BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(67), 2011, Pages 29–44
ISSN 1024–7696

A Generalization of Hardy-Hilbert's Inequality for Non-homogeneous Kernel

Namita Das, Srinibas Sahoo

Abstract. This paper deals with a generalization of Hardy-Hilbert's inequality for non-homogeneous kernel by considering sequences (s_n), (t_n), the functions ϕ_p, ϕ_q and parameter λ. This inequality generalizes both Hardy-Hilbert's inequality and Mulholland's inequality, which includes most of the recent results of this type. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

Mathematics subject classification: 26D15.
Keywords and phrases: Hardy-Hilbert's inequality; Mulholland's inequality, β-function, Hölder's inequality.

1 Introduction

If $a_n, b_n \geq 0$ satisfy $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, then the well known Hilbert's inequality (see [1]) is given by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}} \tag{1}$$

and an equivalent form is given by

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2\sum_{n=1}^{\infty} a_n^2, \tag{2}$$

where the constant factors π and π^2 are the best possible. In 1925, Hardy [2] gave some extensions of (1) and (2) by introducing the (p,q)-parameters as: if $p > 1, \frac{1}{p}+\frac{1}{q} = 1$, $a_n, b_n \geq 0$ satisfy $\sum_{n=1}^{\infty} a_n^p < \infty$ and $\sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q} \tag{3}$$

and an equivalent form is given by

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty} a_n^p, \tag{4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $[\frac{\pi}{\sin(\pi/p)}]^p$ are the best possible. Inequality (3) is called Hardy-Hilbert's inequality and is important in analysis and its applications (cf. Mintrinovic et al. [4]). Recently many generalizations and refinements of these inequalities were also obtained. Some of them are given in [5–15].

If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} \frac{1}{m}\, a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} \frac{1}{n}\, b_n^q < \infty$, then the Mulholland's inequality (cf. [1,3]) is given by

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=2}^{\infty} \frac{1}{n}\, a_n^p\right\}^{1/p} \left\{\sum_{n=2}^{\infty} \frac{1}{n}\, b_n^q\right\}^{1/q}; \tag{5}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Replacing a_m with ma_m and b_n with nb_n we have the following inequality:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0$ satisfy $0 < \sum_{n=2}^{\infty} n^{p-1} a_n^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1} b_n^q < \infty$, then the inequality

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=2}^{\infty} n^{p-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^{\infty} n^{q-1} b_n^q\right\}^{\frac{1}{q}} \tag{6}$$

holds, where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The inequality (6) is also referred to as Mulholland's inequality. Some generalizations of these inequalities are given in [16,17].

Most of the recent generalizations of inequalities (1) and (3) (cf. [5–15]) estimate the upper bounds of the double sum of the form $\sum\sum K(m,n)a_m b_n$, where the kernel $K(m,n)$ is homogeneous in m and n. In this paper, we give a generalization of Hardy-Hilbert's inequality for non-homogeneous kernel $K(m,n) = (s_m + t_n)^{-1}$ by considering the sequences (s_n), (t_n), the functions ϕ_p, ϕ_q and parameter λ. This inequality generalizes both Hardy-Hilbert's inequality and Mulholland's inequality, from which all the inequalities given in [5–17] are obtained as particular cases. As applications, the equivalent form, some particular results and a generalized Hardy-Littlewood inequality are established.

2 Some Lemmas

We first set the following notations. Suppose $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and ϕ_r $(r = p, q)$ is a function of r such that $0 < \phi_r < \lambda$ $(r = p, q)$. Let $m_0, n_0 \in \mathbb{N}$ and $s(x), t(x)$ are differentiable strictly increasing functions in (m_0-1,∞) and (n_0-1,∞), respectively, such that $s((m_0 - 1)+) = t((n_0 - 1)+) = 0$ and $s(\infty) = t(\infty) = \infty$, $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in $(m_0 - 1, \infty)$ and $(n_0 - 1, \infty)$, respectively. We write $s(m) = s_m$, $s'(m) = s'_m$, $t(n) = t_n$ and $t'(n) = t'_n$.

We need the formula of the $\beta-$function as (cf. Wang et al. [18]):

$$B(p,q) = \int_0^{\infty} \frac{1}{(1+u)^{p+q}} u^{p-1} du = B(q,p) \ \ (p,q > 0). \tag{7}$$

Lemma 1. *Define the weight functions $\omega_\lambda(s,t,p,m)$ and $\omega_\lambda(t,s,q,n)$ as*

$$\omega_\lambda(s,t,p,m)=\sum_{n=n_0}^{\infty}\frac{1}{(s_m+t_n)^\lambda}\frac{t'_n}{(t_n)^{1-\phi_p}},\ (m\geq m_0); \tag{8}$$

$$\omega_\lambda(t,s,q,n)=\sum_{m=m_0}^{\infty}\frac{1}{(s_m+t_n)^\lambda}\frac{s'_m}{(s_m)^{1-\phi_q}},\ (n\geq n_0). \tag{9}$$

Then

$$\omega_\lambda(s,t,p,m)<B(\phi_p,\lambda-\phi_p)(s_m)^{\phi_p-\lambda},\ (m\geq m_0); \tag{10}$$

$$\omega_\lambda(t,s,q,n)<B(\phi_q,\lambda-\phi_q)(t_n)^{\phi_q-\lambda},\ (n\geq n_0). \tag{11}$$

