Some addition theorems for rectifiable spaces

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Abstract. We establish that if a compact Hausdorff space $B$ with the cardinality less than $2^{<\omega}$ is represented as the union of two non-locally compact rectifiable subspaces $X$ and $Y$, then $X$, $Y$ and $B$ are separable and metrizable.

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1 Introduction

It is well-known that if the cardinality of a compact topological group $X$ does not exceed $2^{<\omega}$ and the continuum hypothesis is satisfied, then $X$ is separable and metrizable (see [8]). Extending this result, we show that if the cardinality of a compact Hausdorff space $X$ is less than $2^{<\omega}$, then $X$ cannot be represented as the union of two non-locally compact rectifiable spaces. Recall that every topological group is a rectifiable space (see the definition below). Some other results in this direction are also obtained.

We use the terminology and notations from [12]. A remainder of a Tychonoff space $X$ is the subspace $bX \setminus X$ of a Hausdorff compactification $bX$ of $X$.

A space $X$ is of countable type (respectively, of pointwise countable type) if every compact subspace $P$ (respectively, any point $p$) of $X$ is contained in a compact subspace $F \subset X$ with a countable base of open neighbourhoods in $X$. All metrizable spaces and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type [1, 2, 12].

A famous classical result on duality between properties of spaces and properties of their remainders is the following theorem of M. Henriksen and J. Isbell [14]:

Theorem 1. A Tychonoff space $X$ is of countable type if and only if the remainder in any (in some) Hausdorff compactification of $X$ is Lindelöf.

It follows from this theorem that every remainder of a metrizable space is Lindelöf.
2 Addition theorems for rectifiable spaces

Recall that a space $X$ is rectifiable if there exists $e \in X$ and a homeomorphism $g : X \times X \to X \times X$ such that $g((x, e)) = (x, x)$, for every $x \in X$, and the restriction of $g$ to the subspace $X_e = \{(x, y) : y \in X\}$ is a homeomorphism of $X_e$ onto itself, for every $x \in X$. Every topological group is rectifiable, and every rectifiable space is homogeneous (see [9, 10]).

**Theorem 2.** Suppose that $B$ is a compact Hausdorff space such that $|B| < 2^{\omega_1}$. Suppose further that $B = X \cup Y$, where $X$ and $Y$ are non-locally compact rectifiable spaces. Then the spaces $B$, $X$, and $Y$ are separable and metrizable.

**Proof.** Clearly, $Y$ and $X$ are non-empty, since they are not locally compact. Hence, $B$ is non-empty. By Čech-Pospíšil Theorem ([3, 15], [12], Problem 3.12.11), there exists a point $a \in B$ such that $B$ is first-countable at $a$. Without loss of generality, we may assume that $a \in X$. Put $Z = B \setminus X$ and $H = B \setminus Y$. The spaces $X$ and $Y$ are nowhere locally compact, since they are homogeneous and non-locally compact. It also follows that $Z$ and $H$ are nowhere locally compact. Hence, $X$, $Y$, $Z$, and $H$ are dense in $B$.

Since $X$ is homogeneous and $X$ is first-countable at $a$, it follows that the space $X$ is first-countable. Therefore, $X$ is metrizable, since $X$ is rectifiable [13]. Hence, $X$ is a space of countable type [1], which implies that the remainder $Z$ of $X$ in $B$ is Lindelöf [14].

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), the remainder $H$ of $Y$ in the Hausdorff compactification $B$ of $Y$ is either pseudocompact, or Lindelöf. Notice that $H$ is metrizable in any case, since, obviously $H \subset X$.

**Case 1:** $H$ is pseudocompact.

Then $H$ is compact, since $H$ is metrizable. Therefore, $H$ is closed in $B$. Hence, $Y$ is open in $B$, which implies that $Y$ is locally compact, a contradiction. Thus, Case 1 is impossible.

**Case 2:** $H$ is Lindelöf.

Then $H$ is separable, since $H$ is metrizable. Hence, $B$ is separable, which implies that the Souslin number of $Y$ is countable, since $Y$ is dense in $B$. It also follows that $X$ is separable, since $H$ is dense in $X$.

Lindelöfness of $H$ also implies that $Y$ is a space of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, we can fix a non-empty compact subspace $F$ of $Y$ with a countable base of open neighbourhoods in $Y$. We have: $|F| \leq |B| < 2^{\omega_1}$. Applying one more time the Čech-Pospíšil Theorem [3, 12, 15] we conclude that there exists a point $b \in F$ such that $F$ is first-countable at $b$. Since $F$ has a countable base of open neighbourhoods in $Y$, we can now conclude that the space $Y$ is first-countable at the point $b$. Therefore, the space $Y$ is first-countable, since it is homogeneous. Hence, $Y$ is metrizable, since it is rectifiable. Finally, it follows that $Y$ is separable, since the Souslin number of $Y$ is countable. \[\square\]
A family $\eta$ of non-empty open subsets of a space $X$ is said to be a $\pi$-base of $X$ at a point $a \in X$ if every open neighbourhood of $a$ in $X$ contains some $V \in \eta$ (see [7]).

