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Saddle points with respect to a set

Dorel I. Duca, Liana Lupşa

Abstract. An extension of the concept of saddle point, a continuous property of two functions related to saddle points with respect to a set and a theorem of existence of saddle points with respect to a set are given. The paper ends with an example which shows that the proved theorems are consistent.

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1 Introduction

Let A and B be nonempty sets and $f : A \times B \to \mathbf{R}$ be a function. We remember that a point $(a, b) \in A \times B$ is said to be a saddle point of f on $A \times B$ if

$$f(x, b) \le f(a, b) \le f(a, y), \quad \text{for all}\,(x, y) \in A \times B. \tag{1}$$

The condition (1) is equivalent to

$$\max_{x \in A} \min_{y \in B} f(x, y) = \min_{y \in B} \max_{x \in A} f(x, y). \tag{2}$$

Let us consider a two-person zero-sum game G_f generated by the function f. This means that the first player selects a point x from A and the second player selects a point y from B. As a result of this choise, the second player pays the first one the amount $f(x, y)$. Then a point $(a, b) \in A \times B$ is a solution of the game G_f if and only if it is a saddle point of f on $A \times B$.

The first saddle point theorem was proved by von Neumann [11]. Von Neumann's theorem can be stated as follows: if A and B are finite dimensional simplices and f is a bilinear function on $A \times B$, then f has a saddle point; i.e (2) holds. M. Shiffman [14] seems to have been the first to have considered concave-convex functions in a saddle point theorem. H. Kneser [10], K. Fan [6], and C. Berge [1] (using induction and the method of separating two disjoint convex sets in Euclidian space by a hyperplane) proved saddle point theorems for concave-convex functions that are appropriately semicontinuous in one of the two variables. H. Nikaido [12], on the other hand, using Brouwer's fixed point theorem, proved the existence of a saddle point for a function satisfying the weaker algebraic condition of being quasi-concave-convex, but the stronger topological condition of being continuous in each variable. M. Sion [16] proved a very general saddle point theorem for a function which is quasi-concave and upper semicontinuous in its first variabile and quasi-convex and lower semicontinuous in its second variable in a topological vector space.

Most of the effort has been spent on relaxing the assumptions on the concave-convexity of f and also on the compactness condition for one of the sets A and B. As examples we can give the papers of H. Tuy [17–19], J. Hartung [8], U. Passy and E. Z. Prisman [13], G. H. Greco and C. D. Horvath [7], S. Simons [15], J. Yu and X. Z. Yuan [20], etc.

A little less study was dedicated to the case when the function f is defined on a proper subset M of $A \times B$ (for example P. S. Kenderov and R. E. Lucchetti [9].

This problem arises, for instance, in connection with the following two-player game. The first player wants to choose a strategy $a \in A$ such that his payoff $f(x, y)$ is maximum. This choice depends on the choice $y \in B$ of the second player. Once the leading player chooses some strategies $x \in A$, the "move" of the second player is to choose some y in the set of all the feasible strategies $y \in M_2^x \subseteq B$. Then the value

$$\underline{v} = \max_{x \in A} \min_{y \in M_2^x} f(x, y)$$

is the maximum payoff that can be guaranteed for the first player.

Analogously, for the second player, the value

$$\overline{v} = \min_{y \in B} \max_{x \in M_1^y} f(x, y)$$

is the minimum loss that can be guaranteed for the second player.

Do $\underline{v}$ and $\overline{v}$ exist ? If so, is $\underline{v} = \overline{v}$ equivalent to (1)? Therefore, in this paper we study this problem by means of the notion of *saddle point with respect to a set.*

In Section 2 we give the definition of saddle point with respect to a set and we show that this notion is effectively a generalization of the notion of saddle point. In Section 3 some properties of the function $\underline{f}$ and $\overline{f}$ defined by (4) and (5) are given. The existence of saddle point of bi-$(1_{\mathbb{R}^m}, 1_{\mathbb{R}^n})$ strictly concave-convex functions is studied in Section 4. The paper ends with an example which shows that the proved theorems are consistent.

