phi_q = \lambda \), then by Theorem 2, we get inequalities (17) and (22) are equivalent. By Theorem 3, the constant factor in (17) is best possible, hence the constant factor in (22) is best possible. The theorem is proved. \( \square \)

4 Generalization of Hardy-Hilbert’s Inequality

Theorem 5. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi_r < \lambda(r = p, q), A, B > 0, 0 < \alpha \leq \frac{1}{\phi_q}, 0 < \beta \leq \frac{1}{\phi_r}, a_m, b_n \geq 0 \) satisfy \( 0 < \sum_{m=1}^{\infty} m^\alpha (\phi_r - \lambda + (1-p)\phi_q) + p-1 a_m^p < \infty \) and \( 0 < \sum_{n=1}^{\infty} n^\beta (\phi_q + (1-p)\phi_r) + q-1 b_n^q < \infty \), then the following two equivalent inequalities hold:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^\alpha + Bn^\beta)^{1/\gamma}} < \mu H_\lambda(\phi_p, \phi_q) \left( \sum_{m=1}^{\infty} m^\alpha (\phi_r - \lambda + (1-p)\phi_q) + p-1 a_m^p \right)^{1/\gamma} \times \left( \sum_{n=1}^{\infty} n^\beta (\phi_r - \lambda + (1-q)\phi_r) + q-1 b_n^{q/\gamma} \right)^{1/\gamma} \tag{23}
\]

\[
\sum_{n=1}^{\infty} n^\beta (\phi_r + (1-p)\phi_q) - 1 \left[ \sum_{m=1}^{\infty} \frac{a_m}{(Am^\alpha + Bn^\beta)^{1/\gamma}} \right]^{p} < \mu H_\lambda(\phi_p, \phi_q) \sum_{m=1}^{\infty} m^\alpha (\phi_r - \lambda + (1-p)\phi_q) + p-1 a_m^p \tag{24}
\]

where \( \mu = \left( \frac{A^\alpha \phi_q - \lambda}{B^{\beta p} \phi_r} \right)^{1/\gamma} \left( \frac{B^\beta \phi_q - \lambda}{\alpha A^\alpha} \right)^{1/\gamma} \) and \( H_\lambda(\phi_p, \phi_q) = B^{\beta} (\phi_p, \phi_q) B^{1/\gamma} (\phi_q, \lambda - \phi_q) \). The constant factors \( \mu H_\lambda(\phi_p, \phi_q) \) and \( \mu H_\lambda(\phi_p, \phi_q) \) are the best possible if \( \phi_p + \phi_q = \lambda \).
Proof. Setting $s_m = A m^\alpha$, $t_n = B n^\beta$ in Theorem 1 and Theorem 2, we get both the inequalities (23) and (24) are valid and equivalent. From Theorem 3 and Theorem 4, it follows that the constant factors are the best possible. This completes the proof.

We discuss a number of special cases of inequality (23). Similar inequalities can also be derived from inequality (24).

**Example 1.** Setting $\phi_p = 1 - A_2 p$, $\phi_q = 1 - A_1 q$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $A_1 < \frac{1}{q}$, $A_2 < \frac{1}{p}$, $0 < \alpha \leq \frac{1}{A_1 q}$, $0 < \beta \leq \frac{1}{A_2 p}$, $\lambda > \max \{1 - A_2 p, 1 - A_1 q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))} + p^{-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))} + q^{-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < L \left( \sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))} + p^{-1} a_m^p \right)^{\frac{1}{p}} \times \left( \sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))} + q^{-1} b_n^q \right)^{\frac{1}{q}}$$

(25)

where $L = \left( \frac{A_1^{1-A_2 p-\lambda}}{B_1^{1-A_1 q-\lambda}} \right)^{\frac{1}{p}} \left( \frac{B_1^{1-A_1 q-\lambda}}{A_1^{1-A_2 p-\lambda}} \right)^{\frac{1}{q}} H_\lambda (1 - A_2 p, 1 - A_1 q)$. For $A = B = \alpha = \beta = 1$, we get the result of Brnetic and Pecaric [5, Theorem 2].