Proof. Since $\lambda>0,\ s(x),\ t(x)$ are differentiable, strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in (m_0-1,∞) and (n_0-1,∞), respectively. So

$$\begin{aligned}\omega_\lambda(s,t,p,m)&<\sum_{n=n_0}^{\infty}\int_{n-1}^{n}\frac{1}{(s_m+t(y))^\lambda}\frac{t'(y)}{(t(y))^{1-\phi_p}}dy\\&=\int_{n_0-1}^{\infty}\frac{(t(y))^{\phi_p-1}t'(y)}{(s_m+t(y))^\lambda}dy\\&=(s_m)^{\phi_p-\lambda}\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\phi_p-1}du\qquad\left(\text{setting }\ u=\frac{t(y)}{s_m}\right).\end{aligned}$$

Then by (7), we get (10). Similarly, (11) can be proved. The lemma is proved. □

Lemma 2. *If $\phi_p+\phi_q=\lambda$ and $0<\varepsilon<q\phi_p$, then*

$$\begin{aligned}\sum\nolimits_1:=&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{1}{(s_m+t_n)^\lambda}\times\frac{s'_m}{(s_m)^{1-\phi_q+\frac{\varepsilon}{p}}}\times\frac{t'_n}{(t_n)^{1-\phi_p+\frac{\varepsilon}{q}}}\\&>\frac{1}{\varepsilon(s_{m_0})^\varepsilon}B\left(\phi_p-\frac{\varepsilon}{q},\phi_q+\frac{\varepsilon}{q}\right)-\bigcirc(1).\end{aligned} \tag{12}$$

Proof. Since $\lambda>0,\ s(x),\ t(x)$ are differentiable strictly increasing functions and $\frac{s'(x)}{(s(x))^{1-\phi_q}}$ and $\frac{t'(x)}{(t(x))^{1-\phi_p}}$ are decreasing in (m_0-1,∞) and (n_0-1,∞), respectively, we have

$$
\begin{aligned}
\sum\nolimits_1 &> \int_{m_0}^{\infty}\int_{n_0}^{\infty}\frac{1}{(s(x)+t(y))^{\lambda}}\times\frac{s'(x)}{(s(x))^{1-\phi_q+\frac{\varepsilon}{p}}}\times\frac{t'(y)}{(t(y))^{1-\phi_p+\frac{\varepsilon}{q}}}dxdy\\
&= \int_{m_0}^{\infty}\frac{s'(x)}{(s(x))^{1+\varepsilon}}\left[\int_{\frac{t(n_0)}{s(x)}}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\phi_p-\frac{\varepsilon}{q}-1}du\right]dx \qquad \left(\text{setting } u=\frac{t(y)}{s(x)}\right)\\
&= \int_{m_0}^{\infty}\frac{s'(x)}{(s(x))^{1+\varepsilon}}dx\int_0^{\infty}\frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}}du-\int_{m_0}^{\infty}\frac{s'(x)}{(s(x))^{1+\varepsilon}}\left[\int_0^{\frac{t(n_0)}{s(x)}}\frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}}du\right]dx\\
&> \frac{1}{\varepsilon(s(m_0))^{\varepsilon}}\int_0^{\infty}\frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}}du-\int_{m_0}^{\infty}\frac{s'(x)}{(s(x))^{1+\varepsilon}}\left[\int_0^{\frac{t(n_0)}{s(x)}}u^{\phi_p-\frac{\varepsilon}{q}-1}du\right]dx\\
&= \frac{1}{\varepsilon(s_{m_0})^{\varepsilon}}\int_0^{\infty}\frac{u^{\phi_p-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda}}du-\frac{(t_{n_0})^{\phi_p-\frac{\varepsilon}{q}}}{(u_{m_0})^{\phi_p-\frac{\varepsilon}{q}+\varepsilon}}\left(\phi_p-\frac{\varepsilon}{q}\right)^{-1}\left(\phi_p-\frac{\varepsilon}{q}+\varepsilon\right)^{-1}.
\end{aligned}
$$

Then by (7), (12) is valid. The lemma is proved. □

3 Main Result

Theorem 1. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\phi_r<\lambda\ (r=p,q)$ *and* $a_n,b_n\geq 0$ *satisfy* $0<\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p<\infty$ *and* $0<\sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q<\infty$, *then*

$$
\begin{aligned}
\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_mb_n}{(s_m+t_n)^{\lambda}} &< H_{\lambda}(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q\right\}^{\frac{1}{q}}
\end{aligned}
\tag{13}
$$

where $H_{\lambda}(\phi_p,\phi_q)=B^{\frac{1}{p}}(\phi_p,\lambda-\phi_p)B^{\frac{1}{q}}(\phi_q,\lambda-\phi_q)$.

Proof. By Hölder's inequality with weight (cf. Kuang [19]), we have

$$
\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_mb_n}{(s_m+t_n)^{\lambda}}\\
&=\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{1}{(s_m+t_n)^{\lambda}}\left\{\frac{(t_n)^{(\phi_p-1)/p}(t'_n)^{1/p}}{(s_m)^{(\phi_q-1)/q}(s'_m)^{1/q}}a_m\right\}\left\{\frac{(s_m)^{(\phi_q-1)/q}(s'_m)^{1/q}}{(t_n)^{(\phi_p-1)/p}(t'_n)^{1/p}}b_n\right\}\\
&\leq\left\{\sum_{m=m_0}^{\infty}\left[\sum_{n=n_0}^{\infty}\frac{1}{(s_m+t_n)^{\lambda}}\frac{t'_n}{(t_n)^{1-\phi_p}}\right]\frac{(s_m)^{(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}\ a_m^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=n_0}^{\infty}\left[\sum_{m=m_0}^{\infty}\frac{1}{(s_m+t_n)^{\lambda}}\frac{s'_m}{(s_m)^{1-\phi_q}}\right]\frac{(t_n)^{(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}\ b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Then by (8) and (9), we have