Here is another restriction on a compactum $B$ under which we can obtain even a stronger conclusion:

**Theorem 3.** Suppose that $B$ is a compact Hausdorff space of countable tightness and that $B = \bigcup\{Y_n : n \in \omega = \{0, 1, 2, \ldots\}\}$, where each $Y_n$ is dense in $B$ and is rectifiable. Then $B$ and each $Y_n$ are separable and metrizable.

**Proof.** Take any $y \in Y_n$. Then there exists a countable $\pi$-base $\xi$ of $B$ at $y$, since the tightness of the compactum $B$ is countable (see [18] and [3]). Then $\eta = \{V \cap Y_n : V \in \xi\}$ is a countable $\pi$-base of the subspace $Y_n$ at $y$, since $Y_n$ is dense in $B$. Since $Y_n$ is rectifiable and $Y_n$ has a countable $\pi$-base at $y$, it follows from a result of A. Gul’ko [13] that the space $Y_n$ is metrizable. Therefore, each $Y_n$ has a $\sigma$-disjoint open base. Since $Y_n$ is dense in $B$, it follows that $B$ is first-countable and that $\sigma$-disjoint open bases in the subspaces $Y_n$ can be extended, in a standard way, to a point-countable base in $B$. It remains to use a well-known deep theorem of A.S. Mischenko that every compact Hausdorff space with a point-countable base is separable and metrizable (see [12], Problem 3.12.22(f)).

The next result considerably generalizes Theorem 2.

**Theorem 4.** Suppose that $B$ is a compact Hausdorff space which doesn’t admit a continuous mapping onto the Tychonoff cube $I^{\omega_1}$. Suppose further that $B = X \cup Y$, where $X$ and $Y$ are non-locally compact rectifiable spaces. Then $B$, $X$, and $Y$ are separable and metrizable.

**Proof.** Clearly, $Y$ and $X$ are non-empty. Hence, $B$ is non-empty. Since $B$ cannot be continuously mapped onto the Tychonoff cube $I^{\omega_1}$, it follows from a Theorem of B.E. Shapirovskij (see [18], [3], Theorems 2.2.20 and 3.1.9) that there exists a point $a \in B$ such that $B$ has a countable $\pi$-base at $a$. Without loss of generality, we may assume that $a \in X$. Put $Z = B \setminus X$ and $H = B \setminus Y$. The spaces $X$ and $Y$ are nowhere locally compact, since they are homogeneous and non-locally compact. Clearly, the subspaces $Z$ and $H$ are nowhere locally compact as well. Thus, $X$, $Y$, $Z$, and $H$ are dense in $B$.

Since $X$ has a countable $\pi$-base at $a$ and $X$ is rectifiable, it follows that the space $X$ is metrizable [13]. Hence, $X$ is a space of countable type [1], which implies that the remainder $Z$ of $X$ in $B$ is Lindelöf [14].

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), the remainder $H$ of $Y$ in the compactification $B$ of $Y$ is either pseudocompact, or Lindelöf. Notice that $H$ is metrizable, since $H \subseteq X$.

If $H$ is pseudocompact, then $H$ is compact, since $H$ is metrizable. Therefore, $H$ is closed in $B$ and $Y$ is open in $B$, which implies that $Y$ is locally compact, a contradiction.

Hence, $H$ is Lindelöf. Then $H$ has a countable base, since $H$ is metrizable. Hence, $B$ has a countable $\pi$-base, since $H$ is dense in $B$, which implies that $Y$ also
has a countable $\pi$-base, since $Y$ is dense in $B$. Since the space $Y$ is rectifiable, using again a theorem of A. Gul’ko in [13], we conclude that $Y$ is metrizable. Clearly, $Y$ is separable. It also follows that $X$ is separable, since $H$ is dense in $X$. Therefore, $B$ is separable and metrizable, as the union of two separable metrizable subspaces (see [12], Corollary 3.1.20).

The proof of Theorem 2 obviously contains a proof of the next statement:

**Theorem 5.** Suppose that $B$ is a compact Hausdorff space and that $B = X \cup Y$, where $X$ and $Y$ are non-locally compact spaces. Suppose further that $X$ is metrizable and $Y$ is rectifiable. Then $B$, $X$ and $Y$ are separable and metrizable.

### 3 On $k$-gentle paratopological groups

A group $G$ with a topology $T$ is called a paratopological group if the multiplication $(x, y) \to x \cdot y$ is a continuous mapping of $G \times G$ onto $G$.