**Example 2.** Setting $\phi_r = \frac{1}{r} (r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{2}{p}$, $0 < \beta \leq \frac{2}{q}$, $\lambda > 0$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha \lambda)} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\beta \lambda)} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < \mu B \left( \frac{\alpha}{p}, \frac{\lambda}{q} \right) \left( \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha \lambda)} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{(q-1)(1-\beta \lambda)} b_n^q \right)^{\frac{1}{q}}$$

(26)

where $\mu = \left( A^{\frac{1}{p}} B^{\frac{1}{q}} \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \right)^{-1}$ and the constant factor $\mu B \left( \frac{\lambda}{p}, \frac{\beta}{q} \right)$ is the best possible. For $A = B = \lambda = 1$, $\alpha = \beta$, we get the result of Yang [7]. Setting $\alpha = \beta = 1$, $p = q = 2$, we get the result of Yang [13] and setting $\alpha = \beta = 1$, we get the result of Yang [15].

**Example 3.** Setting $\phi_r = \lambda (1 - \frac{1}{r}) (r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{2}{p}$, $0 < \beta \leq \frac{2}{q}$, $\lambda > 0$ and $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{p-\alpha \lambda - 1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-\beta \lambda - 1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < \mu B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left( \sum_{m=1}^{\infty} m^{p-\alpha \lambda - 1} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{q-\beta \lambda - 1} b_n^q \right)^{\frac{1}{q}}$$

(27)
where \( \mu = \left( A \frac{\lambda}{2} B \frac{\lambda}{2} \alpha \frac{1}{2} \beta \frac{1}{2} \right)^{-1} \) and the constant factor \( \mu B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \) is the best possible.

For \( \lambda = 1, \alpha = \beta \), we get the result of Yang [8]. Setting \( A = B = \alpha = \beta = 1 \), we recover the result of Yang [9].

**Example 4.** Setting \( \phi_r = 1 + \frac{\lambda - 2}{p} (r = p, q) \) in Theorem 5, we have the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, A, B > 0, 0 < \alpha \leq \frac{q}{p+\lambda-2}, 0 < \beta \leq \frac{p}{p+\lambda-2} \), \( \lambda > 2 - \min\{p, q\} \), \( a_m, b_n \geq 0 \) satisfy \( 0 \leq \sum_{m=1}^{\infty} m(p-1)(1-\alpha(q+\lambda-2))a_m^p < \infty \) and \( 0 \leq \sum_{n=1}^{\infty} n(q-1)(1-\beta(p+\lambda-2))b_n^q < \infty \), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < L_1 \left\{ \sum_{m=1}^{\infty} m(p-1)(1-\alpha(q+\lambda-2))a_m^p \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} n(q-1)(1-\beta(p+\lambda-2))b_n^q \right\}^{\frac{1}{2}}
\]

(28)

where the constant factor \( L_1 = \left( A \frac{q+\lambda-2}{2} B \frac{q+\lambda-2}{2} \alpha \frac{1}{2} \beta \frac{1}{2} \right)^{-1} \) \( \times \) \( B \left( \frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \) is the best possible. In particular for \( \alpha = \beta = 1, p = q = 2 \), we get the result of Yang [13].

**Example 5.** Setting \( \phi_r = 1 + (1 - \frac{1}{p}) (\lambda - 2) (r = p, q) \) in Theorem 5, we have the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, A, B > 0, 0 < \alpha \leq \frac{p}{p+\lambda-2}, 0 < \beta \leq \frac{q}{q+\lambda-2}, \lambda > 2 - \min\{p, q\} \), \( a_m, b_n \geq 0 \) satisfy \( 0 \leq \sum_{m=1}^{\infty} m^{(2-\lambda-p)+p-1}a_m^p < \infty \) and \( 0 \leq \sum_{n=1}^{\infty} n^{(2-\lambda-q)+q-1}b_n^q < \infty \), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < L_2 \left\{ \sum_{m=1}^{\infty} m^{(2-\lambda-p)+p-1}a_m^p \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} n^{(2-\lambda-q)+q-1}b_n^q \right\}^{\frac{1}{2}}
\]

(29)

where the constant factor \( L_2 = \left( A \frac{p+\lambda-2}{2} B \frac{q+\lambda-2}{2} \alpha \frac{1}{2} \beta \frac{1}{2} \right)^{-1} \) \( \times \) \( B \left( \frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \) is the best possible. For \( \alpha = \beta = 1 \), we get the result of Yang and Debnath [6]. Setting \( A = B = \lambda = 1, \alpha = \beta \), we recover the result of Yang [7].