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(s_m+t_n)^{\lambda}} \le \left\{\sum_{m=m_0}^{\infty}\omega_{\lambda}(s,t,p,m)\frac{(s_m)^{(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}\,a_m^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=n_0}^{\infty}\omega_{\lambda}(t,s,q,n)\frac{(t_n)^{(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}\,b_n^q\right\}^{\frac{1}{q}}$$

and in view of (10) and (11), it follows that (13) is valid. The theorem is proved. □

Theorem 2. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \phi_r < \lambda$ $(r = p, q)$ *and* $a_n \ge 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p < \infty$, *then we obtain an equivalent inequality of* (13) *as follows:*

$$\sum_{n=n_0}^{\infty}\frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p < [H_{\lambda}(\phi_p,\phi_q)]^p\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p. \tag{14}$$

Proof. Setting $b_n = \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^{p-1}$ and using (13) we obtain

$$\begin{aligned}
0 &< \sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q\\
&= \sum_{n=n_0}^{\infty}\frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty}\frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p\\
&= \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(s_m+t_n)^{\lambda}}\\
&\le H_{\lambda}(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty}\frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}}a_m^p\right\}^{\frac{1}{p}}\\
&\times\left\{\sum_{n=n_0}^{\infty}\frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{15}$$

Hence

$$\begin{aligned}
0 &< \left[\sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q\right]^{\frac{1}{p}} \\
&= \left\{\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}} \left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \qquad (16)\\
&\le H_{\lambda}(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{\phi_p-\lambda+(p-1)(1-\phi_q)}}{(s'_m)^{p-1}} a_m^p\right\}^{\frac{1}{p}} < \infty.
\end{aligned}$$

By using (13) it follows that (15) takes the form of strict inequality; so does (16). Hence we get (14).

On the other hand, if (14) holds, then by Hölder's inequality, we have

$$\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} \\
&= \sum_{n=n_0}^{\infty}\left[\frac{(t'_n)^{1/p}}{(t_n)^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}} \sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]\left[\frac{(t_n)^{(1-\phi_p+(p-1)(\phi_q-\lambda))/p}}{(t'_n)^{1/p}} b_n\right] \\
&\le \left\{\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-\phi_p+(p-1)(\phi_q-\lambda)}}\left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m+t_n)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{\phi_q-\lambda+(q-1)(1-\phi_p)}}{(t'_n)^{q-1}} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (14), (13) yields. Thus it follows that (13) and (14) are equivalent. The theorem is proved. □

Theorem 3. *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\phi_r > 0$ $(r = p, q)$, $\phi_p + \phi_q = \lambda$, $a_n, b_n \ge 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p < \infty$ *and* $0 < \sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} b_n^q < \infty$, *then*

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} \frac{a_m b_n}{(s_m+t_n)^{\lambda}} < B(\phi_p,\phi_q)\left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} b_n^q\right\}^{\frac{1}{q}} \qquad (17)$$

where the constant factor $B(\phi_p, \phi_q)$ *is the best possible.*

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 1, (17) is valid. For $0 < \varepsilon < q\phi_p$, we take

$$\begin{aligned}
\tilde{a}_m &= (s_m)^{-1+\phi_q-\varepsilon/p}\, s'_m \quad (m \ge m_0),\\
\tilde{b}_n &= (t_n)^{-1+\phi_p-\varepsilon/q}\, t'_n \quad (n \ge n_0).
\end{aligned}$$

Since $\frac{s'(x)}{(s(x))^{1+\varepsilon}} = \frac{s'(x)}{(s(x))^{1-\phi_q}} \frac{1}{(s(x))^{\phi_q+\varepsilon}}$ is decreasing in $(m_0 - 1, \infty)$, we have

$$\begin{aligned} \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} \tilde{a}_m^p &= \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \sum_{m=m_0+1}^{\infty} \frac{s'_m}{(s_m)^{1+\varepsilon}} \\ &\leq \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \int_{m_0}^{\infty} \frac{s'(x)}{(s(x))^{1+\varepsilon}} dx \\ &= \frac{1}{\varepsilon}\left[\varepsilon \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \frac{1}{(s_{m_0})^{\varepsilon}}\right]. \end{aligned} \tag{18}$$

Similarly,

$$\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} \tilde{b}_n^q \leq \frac{1}{\varepsilon}\left[\varepsilon \frac{t'_{n_0}}{(t_{n_0})^{1+\varepsilon}} + \frac{1}{(t_{n_0})^{\varepsilon}}\right]. \tag{19}$$

If the constant factor $B(\phi_p, \phi_q)$ in (17) is not the best possible, then there exists a positive constant $K < B(\phi_p, \phi_q)$ such that (17) is still valid if we replace $B(\phi_p, \phi_q)$ by K. In particular by (12), (18) and (19), we have

$$\begin{aligned} &\frac{1}{(s_{m_0})^{\varepsilon}} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \varepsilon \bigcirc(1) \\ &< \varepsilon \sum\nolimits_1 = \varepsilon \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(s_m + t_n)^{\lambda}} \\ &< \varepsilon K \left\{\sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} \tilde{a}_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty} \frac{(t_n)^{q(1-\phi_p)-1}}{(t'_n)^{q-1}} \tilde{b}_n^q\right\}^{\frac{1}{q}} \\ &< K \left\{\varepsilon \frac{s'_{m_0}}{(s_{m_0})^{1+\varepsilon}} + \frac{1}{(s_{m_0})^{\varepsilon}}\right\}^{\frac{1}{p}} \left\{\varepsilon \frac{t'_{n_0}}{(t_{n_0})^{1+\varepsilon}} + \frac{1}{(t_{n_0})^{\varepsilon}}\right\}^{\frac{1}{q}} \end{aligned}$$

and taking $\varepsilon \to 0^+$, we get $B(\phi_p, \phi_q) \leq K$. This contradiction leads to the conclusion that the constant factor $B(\phi_p, \phi_q)$ in (17) is the best possible. The theorem is proved. $\square$