2 Saddle points with respect to a set

Let A and B be nonempty sets and M be a nonempty subset of $A \times B$. We put

$$M_1 = pr_A M = \{x \in A \mid \exists y \in B \text{ such that } (x, y) \in M\},$$

$$M_2 = pr_B M = \{y \in B \mid \exists x \in A \text{ such that } (x, y) \in M\}.$$

For each $x \in M_1$ we denote by

$$M_2^x = \{y \in M_2 \,|\, (x, y) \in M\} \subseteq B$$

and for each $y \in M_2$ we denote by

$$M_1^y = \{x \in M_1 \,|\, (x, y) \in M\} \subseteq A.$$

Throughout the paper, M_1, M_2, M_2^x M_1^y, where $x \in M_1$ and $y \in M_2$, will always have this meaning.

Definition. *Let A and B be nonempty sets, M be a nonempty subset of $A \times B$ and $f : M \to \mathbf{R}$ be a function. A point $(a, b) \in M$ is called a saddle point of f with respect to M if*

$$f(x, b) \leq f(a, b) \leq f(a, y), \tag{3}$$

for all $x \in M_1^b$ and all $y \in M_2^a$.

Example 1. Let

$$M = \{(x, y) \in [0, 1] \times [0, 1] : \ x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R},$$

and $f : M \to \mathbb{R}$ be defined by $f(x, y) = -x^2 + y^4$, for all $(x, y) \in M$. For $(a, b) = (0, 0) \in M$, we have $f(x, 0) = -x^2 \leq 0 = f(0, 0) \leq y^4 = f(0, y)$, for all $x \in M_1^0 = \{0\}$, and $y \in M_2^0 = [0, 1]$. It follows that $(a, b) = (0, 0)$ is a saddle point of f with respect to M.$\diamond$

Example 2. Let $A = [1, 3]$, $B = [1, 2]$, $M = \{(x, 2) | x \in [1, 3]\} \bigcup \{(2, y) | y \in [1, 2]\}$ and $f : M \to \mathbf{R}$, $f(x, y) = x \cdot y$, for all $(x, y) \in M$. Because for the point $(a, b) = (2, 1) \in M$ we have $f(x, 1) = x \leq 2 = f(2, 1) \leq 2y = f(2, y)$, for all $x \in M_1^b = \{2\}$, and $y \in M_2^a = [1, 2]$, this point is a saddle point of f with respect to M.$\diamond$

If $M = A \times B$, then $M_1 = A$, $M_2 = B$ and, for each $x \in M_1$ and each $y \in M_2$, we have $M_x = A$ and $M_y = B$. It follows that if $f : A \times B \to \mathbf{R}$ is a function, then condition (3) is equivalent to condition (1). Hence the notion of saddle point with respect to a set is a generalization of the notion of saddle point.

Remark 1. If $f : A \times B \to \mathbf{R}$ is a function and $(a, b) \in A \times B$ is a saddle point of f, then (a, b) is also a saddle point of f with respect to M, for each subset M of $A \times B$ which has the property that $(a, b) \in M$.

Usually, the converse is not true, as it can be seen below:

Example 3. Let $A = [1, 3]$, $B = [1, 2]$, $M = \{(x, 2) \,|\, x \in [1, 3]\} \bigcup \{(2, y) \,|\, y \in [1, 2]\}$ and $f : A \times B \to \mathbf{R}$, $f(x, y) = x \cdot y$, for all $(x, y) \in A \times B$. Then, the point $(a, b) = (2, 1) \in M$ is a saddle point of f with respect to M (see Example 1), but $(a, b) = (2, 1) \in A \times B$ is not a saddle point of f (in the classical sense) because $f(3, 2) = 6 \not\leq 2 = f(2, 1)$.$\diamond$