**Example 6.** Setting \( \phi_r = \frac{\lambda+1}{2} + \frac{1}{r} (r = p, q) \), in Theorem 5, we have the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, A, B > 0, 0 < \alpha \leq \frac{p}{\lambda+1} + \frac{1}{q} \), \( 0 < \beta \leq \frac{p}{\lambda+1} + \frac{1}{q} \), \( \lambda > 1 - 2 \min\{\frac{1}{p}, \frac{1}{q}\} \), \( a_m, b_n \geq 0 \) satisfy \( 0 \leq \sum_{m=1}^{\infty} m^{p-1+\alpha(2-\lambda-p)/2}a_m^p < \infty \) and \( 0 \leq \sum_{n=1}^{\infty} n^{q-1+\beta(2-\lambda-q)/2}b_n^q < \infty \), then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(A m^\alpha + B n^\beta)^\lambda} < L_3 \left\{ \sum_{m=1}^{\infty} m^{p-1+\alpha(2-\lambda-p)/2}a_m^p \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} n^{q-1+\beta(2-\lambda-q)/2}b_n^q \right\}^{\frac{1}{2}}
\]

(30)

where the constant factor \( L_3 = \left( A \frac{\lambda+1}{2} + \frac{1}{p} B \frac{\lambda+1}{2} + \frac{1}{q} \alpha \frac{1}{2} \beta \frac{1}{2} \right)^{-1} \) \( \times \) \( B \left( \frac{\lambda+1}{2} + \frac{1}{p}, \frac{\lambda+1}{2} + \frac{1}{q} \right) \) is the best possible. Setting \( A = B = \lambda = 1, \alpha = \beta \), we recover the result of Yang [7].
Example 7. Setting $\phi_p = \lambda(\frac{1}{n} + (1 - \frac{1}{p})(\alpha - 2))$, $\phi_q = \lambda(\frac{1}{n} + (1 - \frac{1}{q})(\beta - 2))$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A$, $B$, $\lambda > 0$, $2 - p < \alpha \leq 2 + p(\frac{1}{q} - 1)$, $2 - q < \beta \leq 2 + q(\frac{1}{p} - 1)$, $a_m$, $b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p \ll \alpha$ and $0 < \sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q \ll \beta$, then

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^\lambda} < L_4 \left( \sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q \right)^{\frac{1}{q}}
$$

where the constant factor $L_4 = \left( \frac{A^{\lambda(p+\alpha-2)p}}{B^{\lambda(q+\beta-2)q}} \right)^{\frac{1}{p}}$, $B^{\lambda(p+\alpha-2)p}$ is the best possible. For $A = B = \lambda = 1, \alpha = \beta$, we get the result of Yang [12].

Remark 1. Setting (i) $\phi_r = \frac{1}{p} (r = p, q)$, (ii) $\phi_r = 1 - \frac{1}{r} (r = p, q)$, (iii) $\phi_r = \frac{\lambda+1}{2} - \frac{1}{r} (r = p, q)$ in Theorem 5, we get new inequalities.

Remark 2. Taking $\alpha = \beta$, $A = B = 1$, $\phi_r = \frac{\alpha}{\beta}$ $(r = p, q)$ in (23), we get the result of Yang [10].

Remark 3. Taking $s_m = t_m = u(m)$ in Theorem 3, we get the result of Yang [14].

For other appropriate values of $\lambda, \phi_p, \phi_q$ and suitably choosing sequences $s_m$ and $t_n$ in Theorem 1 and Theorem 3, one can obtain many new inequalities.