Corollary 1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $a_m \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} (s'_m)^{1-r} a_m^r < \infty$ $(r = p, q)$, *then*

$$\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=m_0}^{\infty} (s'_m)^{1-p} a_m^p\right)^{\frac{1}{p}} \left(\sum_{m=m_0}^{\infty} (s'_m)^{1-q} a_m^q\right)^{\frac{1}{q}} \tag{20}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ *is the best possible.*

Proof. Taking $a_n = b_n, s_n = t_n, \lambda = 1, \phi_r = \frac{1}{r}$ $(r = p,\ q)$ in (17), we get (20). The corollary is proved. $\square$

Corollary 2. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ *and* $a_m \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{r}{2}-1}}{(s'_m)^{r-1}} a_m^r < \infty$ $(r = p, q)$*, then*

$$\sum_{m=m_0}^{\infty} \sum_{n=m_0}^{\infty} \frac{a_m a_n}{s_m + s_n} < \pi \left(\sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{p}{2}-1}}{(s'_m)^{p-1}} a_m^p \right)^{\frac{1}{p}} \left(\sum_{m=m_0}^{\infty} \frac{(s_m)^{\frac{q}{2}-1}}{(s'_m)^{q-1}} a_m^q \right)^{\frac{1}{q}} \tag{21}$$

where the constant factor π *is the best possible.*

Proof. Taking $a_n = b_n,\ s_n = t_n,\ \lambda = 1,\ \phi_r = \frac{1}{2}\ (r = p, q)$ in (17), we get (21). The corollary is proved. □

Theorem 4. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \phi_r > 0\ (r = p, q), \phi_p + \phi_q = \lambda,\ a_n \geq 0$ *satisfy* $0 < \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p < \infty$*, then we obtain an equivalent inequality of* (17) *as follows:*

$$\sum_{n=n_0}^{\infty} \frac{t'_n}{(t_n)^{1-p\phi_p}} \left[\sum_{m=m_0}^{\infty} \frac{a_m}{(s_m + t_n)^{\lambda}} \right]^p < [B(\phi_p, \phi_q)]^p \sum_{m=m_0}^{\infty} \frac{(s_m)^{p(1-\phi_q)-1}}{(s'_m)^{p-1}} a_m^p \tag{22}$$

where the constant factor $[B(\phi_p, \phi_q)]^p$ *is the best possible.*

Proof. Since $\phi_p + \phi_q = \lambda$, then by Theorem 2, we get inequalities (17) and (22) are equivalent. By Theorem 3, the constant factor in (17) is best possible, hence the constant factor in (22) is best possible. The theorem is proved. □

4 Generalization of Hardy-Hilbert's Inequality

Theorem 5. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \phi_r < \lambda (r = p, q),\ A, B > 0,\ 0 < \alpha \leq \frac{1}{\phi_q}$, $0 < \beta \leq \frac{1}{\phi_p}$, $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q)+p-1} a_m^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\beta(\phi_q - \lambda + (1-q)\phi_p)+q-1} b_n^q < \infty$*, then the following two equivalent inequalities hold:*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha} + Bn^{\beta})^{\lambda}} < \mu H_{\lambda}(\phi_p, \phi_q) \left\{ \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q)+p-1} a_m^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=1}^{\infty} n^{\beta(\phi_q - \lambda + (1-q)\phi_p)+q-1} b_n^q \right\}^{\frac{1}{q}}; \tag{23}$$

$$\sum_{n=1}^{\infty} n^{\beta(\phi_p + (1-p)(\phi_q - \lambda))-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(Am^{\alpha} + Bn^{\beta})^{\lambda}} \right]^p < [\mu H_{\lambda}(\phi_p, \phi_q)]^p \sum_{m=1}^{\infty} m^{\alpha(\phi_p - \lambda + (1-p)\phi_q)+p-1} a_m^p \tag{24}$$

where $\mu = \left(\frac{A^{\phi_p - \lambda}}{\beta B^{\phi_p}} \right)^{\frac{1}{p}} \left(\frac{B^{\phi_q - \lambda}}{\alpha A^{\phi_q}} \right)^{\frac{1}{q}}$ *and* $H_{\lambda}(\phi_p, \phi_q) = B^{\frac{1}{p}}(\phi_p, \lambda - \phi_p) B^{\frac{1}{q}}(\phi_q, \lambda - \phi_q)$*. The constant factors* $\mu H_{\lambda}(\phi_p, \phi_q)$ *and* $[\mu H_{\lambda}(\phi_p, \phi_q)]^p$ *are the best possible if* $\phi_p + \phi_q = \lambda$*.*

Proof. Setting $s_m = Am^\alpha$, $t_n = Bn^\beta$ in Theorem 1 and Theorem 2, we get both the inequalities (23) and (24) are valid and equivalent. From Theorem 3 and Theorem 4, it follows that the constant factors are the best possible. This completes the proof. □

We discuss a number of special cases of inequality (23). Similar inequalities can also be derived from inequality (24).