3 Some properties of the functions $\underline{f}$ and $\overline{f}$

If $f : M \to \mathbf{R}$ is a continuous function and M is a compact nonempty subset of $\mathbf{R}^m \times \mathbf{R}^n$, we consider the functions $\underline{f} : M_1 \to \mathbf{R}$ and $\overline{f} : M_2 \to \mathbf{R}$ defined by

$$\underline{f}(x) \;=\; \min\{f(x, y) | y \in M_2^x\}, \text{ for all } x \in M_1, \tag{4}$$

$$\overline{f}(y) \;=\; \max\{f(x, y) | x \in M_1^y\}, \text{ for all } y \in M_2. \tag{5}$$

The following assertion holds.

Theorem 1. If $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is a nonempty set, $f : M \to \mathbf{R}$ is a function and $(a, b) \in M$ is a saddle point of f with respect to M, then

$$\underline{f}(a) \,=\, \overline{f}(b) \,=\, f(a, b). \tag{6}$$

Proof. From $f(x, b) \leq f(a, b)$, for all $x \in M_1^b$, we conclude that $f(a, b) = \max\{f(x, b) \,|\, x \in M_1^b\} = \overline{f}(b)$, and from $f(a, b) \leq f(a, y)$, for all $y \in M_2^a$, we deduce that $f(a, b) = \min\{f(a, y) \,|\, y \in M_2^a\} = \underline{f}(a)$. Hence (6) is true. □

In the case when $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ are nonempty compact sets,

$$M \,=\, A \times B \,\subseteq\, \mathbf{R}^m \times \mathbf{R}^n$$

and $f : M \to \mathbf{R}$ is a continuous function, then the functions $\underline{f}$ and $\overline{f}$ are also continuous on $M_1 = A$, respectively, on $M_2 = B$. If M is not a cartesian product, this property is not true, as seen in the following example.

Example 4. Let $M = (\{0\} \times [0,\, 1]) \bigcup ([0,\, 3] \times \{1\})$ and $f : M \to \mathbf{R}$ be the function given by $f(x, y) = \ln(11 - x + y^2)$, for all $(x, y) \in M$. We have $M_1 = [0, 3]$, $M_2 = [0, 1]$,

$$M_2^x = \begin{cases} [0,\, 1], & \text{if } x = 0 \\ \{1\}, & \text{if } x \in]0, 3] \end{cases}, \quad M_1^y = \begin{cases} \{0\}, & \text{if } y \in [0,\, 1[\\ [1,\, 3], & \text{if } y = 1 \end{cases}.$$

The function $\underline{f} : M_1 \to \mathbf{R}$, given by

$$\underline{f}(x) = \begin{cases} \ln 11, & \text{if } x = 0 \\ \ln(12 - x), & \text{if } x \in]0, 3] \end{cases}$$

is not continuous. Moreover, $\max\{\underline{f}(x) | x \in M_1\}$ does not exist. But the function f has a saddle point with respect to M; this point is $(0, 0)$.⋄

Remark 2. Let $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ be nonempty sets, M a nonempty subset of $A \times B$, $f : M \to \mathbf{R}$ be a function and $(a, b) \in M$. The following statements are true:

i) If $a \in M_1$ is a maximum point of $\underline{f}$ and $M_1^b = \{a\}$, then (a, b) is a saddle point of f with respect to M.

ii) If $b \in M_2$ is a minimum point of $\overline{f}$ and $M_2^a = \{b\}$, then (a, b) is a saddle point of f with respect to M.

Theorem 2. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a nonempty compact set, $a \in M_1$, $b \in M_2$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) $M_2^a = M_2$,

ii) b is a minimum point of the function $f(a, \cdot) : M_2 \to \mathbf{R}$,

iii) there exists a real number $\delta > 0$ such that

$$M_1 \bigcap B(a, \delta) \subseteq M_1^b,$$

then the function $\underline{f} : M_1 \to \mathbf{R}$ *defined by*

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\}, \text{ for all } x \in M_1,$$

is continuous at a.