5 Generalization of Mulholland’s Inequality

Theorem 6. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi_r \leq 1 (r = p, q)$, $\lambda > \max\{\phi_p, \phi_q\}$, $\alpha, \beta > 0$, and $a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{\phi_p-\lambda+(p-1)(p-1)\phi_p} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{\phi_q-\lambda+(q-1)(1-\phi_q)} b_n^q < \infty$, then the following two equivalent inequalities holds:

$$
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(ln m^{\alpha} n^{\beta})^\lambda} < \eta H_\lambda(\phi_p, \phi_q) \left( \sum_{m=2}^{\infty} m^{\phi_p-\lambda+(p-1)(p-1)\phi_p} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=2}^{\infty} n^{\phi_q-\lambda+(q-1)(1-\phi_q)} b_n^q \right)^{\frac{1}{q}},
$$

and

$$
\sum_{n=2}^{\infty} \left[ (ln n)^{\phi_p-1+(p-1)(\lambda-\phi_q)} \sum_{m=2}^{\infty} \frac{a_m}{(ln m^{\alpha} n^{\beta})^\lambda} \right]^p n < \eta H_\lambda(\phi_p, \phi_q)^p \sum_{m=2}^{\infty} m^{\phi_p-\lambda+(p-1)(p-1)\phi_p} a_m^p
$$

where

$$
\eta = \left( \frac{\alpha^{\phi_p-\lambda}}{\beta^{\phi_q-\lambda}} \right)^{\frac{1}{p}} \left( \frac{\beta^{\phi_q-\lambda}}{\alpha^{\phi_p-\lambda}} \right)^{\frac{1}{q}}
$$

and $H_\lambda(\phi_p, \phi_q) = B^\frac{1}{p}(\phi_p, \lambda - \phi_p)B^\frac{1}{q}(\phi_q, \lambda - \phi_q)$.

The constant factors $\eta H_\lambda(\phi_p, \phi_q)$ and $[\eta H_\lambda(\phi_p, \phi_q)]^p$ are the best possible if $\phi_p + \phi_q = \lambda$. 
Proof. Setting \( s_m = \ln m^\alpha \), \( t_n = \ln n^\beta \) in Theorem 1 and Theorem 2, we get both the inequalities (32) and (33) are valid and equivalent. The constant factors are the best possible obtained from Theorem 3 and Theorem 4. This completes the proof. \( \square \)

Example 8. Setting \( \phi_r = \frac{1}{r} (r = p, q) \) in Theorem 6, we obtain the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, \beta > 0, \lambda > 0 \), then

\[
0 < \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{1-\lambda} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{1-\lambda} b_n^q < \infty,
\]

where the constant factor

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \frac{1}{\alpha^\lambda \beta^\beta \lambda^\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{(p-1)(1-\lambda)} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}}
\]

(34)

In particular for \( \alpha = \beta = \lambda = 1 \), we get the result of Yang [16, Theorem 2.1].

Example 9. Setting \( \phi_r = \frac{1}{r} (r = p, q) \) in Theorem 6, we have the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, \beta > 0, 0 < \lambda \leq \min\{p, q\} \), then

\[
0 < \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{p(1-\lambda)} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{(q-1)(1-\lambda)} b_n^q < \infty,
\]

where the constant factor

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \frac{1}{\alpha^\lambda \beta^\beta \lambda^\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{(p-1)(1-\lambda)} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}}
\]

(35)

Example 10. Setting \( \phi_r = \lambda (1-\frac{1}{r}) (r = p, q) \) in Theorem 6, we have the following inequality: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, \beta > 0, 0 < \lambda \leq \min\{p, q\} \), then

\[
0 < \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{p\lambda-1} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{q\lambda-1} b_n^q < \infty,
\]

where the constant factor

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \frac{1}{\alpha^\lambda \beta^\beta \lambda^\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{p\lambda-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{q\lambda-1} b_n^q \right\}^{\frac{1}{q}}
\]

(36)

where the constant factor

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \frac{1}{\alpha^\lambda \beta^\beta \lambda^\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{p\lambda-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{q\lambda-1} b_n^q \right\}^{\frac{1}{q}}
\]

(37)

In particular for \( \alpha = \beta = \lambda = 1 \), it reduces to

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^\lambda} < \frac{\pi}{\sin \frac{\pi}{p}} \left\{ \sum_{m=2}^{\infty} m^{p-1} (\ln m)^{p-2} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{q-2} b_n^q \right\}^{\frac{1}{q}}
\]
and we obtain a new inequality in \((p, q)\)-parameter form other than (6), with the same best constant factor.