Example 1. Setting $\phi_p = 1 - A_2 p$, $\phi_q = 1 - A_1 q$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $A_1 < \frac{1}{q}$, $A_2 < \frac{1}{p}$, $0 < \alpha \le \frac{1}{1-A_1 q}$, $0 < \beta \le \frac{1}{1-A_2 p}$, $\lambda > \max\{1 - A_2 p, 1 - A_1 q\}$, $a_m, b_n \ge 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^\alpha + Bn^\beta)^\lambda} < L\left\{\sum_{m=1}^{\infty} m^{\alpha(2-p-\lambda+p(A_1-A_2))+p-1} a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty} n^{\beta(2-q-\lambda+q(A_2-A_1))+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{25}$$

where $L = \left(\frac{A^{1-A_2 p-\lambda}}{\beta B^{1-A_2 p}}\right)^{\frac{1}{p}} \left(\frac{B^{1-A_1 q-\lambda}}{\alpha A^{1-A_1 q}}\right)^{\frac{1}{q}} H_\lambda(1 - A_2 p, 1 - A_1 q)$. For $A = B = \alpha = \beta = 1$, we get the result of Brnetic and Pecaric [5, Theorem 2].

Example 2. Setting $\phi_r = \frac{\lambda}{r}\,(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \le \frac{q}{\lambda}$, $0 < \beta \le \frac{p}{\lambda}$, $\lambda > 0$, $a_m, b_n \ge 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\beta\lambda)} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^\alpha + Bn^\beta)^\lambda} < \mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha\lambda)} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\beta\lambda)} b_n^q\right\}^{\frac{1}{q}} \tag{26}$$

where $\mu = \left(A^{\frac{\lambda}{q}} B^{\frac{\lambda}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1}$ and the constant factor $\mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. For $A = B = \lambda = 1$, $\alpha = \beta$, we get the result of Yang [7]. Setting $\alpha = \beta = 1$, $p = q = 2$, we get the result of Yang [13] and setting $\alpha = \beta = 1$, we get the result of Yang [15].

Example 3. Setting $\phi_r = \lambda(1 - \frac{1}{r})\,(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \le \frac{p}{\lambda}$, $0 < \beta \le \frac{q}{\lambda}$, $\lambda > 0$ and $a_m, b_n \ge 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{p-\alpha\lambda-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-\beta\lambda-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^\alpha + Bn^\beta)^\lambda} < \mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=1}^{\infty} m^{p-\alpha\lambda-1} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q-\beta\lambda-1} b_n^q\right\}^{\frac{1}{q}} \tag{27}$$

where $\mu = \left(A^{\frac{\lambda}{p}} B^{\frac{\lambda}{q}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1}$ and the constant factor $\mu B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. For $\lambda = 1, \alpha = \beta$, we get the result of Yang [8]. Setting $A = B = \alpha = \beta = 1$, we recover the result of Yang [9].

Example 4. Setting $\phi_r = 1 + \frac{\lambda-2}{r}$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{q}{q+\lambda-2}$, $0 < \beta \leq \frac{p}{p+\lambda-2}$, $\lambda > 2 - \min\{p, q\}$, a_m, $b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\alpha(q+\lambda-2))} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\beta(p+\lambda-2))} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_1\left\{\sum_{m=1}^{\infty} m^{(p-1)(1-\alpha(q+\lambda-2))} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\beta(p+\lambda-2))} b_n^q\right\}^{\frac{1}{q}} \tag{28}$$

where the constant factor $L_1 = \left(A^{\frac{q+\lambda-2}{q}} B^{\frac{p+\lambda-2}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible. In particular for $\alpha = \beta = 1$, $p = q = 2$, we get the result of Yang [13].

Example 5. Setting $\phi_r = 1 + (1 - \frac{1}{r})(\lambda - 2)$ $(r = p, q)$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq \frac{p}{p+\lambda-2}$, $0 < \beta \leq \frac{q}{q+\lambda-2}$, $\lambda > 2 - \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\alpha(2-\lambda-p)+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\beta(2-\lambda-q)+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_2\left\{\sum_{m=1}^{\infty} m^{\alpha(2-\lambda-p)+p-1} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\beta(2-\lambda-q)+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{29}$$

where the constant factor $L_2 = \left(A^{\frac{p+\lambda-2}{p}} B^{\frac{q+\lambda-2}{q}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible. For $\alpha = \beta = 1$, we get the result of Yang and Debnath [6]. Setting $A = B = \lambda = 1$, $\alpha = \beta$, we recover the result of Yang [7].

Example 6. Setting $\phi_r = \frac{\lambda-1}{2} + \frac{1}{r}$ $(r = p, q)$, in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A, B > 0$, $0 < \alpha \leq (\frac{\lambda-1}{2} + \frac{1}{q})^{-1}$, $0 < \beta \leq (\frac{\lambda-1}{2} + \frac{1}{p})^{-1}$, $\lambda > 1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{p-1+\alpha(2-p\lambda-p)/2} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-1+\beta(2-q\lambda-q)/2} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\lambda}} < L_3\left\{\sum_{m=1}^{\infty} m^{p-1+\alpha(2-p\lambda-p)/2} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q-1+\beta(2-q\lambda-q)/2} b_n^q\right\}^{\frac{1}{q}} \tag{30}$$

where the constant factor $L_3 = \left(A^{\frac{\lambda-1}{2}+\frac{1}{q}} B^{\frac{\lambda-1}{2}+\frac{1}{p}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{\lambda-1}{2} + \frac{1}{p}, \frac{\lambda-1}{2} + \frac{1}{q}\right)$ is the best possible. Setting $A = B = \lambda = 1$, $\alpha = \beta$, we recover the result of Yang [7].