Proof. We begin the proof by noticing that, for each $x \in M_1$, $y \in M_2$, the sets M_2^x and M_1^y are compact. In order to show that $\underline{f}$ is continuous at a, let $\varepsilon > 0$. Then, by the continuity of f on the compact M, there exists a positive real number δ_ε such that

$$|f(x,y) - f(u,v)| < \frac{\varepsilon}{2},$$

for each (x,y), $(u,v) \in M$ with

$$||(x,y) - (u,v)|| < \delta_\varepsilon.$$

Let $\tilde{\delta} = \min\{\delta_\varepsilon, \delta\}$ and $x \in M_1 \bigcap B(a, \tilde{\delta})$. Then, for each $y \in M_2^x$, we have $y \in M_2^a$, because $M_2^x \subseteq M_2 = M_2^a$. Also, for each $y \in M_2^x$, we have

$$|f(x,y) - f(a,y)| < \frac{\varepsilon}{2}, \tag{7}$$

because

$$||(x,y) - (a,y)|| < \tilde{\delta}.$$

It follows that

$$f(x,y) > f(a,y) - \frac{\varepsilon}{2} \geq \min\{f(a,v) | v \in M_2^a\} - \frac{\varepsilon}{2} = \underline{f}(a) - \frac{\varepsilon}{2},$$

for each $y \in M_2^x$. Since M_2^x is compact and f is continuous, we have that

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\} \geq \underline{f}(a) - \frac{\varepsilon}{2} > \underline{f}(a) - \varepsilon,$$

i.e.

$$\underline{f}(x) - \underline{f}(a) > -\varepsilon, \text{ for each } x \in M_1 \bigcap B(a, \tilde{\delta}). \tag{8}$$

On the other hand, from

$$M_1 \bigcap B(a, \tilde{\delta}) \subseteq M_1^b,$$

we deduce that $x \in M_1 \bigcap B(a, \tilde{\delta})$ implies $x \in M_1^b$, i.e. $b \in M_2^x$. Then, by (7), it follows that

$$|f(x,b) - f(a,b)| < \frac{\varepsilon}{2},$$

for each $x \in M_1 \bigcap B(a, \tilde{\delta})$. Hence

$$f(x,b) < f(a,b) + \frac{\varepsilon}{2},$$

for each $x \in M_1 \bigcap B(a, \tilde{\delta})$.

It follows that, for each $x \in M_1 \bigcap B(a, \tilde{\delta})$, we have

$$\underline{f}(x) = \min\{f(x,y) | y \in M_2^x\} \leq f(x,b) < f(a,b) + \frac{\varepsilon}{2} =$$

$$= \min\{f(a,y) | y \in M_2^a\} + \frac{\varepsilon}{2} = \underline{f}(a) + \frac{\varepsilon}{2},$$

by hypothesis ii). Consequently,

$$\underline{f}(x) - \underline{f}(a) < \varepsilon, \text{ for all } x \in M_1 \bigcap B(a, \tilde{\delta}). \tag{9}$$

By (8) and (9), the theorem follows. □

By analogy we have:

Theorem 3. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a nonempty compact set, $b \in M_2$, $a \in M_1$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) $M_1^b = M_1$,

ii) $a \in M_1$ is a maximum point of the function $f(\cdot, b) : M_1 \to \mathbf{R}$,

iii) there exists a real number $\delta > 0$ such that

$$M_2 \bigcap B(b, \delta) \subseteq M_2^a,$$

then the function $\overline{f} : M_2 \to \mathbf{R}$ defined by

$$\overline{f}(y) = \max\{f(x,y) | x \in M_1^y\}, \text{ for all } y \in M_2,$$

is continuous at b.