**Example 11.** Setting \(\phi_r = 1 + (1 - \frac{1}{p})(\lambda - 2)(r = p, q)\) in Theorem 6, we have the following inequality: If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, \beta > 0, 2 - \min\{p, q\} < \lambda \leq 2, a_m, b_n \geq 0\) satisfy \(0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_p^p < \infty\) and \(0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_q^q < \infty\), then

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^{\lambda}} < \eta k_{\lambda}(p) \left\{ \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_p^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_q^q \right\}^{\frac{1}{q}} \tag{38}
\]

where \(\eta = \frac{2 - \lambda - \frac{1}{p} \beta}{\frac{1}{p} + \frac{1}{q} \beta - \lambda - 2}, k_{\lambda}(p) = B\left(\frac{p + \lambda - 2}{p}, \frac{q + \lambda - 2}{q}\right)\) and the constant factor \(\eta k_{\lambda}(p)\) is the best possible.

In particular for \(\alpha = \beta = 1\), we get

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^{\lambda}} < k_{\lambda}(p) \left\{ \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_p^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_q^q \right\}^{\frac{1}{q}} \tag{39}
\]

where the constant factor \(k_{\lambda}(p)\) is the best possible. For \(\lambda = 1\), it reduces to the result of Yang [16, Theorem 2.1]. Replacing \(a_m, b_n\) by \(\frac{a_m}{m^{\alpha}}, \frac{b_n}{n^{\beta}}\) respectively, we get the result of Yang and Debnath [17, Theorem 1]).

**Example 12.** Setting \(\phi_r = \frac{\lambda - 1}{2} + \frac{1}{r} (r = p, q)\) in Theorem 6, we have the following inequality: If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, \beta > 0, 1 - 2 \min\{\frac{1}{p}, \frac{1}{q}\} < \lambda < 1 + 2 \min\{\frac{1}{p}, \frac{1}{q}\}, a_m, b_n \geq 0\) satisfy \(0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2} a_p^p < \infty\) and \(0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{g(1-\lambda)/2} b_q^q < \infty\), then

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^{\lambda}} < \eta \tilde{k}_{\lambda}(p) \left\{ \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2} a_p^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{g(1-\lambda)/2} b_q^q \right\}^{\frac{1}{q}} \tag{40}
\]

where \(\eta = \alpha^{\frac{\lambda - 1}{2} - \frac{1}{p} \beta} \frac{\lambda - 1}{2} - \frac{1}{q}, \tilde{k}_{\lambda}(p) = B\left(\frac{\lambda - 1}{2} + \frac{1}{p}, \frac{\lambda - 1}{2} + \frac{1}{q}\right)\) and the constant factor \(\eta \tilde{k}_{\lambda}(p)\) is the best possible. In particular for \(\alpha = \beta = \lambda = 1\), it reduces to (6).

**Remark 4.** Setting (i) \(\phi_r = 1 - \frac{1}{r} (r = p, q)\), (ii) \(\phi_r = 1 + \frac{\lambda - 2}{r} (r = p, q)\), (iii) \(\phi_r = \frac{\lambda + 1}{r} - \frac{1}{r} (r = p, q)\) in Theorem 6, we get new inequalities.
6 Applications

In this section, we will give the generalizations of Hardy-Littlewood’s inequality. Let \( f \in L^2(0, 1) \) and \( f(x) \neq 0 \). If

\[
a_n = \int_0^1 x^n f(x) \, dx, \quad n = 0, 1, 2, 3, \ldots
\]

then we have the Hardy-Littlewood’s inequality (see [1]) of the form

\[
\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x) \, dx
\]  