Example 7. Setting $\phi_p = \lambda(\frac{1}{\alpha} + (1 - \frac{1}{p})(\alpha - 2))$, $\phi_q = \lambda(\frac{1}{\beta} + (1 - \frac{1}{q})(\beta - 2))$ in Theorem 5, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, A, B, $\lambda > 0$, $2 - p < \alpha \leq 2 + p(\frac{\lambda-1}{\lambda})$, $2 - q < \beta \leq 2 + q(\frac{\lambda-1}{\lambda})$, a_m, $b_n \geq 0$ satisfy $0 < \sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^\alpha + Bn^\beta)^\lambda} < L_4 \left\{\sum_{m=1}^{\infty} m^{\lambda(2-\alpha-p)+p-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{\lambda(2-\beta-q)+q-1} b_n^q\right\}^{\frac{1}{q}} \tag{31}$$

where the constant factor $L_4 = \left(A^{\frac{\lambda(p+\alpha-2)}{p\alpha}} B^{\frac{\lambda(q+\beta-2)}{q\beta}} \alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}\right)^{-1} \times B\left(\frac{\lambda(p+\alpha-2)}{p\alpha}, \frac{\lambda(q+\beta-2)}{q\beta}\right)$ is the best possible. For $A = B = \lambda = 1, \alpha = \beta$, we get the result of Yang [12].

Remark 1. Setting **(i)** $\phi_r = \frac{1}{r}$ $(r = p, q)$, **(ii)** $\phi_r = 1 - \frac{1}{r}$ $(r = p, q)$, **(iii)** $\phi_r = \frac{\lambda+1}{2} - \frac{1}{r}$ $(r = p, q)$ in Theorem 5, we get new inequalities.

Remark 2. Taking $\alpha = \beta$, $A = B = 1$, $\phi_r = \frac{\varphi_r}{\alpha}$ $(r = p, q)$ in (23), we get the result of Yang [10].

Remark 3. Taking $s_m = t_m = u(m)$ in Theorem 3, we get the result of Yang [14].

For other appropriate values of λ, ϕ_p, ϕ_q and suitably choosing sequences s_m and t_n in Theorem 1 and Theorem 3, one can obtain many new inequalities.

5 Generalization of Mulholland's Inequality

Theorem 6. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi_r \leq 1$ $(r = p, q)$, $\lambda > \max\{\phi_p, \phi_q\}$, $\alpha, \beta > 0$ and $a_n, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{\phi_q-\lambda+(q-1)(1-\phi_p)} b_n^q < \infty$, then the following two equivalent inequalities holds:*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \eta H_\lambda(\phi_p, \phi_q) \left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{\phi_q-\lambda+(q-1)(1-\phi_p)} b_n^q\right\}^{\frac{1}{q}}; \tag{32}$$

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\phi_p-1+(p-1)(\lambda-\phi_q)}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{(\ln m^\alpha n^\beta)^\lambda}\right]^p < [\eta H_\lambda(\phi_p, \phi_q)]^p \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{\phi_p-\lambda+(p-1)(1-\phi_q)} a_m^p \tag{33}$$

where $\eta = \left(\frac{\alpha^{\phi_p-\lambda}}{\beta^{\phi_p}}\right)^{\frac{1}{p}} \left(\frac{\beta^{\phi_q-\lambda}}{\alpha^{\phi_q}}\right)^{\frac{1}{q}}$ and $H_\lambda(\phi_p, \phi_q) = B^{\frac{1}{p}}(\phi_p, \lambda - \phi_p) B^{\frac{1}{q}}(\phi_q, \lambda - \phi_q)$. The constant factors $\eta H_\lambda(\phi_p, \phi_q)$ and $[\eta H_\lambda(\phi_p, \phi_q)]^p$ are the best possible if $\phi_p + \phi_q = \lambda$.

Proof. Setting $s_m = \ln m^\alpha$, $t_n = \ln n^\beta$ in Theorem 1 and Theorem 2, we get both the inequalities (32) and (33) are valid and equivalent. The constant factors are the best possible obtained from Theorem 3 and Theorem 4. This completes the proof. □

Example 8. Setting $\phi_r = \frac{1}{r}$ $(r = p, q)$ in Theorem 6, we obtain the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0, \lambda > \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_n^q < \infty$, then

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} < \eta H_\lambda\left(\frac{1}{p}, \frac{1}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}. \tag{34}$$

In particular for $\alpha = \beta = \lambda = 1$, we get the result of Yang [16, Theorem 2.1].

Example 9. Setting $\phi_r = \frac{\lambda}{r}$ $(r = p, q)$ in Theorem 6, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0, 0 < \lambda \leq \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{(p-1)(1-\lambda)} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} &< \frac{1}{\alpha^{\frac{\lambda}{q}}\beta^{\frac{\lambda}{p}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{(p-1)(1-\lambda)} a_m^p\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{(q-1)(1-\lambda)} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{35}$$

where the constant factor $\frac{1}{\alpha^{\lambda/q}\beta^{\lambda/p}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible. In particular for $\alpha = \beta = \lambda = 1$, we get the result of Yang [16, Theorem 2.1].