Remark 3. Let $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ be nonempty sets, M be a nonempty subset of $A \times B$, $f : M \to \mathbf{R}$ be a function and $(a, b) \in M$. If $M = A \times B$, then the conditions i) and iii) are self satisfied; therefore, in this case we obtain the classical theorem with respect to the continuity of the functions $\underline{f}$ and $\overline{f}$.

4 Saddle points of bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ strictly concave-convex functions

First we recall the notions of bi-(φ, ψ) convex set (see [3]) and bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ concave-convex function (see [4] and [5]).

Definition. *Let $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ and $\psi : \mathbf{R}^n \to \mathbf{R}^n$ be two maps. A subset M of $\mathbf{R}^m \times \mathbf{R}^n$ is said to be bi-(φ, ψ) convex either if $M = \emptyset$ or, if for every (x, y), (x, v), (u, y) of M and every $t \in [0, 1]$ we have*

$$(\varphi(x), (1-t)\psi(y) + t\psi(v)) \in M \tag{10}$$

and

$$((1-t)\varphi(x) + t\varphi(u), \psi(y)) \in M. \tag{11}$$

We remark that if $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex (in the classical sense), then M is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex. The converse is not necessarily true (see [4]).

Example 5. The set $M = \{(x, y) \in [0,1] \times [0,1] \, : \, x \le y^2\} \subseteq \mathbb{R} \times \mathbb{R}$, is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex, but not convex. Indeed, let (x, y), (x, v), $(u, v) \in M$, and $t \in [0, 1]$. Then $0 \le x \le y^2 \le 1$, $0 \le x \le v^2 \le 1$, $0 \le u \le v^2 \le 1$. It follows that $(x, (1-t)y + tv) \in [0,1] \times [0,1]$, $(1-t)x + tu, y) \in [0,1] \times [0,1]$, and

$$((1-t)y + tv)^2 = (1-t)^2 y^2 + 2t(1-t)yv + t^2 v^2 \ge (1-t)^2 x + 2t(1-t)x + t^2 x = x,$$

$(1-t)x + tu \le (1-t)v^2 + tv^2 = v^2$. Hence $(1_{\mathbb{R}}(x), (1-t)1_{\mathbb{R}}(y) + t1_{\mathbb{R}}(v)) = (x, (1-t)y + tv) \in M$ and $((1-t)\, 1_{\mathbb{R}}\,(x) + t1_{\mathbb{R}}\,(u)\,, 1_{\mathbb{R}}\,(y)) = ((1-t)\,x + tu, y) \in M$. On the other hand, $(1 - 1/2)\,(0,0) + (1/2)\,(1,1) = (1/2, 1/2) \notin M$. Consequently, the set M is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex but not convex.$\diamond$

Definition. *Let $M \subseteq \mathbf{R}^m \times \mathbf{R}^n$ be a bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex set. A function $f : M \to \mathbf{R}$ is said to be bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ concave-convex (strictly concave-convex) if for every $x \in M_1$ the function $f(x, \cdot) : M_x \to \mathbf{R}$ is convex (strictly convex) and for every $y \in M_2$ the function $f(\cdot, y) : M_y \to \mathbf{R}$ is concave (strictly concave).*

Example 6. Let $M = \{(x, y) \in [0,1] \times [0,1] : \, x \le y^2\} \subseteq \mathbb{R} \times \mathbb{R}$, and $f : M \to \mathbb{R}$ be defined by $f\,(x, y) = -x^2 + y^4$, for all $(x, y) \in M$. The set M is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ convex (see Example 5). One can easily show that f is bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ strictly concave-convex (hence bi-$(1_{\mathbb{R}}, 1_{\mathbb{R}})$ concave-convex).$\diamond$

More properties of them can be found in Refs. [3–5].