(41)

where the constant factor \( \pi \) is the best possible. Yang [11] gave a generalization of (41) for \( p \geq 2 \) as

\[
\left( \sum_{n=0}^{\infty} a_n^p \right)^{\frac{1}{p+1}} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{n=0}^{\infty} a_n^{p(p-1)} \right)^{\frac{1}{p}} \int_0^1 f^2(x) \, dx.
\]  

(42)

**Theorem 7.** Let \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), \( f \in L^2(0, 1) \), \( f(x) \neq 0 \) and

\[
a_n = \left( s_n' \right)^{\frac{1}{p}} \int_0^1 x^{s_n-\frac{1}{2}} f(x) \, dx, \quad n \geq m_0.
\]

If \( 0 < \sum_{n=m_0}^{\infty} (s_n')^{2-p} d_n^{p(p-1)} < \infty \), then

\[
\left( \sum_{n=m_0}^{\infty} a_n^p \right)^{\frac{1}{p+1}} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{n=m_0}^{\infty} (s_n')^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \int_0^1 f^2(x) \, dx.
\]  

(43)

**Proof.** Applying Schwartz inequality, we have

\[
\left( \sum_{n=m_0}^{\infty} a_n^p \right)^2 = \left( \sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} \int_0^1 x^{s_n-\frac{1}{2}} f(x) \, dx \right)^2
\]

\[
\leq \left\{ \int_0^1 \left( \sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} x^{s_n-\frac{1}{2}} \right) f(x) \, dx \right\}^2
\]

\[
\leq \int_0^1 \left( \sum_{n=m_0}^{\infty} a_n^{p-1}(s_n')^{\frac{1}{p}} x^{s_n-\frac{1}{2}} \right)^2 \, dx \int_0^1 f^2(x) \, dx
\]

\[
= \left\{ \sum_{n=m_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_n^{p-1}(s_n')^{\frac{1}{p}} a_m^{p-1}(s_m')^{\frac{1}{p}}}{s_n + s_m} \right\} \int_0^1 f^2(x) \, dx.
\]  

(44)
A GENERALIZATION OF HARDY-HILBERT’S INEQUALITY

Since \( f(x) \neq 0, \ s'_n > 0 \). So, \( a_n \neq 0 \). Hence it is impossible to get equality in (44). Again by Corollary 1, we have

\[
\sum_{n=m_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_n^{p-1}(s'_n)^{\frac{1}{p}} a_m^{p-1}(s'_m)^{\frac{1}{p}}}{s_n + s_m} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{n=m_0}^{\infty} (s'_n)^{1-p} a_n^{p(p-1)} s'_n \right)^{\frac{1}{p}} \left( \sum_{n=m_0}^{\infty} (s'_n)^{1-q} a_n^{q(p-1)} (s'_n)^{\frac{2}{q}} \right)^{\frac{1}{q}}
\]

\[
= \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{n=m_0}^{\infty} (s'_n)^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \left( \sum_{n=m_0}^{\infty} a_n^p \right)^{\frac{1}{q}}.
\]

Hence we obtain the inequality (43). This complete the proof of the theorem.

**Theorem 8.** Let \( p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \ f \in L^2(0,1), \ f(x) \neq 0 \) and

\[
a_n = \frac{(s'_n)^{\frac{1}{p}}}{(s_n)^{\frac{1}{p} - \frac{1}{2}}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx, \quad n \geq m_0.
\]

If

\[
0 < \sum_{n=m_0}^{\infty} \left( \frac{s'_n}{s_n} \right)^{2-p} a_n^{p(p-1)} < \infty
\]

then

\[
\left( \sum_{n=m_0}^{\infty} a_n^p \right)^{1+\frac{1}{p}} \pi \left( \sum_{n=m_0}^{\infty} \left( \frac{s'_n}{s_n} \right)^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \int_0^1 f^2(x)dx.
\]

**Proof.** Proceeding as in Theorem 7 and using Corollary 2, the proof of the theorem follows.

**Remark 5.** For \( s_n = n \), (43) becomes (42). Taking \( p = 2 \) in Theorem 7 and Theorem 8, we get

\[
\sum_{n=m_0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)dx
\]

which reduces to (41) for \( s_n = n \).

**References**


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