Example 10. Setting $\phi_r = \lambda(1 - \frac{1}{r})$ $(r = p, q)$ in Theorem 6, we have the following inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $\beta > 0$, $0 < \lambda \leq \min\{p, q\}$, $a_m, b_n \geq 0$ satisfy $0 < \sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-\lambda-1} a_m^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-\lambda-1} b_n^q < \infty$, then

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln m^\alpha n^\beta)^\lambda} &< \frac{1}{\alpha^{\frac{\lambda}{p}}\beta^{\frac{\lambda}{q}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-\lambda-1} a_m^p\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-\lambda-1} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{36}$$

where the constant factor $\frac{1}{\alpha^{\lambda/p}\beta^{\lambda/q}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is the best possible.

In particular for $\alpha = \beta = \lambda = 1$, it reduces to

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{(\ln mn)^\lambda} < \frac{\pi}{\sin\frac{\pi}{p}}\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p-2} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q-2} b_n^q\right\}^{\frac{1}{q}} \tag{37}$$

and we obtain a new inequality in (p,q)-parameter form other than (6), with the same best constant factor.

Example 11. Setting $\phi_r = 1+(1-\frac{1}{r})(\lambda-2)\,(r=p,q)$ in Theorem 6, we have the following inequality: If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\beta>0$, $2-\min\{p,q\}<\lambda\le 2$, $a_m, b_n \ge 0$ satisfy $0<\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p<\infty$ and $0<\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q<\infty$, then

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(\ln m^{\alpha}n^{\beta})^{\lambda}} < \eta k_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{38}$$

where $\eta = \alpha^{\frac{2-\lambda-p}{p}}\beta^{\frac{2-\lambda-q}{q}}$, $k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ and the constant factor $\eta k_{\lambda}(p)$ is the best possible.

In particular for $\alpha=\beta=1$, we get

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{(\ln mn)^{\lambda}} < k_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{1-\lambda}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}} \tag{39}$$

where the constant factor $k_{\lambda}(p)$ is the best possible. For $\lambda=1$, it reduces to the result of Yang [16, Theorem 2.1]. Replacing a_m, b_n by $\frac{a_m}{m^r}, \frac{b_n}{n^s}$ respectively, we get the result of Yang and Debnath [17, Theorem 1]).

Example 12. Setting $\phi_r = \frac{\lambda-1}{2}+\frac{1}{r}\ (r=p,q)$ in Theorem 6, we have the following inequality: If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\beta>0$, $1-2\min\{\frac{1}{p},\frac{1}{q}\}<\lambda<1+2\min\{\frac{1}{p},\frac{1}{q}\}$, $a_m, b_n\ge 0$ satisfy $0<\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2}a_m^p<\infty$ and $0<\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q(1-\lambda)/2}b_n^q<\infty$, then

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}&\frac{a_m b_n}{(\ln m^{\alpha}n^{\beta})^{\lambda}}\\ &< \eta\tilde{k}_{\lambda}(p)\left\{\sum_{m=2}^{\infty} m^{p-1}(\ln m)^{p(1-\lambda)/2}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty} n^{q-1}(\ln n)^{q(1-\lambda)/2}b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{40}$$

where $\eta = \alpha^{\frac{1-\lambda}{2}-\frac{1}{q}}\beta^{\frac{1-\lambda}{2}-\frac{1}{p}}$, $\tilde{k}_{\lambda}(p) = B\left(\frac{\lambda-1}{2}+\frac{1}{p}, \frac{\lambda-1}{2}+\frac{1}{q}\right)$ and the constant factor $\eta\tilde{k}_{\lambda}(p)$ is the best possible. In particular for $\alpha=\beta=\lambda=1$, it reduces to (6).

Remark 4. Setting **(i)** $\phi_r = 1-\frac{1}{r}\ (r=p,q)$, **(ii)** $\phi_r = 1+\frac{\lambda-2}{r}\ (r=p,q)$, **(iii)** $\phi_r=\frac{\lambda+1}{2}-\frac{1}{r}\ (r=p,q)$ in Theorem 6, we get new inequalities.

6 Applications

In this section, we will give the generalizations of Hardy-Littlewood's inequality. Let $f \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)dx, \qquad n = 0,1,2,3,\dots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)dx \tag{41}$$

where the constant factor π is the best possible. Yang [11] gave a generalization of (41) for $p \geq 2$ as

$$\left(\sum_{n=0}^{\infty} a_n^p\right)^{1+\frac{1}{p}} < \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=0}^{\infty} a_n^{p(p-1)}\right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{42}$$

Theorem 7. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^2(0,1)$, $f(x) \neq 0$ and*

$$a_n = (s'_n)^{\frac{1}{p}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx, \qquad n \geq m_0.$$

If $0 < \sum_{n=m_0}^{\infty} (s'_n)^{2-p} a_n^{p(p-1)} < \infty$, *then*

$$\left(\sum_{n=m_0}^{\infty} a_n^p\right)^{1+\frac{1}{p}} < \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s'_n)^{2-p} a_n^{p(p-1)}\right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{43}$$

Proof. Applying Schwartz inequality, we have

$$\begin{aligned}
\left(\sum_{n=m_0}^{\infty} a_n^p\right)^2 &= \left(\sum_{n=m_0}^{\infty} a_n^{p-1} (s'_n)^{\frac{1}{p}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx\right)^2 \\
&= \left\{\int_0^1 \left(\sum_{n=m_0}^{\infty} a_n^{p-1} (s'_n)^{\frac{1}{p}} x^{s_n - \frac{1}{2}}\right) f(x)dx\right\}^2 \\
&\leq \int_0^1 \left(\sum_{n=m_0}^{\infty} a_n^{p-1} (s'_n)^{\frac{1}{p}} x^{s_n - \frac{1}{2}}\right)^2 dx \int_0^1 f^2(x)dx \\
&= \left\{\sum_{n=m_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_n^{p-1} (s'_n)^{\frac{1}{p}} a_m^{p-1} (s'_m)^{\frac{1}{p}}}{s_n + s_m}\right\} \int_0^1 f^2(x)dx.
\end{aligned} \tag{44}$$