Theorem 4. *Let M be a compact nonempty subset of $\mathbf{R}^m \times \mathbf{R}^n$, $(a, b) \in M$ and $f : M \to \mathbf{R}$ be a continuous function. If*

i) the set M is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ convex,

ii) the function f is bi-$(1_{\mathbf{R}^m}, 1_{\mathbf{R}^n})$ strictly concave-convex,

iii) a is a maximum point of the function $\underline{f} : M_1 \to \mathbf{R}$ defined by

$$\underline{f}(x) \;=\; \min\{f(x, y) | y \in M_2^x\}, \; \textit{for all } x \in M_1,$$

iv) b is a minimum point of the function $f(a, \cdot) : M_a \to \mathbf{R}$,

v) $M_2^a = M_2$,

vi) there is a real number $\delta > 0$ such that

$$B(b, \delta) \bigcap M_2 \subseteq M_2^x, \; \textit{for all } \; x \in M_1^b,$$

then (a, b) is a saddle-point of f with respect to M.

Proof. From hypothesis *iv*), we have

$$f(a, b) \le f(a, y), \quad \text{for all } y \in M_2^a.$$

Let us show that

$$f(a, b) \ge f(x, b), \quad \text{for all } x \in M_1^b. \tag{12}$$

Assume, by contradiction, that there exists a point $\tilde{x} \in M_1^b$ such that

$$f(a,b) \;<\; f(\tilde{x},b).$$

Then, by the continuity of the function $f(\tilde{x},\cdot) : M_2^{\tilde{x}} \to \mathbf{R}$, there exists an open neighborhood V of the point b such that

$$f(a,b) < f(\tilde{x},y),\ \text{ for all } y \in V \bigcap M_2^{\tilde{x}}.$$

Without loss of generality, we can suppose that

$$V \subseteq B(b;\delta). \tag{13}$$

Since $\tilde{x} \in M_1^b$, $vi)$ and (13) imply

$$V \bigcap M_2 \subseteq B(b;\delta) \bigcap M_2 \subseteq M_2^{\tilde{x}}. \tag{14}$$

From $ii)$ and $iv)$, we deduce that

$$f(a,b) < f(a,y), \quad \text{for all } y \in M_2^a \setminus \{b\},$$

and hence

$$f(a,b) < f(a,y), \quad \text{for all } y \in M_2^a \setminus V. \tag{15}$$

On the other hand, from $iii)$ and $iv)$ we have

$$f(a,b) = \min\{f(a,y) | y \in M_2^a\} = \underline{f}(a) \geq \underline{f}(\tilde{x}) = \min\{f(\tilde{x},y) | y \in M_2^{\tilde{x}}\},$$

and hence there is a point $\tilde{y} \in M_2^{\tilde{x}}$ such that

$$f(a,b) \geq f(\tilde{x},\tilde{y}).$$

Consequently, $\tilde{y} \notin V$ and hence $\tilde{y} \in M_2^{\tilde{x}} \setminus V$.

Then, from $v)$, we obtain that

$$\tilde{y} \in M_2^{\tilde{x}} \setminus V \subseteq M_2^a \setminus V.$$

Since $M_2^a \setminus V$ is nonempty and compact and the function $f(a,\cdot) : M_2^a \to \mathbf{R}$ is continuous, then there exists $\min\{f(a,y) | y \in M_2^a \setminus V\}$ and, by (15),

$$\min\{f(a,y) | y \in M_2^a \setminus V\} \;>\; f(a,b).$$

Let

$$\varepsilon = \min\{f(a,y) | y \in M_2^a \setminus V\} \;-\; f(a,b).$$

Then, by the continuity of f and the compactness of M, there exists a real number $\mu > 0$ such that

$$|f(x,y) \;-\; f(u,v)| \;<\; \varepsilon, \tag{16}$$

for all $(x,y),(u,v)\in M$ with

$$||(x,y)-(u,v)||<\mu.$$

Let

$$t=\min\left\{\frac{3}{4},\frac{\delta}{2||\tilde{x}-a||},\frac{\mu}{2||\tilde{x}-a||}\right\},$$

and

$$x^*=(1-t)a+t\tilde{x}. \tag{17}$$

Obviously $0<t<1$. We will show that

$$f(x^*,y)>f(a,b),\quad \text{for all } y\in M_2^{x^*}. \tag{18}$$

Let $y\in M_2^{x^*}$. We distinguish two cases.