Since $f(x) \neq 0,\ s_n' > 0$. So, $a_n \neq 0$. Hence it is impossible to get equality in (44). Again by Corollary 1, we have

$$\sum_{n=m_0}^{\infty} \sum_{m=m_0}^{\infty} \frac{a_n^{p-1}(s_n')^{\frac{1}{p}} a_m^{p-1}(s_m')^{\frac{1}{p}}}{s_n + s_m}$$
$$\leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s_n')^{1-p} a_n^{p(p-1)} s_n' \right)^{\frac{1}{p}} \left(\sum_{n=m_0}^{\infty} (s_n')^{1-q} a_n^{q(p-1)} (s_n')^{\frac{q}{p}} \right)^{\frac{1}{q}}$$
$$= \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{n=m_0}^{\infty} (s_n')^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \left(\sum_{n=m_0}^{\infty} a_n^p \right)^{\frac{1}{q}}.$$

Hence we obtain the inequality (43). This complete the proof of the theorem. □

Theorem 8. *Let* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ f \in L^2(0,1),\ f(x) \neq 0$ *and*

$$a_n = \frac{(s_n')^{\frac{1}{p}}}{(s_n)^{\frac{1}{p}-\frac{1}{2}}} \int_0^1 x^{s_n - \frac{1}{2}} f(x)dx, \qquad n \geq m_0.$$

If

$$0 < \sum_{n=m_0}^{\infty} \left(\frac{s_n'}{s_n}\right)^{2-p} a_n^{p(p-1)} < \infty$$

then

$$\left(\sum_{n=m_0}^{\infty} a_n^p\right)^{1+\frac{1}{p}} < \pi \left(\sum_{n=m_0}^{\infty} \left(\frac{s_n'}{s_n}\right)^{2-p} a_n^{p(p-1)} \right)^{\frac{1}{p}} \int_0^1 f^2(x)dx. \tag{45}$$

Proof. Proceeding as in Theorem 7 and using Corollary 2, the proof of the theorem follows. □

Remark 5. For $s_n = n$, (43) becomes (42). Taking $p = 2$ in Theorem 7 and Theorem 8, we get

$$\sum_{n=m_0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)dx \tag{46}$$

which reduces to (41) for $s_n = n$.

References

[1] Hardy G. H., Littlewood J. E., Polya G. *Inequalities*, Cambridge University Press, Cambrige, 1952.

[2] Hardy G. H. *Note on a theorem of Hilbert concerning series of positive terms.* Proc. London Math. Soc., 1925, **23(2)**, 45–46.

[3] Mulholland H. P. *A Further Generalization of Hilbert Double Series Theorem.* J. London Math. Soc., 1931, **6**, 100–106.

[4] MINTRINOVIC D. S., PECARIC J. E., FINK A. M. *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.

[5] BRNETIC I., PECARIC J. E. *Generalization of inequalities of Hardy-Hilbert type,* Math. Ineq. and Appl., 2004, **7**, No. 2, 217–225.

[6] YANG B., DEBNATH L. *On the extended Hardy-Hilbert's inequality,* J. Math. Anal. Appl., 2002, **272**, 187–199.

[7] YANG B. *On an extension of Hardy-Hilbert's inequality.* Chinese Annals of Math., 2002, **23A**(2), 247–254.

[8] YANG B. *On generalizations Hilbert's inequality.* RGMIA Research Report Collection, 2003, **6(2)**, Article 16.

[9] YANG B. *On new extensions of Hilbert's inequality.* Acta Math. Hungar. 2004, **104(3)**, 293–301.

[10] YANG B. *On the best extension of Hardy-Hilbert's inequality with two parameters.* J. Ineq. Pure and Appl. Math., 2005, **6(3)**, Art. 81.

[11] YANG B. *On a refinement of Hardy-Hilbert's inequality and its applications.* North. Math. J., 2000, **16(3)**, 279–286.

[12] YANG B. *On an Extended Hardy-Hilbert's Inequality and Some Reversed Form.* Int. Math. Forum, 2006, **1**, No. 36, 1905–1912.

[13] YANG B. *On New Generalization of Hilbert's inequality,* J. Math. Anal. Appl., 2000, **248**, No. 1, 29–40.

[14] YANG B. *On a relation between Hardy-Hilbert's inequality and Mulholland's inequality,* Acta Math. Sinica, 2006, **49**, No. 3, 559–566.

[15] YANG B. *On an extension of Hardy-Hilbert's inequality.* KYUNGPOOK Math. J., 2006, **46**, 425–431.

[16] YANG B. *On a new inequality similar to Hardy-Hilbert's inequality.* Math. Ineq. Appl., 2003, **6**, No. 1, 37–44.

[17] YANG B., DEBNATH L. *A Generalization of Mulholland's inequality,* Int. J. Math. Math. Sci., 2003, **56**, 3591–3597.

[18] WANG Z., GUO D. *An Introduction to Special Functions*, Science Press, Beijing, 1979.

[19] KUANG J. *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, 2004.

NAMITA DAS
P.G. Department of Mathematics
Utkal University, Vani Vihar
Bhubaneswar, Orissa, India 751004
E-mail: *namitadas440@yahoo.co.in*

Received October 15, 2010

SRINIBAS SAHOO
Department of Mathematics
Banki Autonomous College, Banki
Orissa, India 754008
E-mail: *sahoosrinibas@yahoo.co.in*