Case 1) If $y\in V$, by (14), we have

$$y\in V\bigcap M_2^{\tilde{x}},$$

i.e. $(\tilde{x},y)\in M$. Then

$$f(\tilde{x},y)>f(a,b).$$

Also, from v) we have $(a,y)\in M$. In view of i) and (17), we get $(x^*,y)\in M$.

Since f is bi-$(1_{\mathbf{R}^m},1_{\mathbf{R}^n})$ strictly concave-convex, we have

$$f(x^*,y)=f((1-t)a+t\tilde{x},y)>(1-t)f(a,y)+tf(\tilde{x},y)\geq$$

$$\geq(1-t)f(a,b)+tf(a,b)=f(a,b),$$

because

$$f(a,y)\geq\min\{f(a,v)|v\in M_2^a\}=\underline{f}(a)=f(a,b).$$

Case 2) If $y\in M_2^{x^*}\setminus V$, from

$$||(x^*,y)-(a,y)||\;=\;||x^*-a||\;<\;\mu$$

and (16), we have

$$|f(x^*,y)-f(a,y)|<\varepsilon.$$

Then

$$f(x^*,y)=f(x^*,y)-f(a,y)+f(a,y)>-\varepsilon+f(a,y)=$$

$$=-\min\{f(a,v)|v\in M_2^a\setminus V\}\;+\;f(a,b).+f\,(a,y)\geq f(a,b).$$

Consequently, (18) is true.

By (18), it follows that

$$\underline{f}(x^*)=\min\{f(x^*,y)|y\in M_2^{x^*}\}>f(a,b)=\underline{f}(a)$$

which contradicts hypothesis iii). Then (12) is true and hence (a,b) is a saddle-point of f with respect to M. $\square$

Example 7. For

$$M = \{(x,y) \in [0,1] \times [0,1] : \ x \leq y^2\} \subseteq \mathbb{R} \times \mathbb{R},$$

$(a,b) = (0,0) \in M$, and $f : M \to \mathbb{R}$ defined by

$$f(x,y) = -x^2 + y^4, \text{ for all } (x,y) \in M,$$

the hypotheses of Theorem 4 are satisfied.

Indeed, the set M is compact nonempty and f is continuous. Moreover,

i) The set M is bi-$(1_{\mathbf{R}}, 1_{\mathbf{R}})$ convex (see Example 5), hence *i*) holds.

ii) The function f is bi-$(1_{\mathbf{R}}, 1_{\mathbf{R}})$ strictly concave-convex (see Example 6), hence *ii*) is true.

iii) For each $x \in M_1 = [0,1]$, we have

$$\underline{f}(x) = \min\{-x^2 + y^4 | y \in M_2^x\} = -x^2 + x^2 = 0,$$

hence $a = 0$ is a maximum point of $\underline{f}$ on M_1.

iv) For each $y \in M_2^0 = [0,1]$, we have

$$f(0,y) = y^2 \geq 0 = f(0,0),$$

hence *iv*) is true.

v) Since $M_2^0 = M_2 = [0,1]$, the hypothesis *v*) is satisfied.

vi) For each $x \in M_1^0 = \{0\}$, and $\delta > 0$ we have

$$B(0,\delta) \cap M_2 \subseteq M_2^x = M_2^0 = [0,1],$$

because $M_2 = [0,1]$, hence *vi*) holds.

Then, in view of Theorem 4, the point $(a,b) = (0,0)$ is a saddle point of f with respect to M (see Example 1).$\diamond$

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DOREL I. DUCA, LIANA LUPŞA
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
str. M. Kogălniceanu nr. 1
400084 Cluj-Napoca, România
E-mail: *dduca@math.ubbcluj.ro, llupsa@math.ubbcluj.ro*

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