Let us call a mapping $f$ of a space $X$ into a space $Y$ $k$-gentle if for every compact subset $F$ of $X$ the image $f(F)$ is also compact.

A group $G$ with a topology will be called $k$-gentle if the inverse mapping $x \to x^{-1}$ is $k$-gentle.

**Proposition 1.** Suppose that $B$ is a compact Hausdorff space in which any non-empty $G_\delta$-subspace has a point of countable character in this subspace. Suppose further that $B = X \cup Y$, where each $Z \in \{X, Y\}$ is a space with the following properties:

- the space $Z$ is not locally compact;
- if the space $Z$ contains some point of countable character in $Z$, then the space $Z$ is metrizable;
- if $bZ$ is a Hausdorff compactification of $Z$, then the remainder $bZ \setminus Z$ is either pseudocompact or Lindelöf.

Then $B$, $X$, and $Y$ are separable and metrizable.

**Proof.** Clearly, $Y$ and $X$ are non-empty, since they are not locally compact. Hence, $B$ is non-empty. Moreover, the sets $X$ and $Y$ are dense in $B$. Thus, $B$ is a compactification of the subspaces $X$ and $Y$. There exists a point $a \in B$ such that $B$ is first-countable at $a$. Without loss of generality, we may assume that $a \in X$. Then the space $X$ is metrizable.

If $b \in X \cap Y$, for some $b$, then the space $Y$ is metrizable, as a space with the countable character at $b$. In this case the proof is complete.

Assume that $X \cap Y = \emptyset$. Clearly, $X$ is a space of countable type [1], since $X$ is metrizable. It follows that the remainder $Y$ of $X$ in $B$ is Lindelöf [14].

Clearly, the space $X$ is a remainder of the space $Y$ in $B$. Hence, $X$ is either pseudocompact or Lindelöf.

**Case 1:** $X$ is pseudocompact.
Then $X$ is compact, since $X$ is metrizable. Therefore, $X$ is closed in $B$, a contradiction. Thus, Case 1 is impossible.

**Case 2:** $X$ is Lindelöf.

Then $X$ is separable, since $X$ is metrizable. Hence, $B$ is separable, which implies that the Souslin number of $Y$ is countable, since $Y$ is dense in $B$. Lindelöfness of $X$ also implies that $Y$ is a space of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, we can fix a non-empty compact subspace $F$ of $Y$ with a countable base of open neighbourhoods in $Y$. By the assumptions, there exists a point $c \in F$ such that $B$ is first-countable at $c$. It follows that the space $Y$ is first-countable at $c$. Thus $Y$ is a metrizable space with a countable Souslin number. Hence, $X$ and $Y$ are separable and metrizable. It follows that $B$ is separable and metrizable, since $B$ is compact and Hausdorff ([12], Corollary 3.1.20).

**Corollary 6.** Suppose that $B$ is a compact Hausdorff space such that $|B| < 2^{\omega_1}$. Suppose further that $B = X \cup Y$, where $X$ and $Y$ are non-locally compact $k$-gentle paratopological groups. Then $B$, $X$, $Y$ are separable, metrizable spaces, and $X$, $Y$ are topological groups.

In view of Proposition 1, this statement follows from the next proposition:

**Proposition 2.** Let $G$ be a Hausdorff $k$-gentle paratopological group such that $G$ is first-countable at some point. Then:

1) the space $G$ is metrizable;

2) $G$ is a topological group;

3) any remainder of $G$ in a Hausdorff compactification $bG$ of $G$ is either pseudocompact or Lindelöf.

**Proof.** Since $G$ is a homogeneous space, the space $G$ is first-countable. Every first-countable Hausdorff space is a $k$-space ([12], Theorem 3.3.20). Hence $G$ is a $k$-space. It is obvious that if a $k$-gentle paratopological group is a $k$-space, then this paratopological group is a topological group. Hence, 2) holds. Every first-countable topological group is metrizable (see [8]). Therefore, 1) holds. By the Dichotomy Theorem for remainders of topological groups (see [4, 5]), since $G$ is a topological group, any remainder of $G$ in a Hausdorff compactification of $G$ is either pseudocompact or Lindelöf. Therefore, 3) holds.

**Example 7.** Let $X_1$ be the space of all rational numbers of the interval $I = [0, 1]$. Clearly, $X_1$ is homeomorphic to a topological group. The space $Y_1 = I \setminus X_1$ is also homeomorphic to a topological group. Take also the topological group $D^{\omega_1}$. Put $B = I \times D^{\omega_1}$, $X = X_1 \times D^{\omega_1}$ and $Y = Y_1 \times D^{\omega_1}$. Then $X$ and $Y$ are dense non-metrizable nowhere locally compact topological groups, $B$ is a homogeneous compact Hausdorff space with the cardinality $2^{\omega_1}$, and $B = X \cup Y$. The space $B$ admits a continuous mapping onto $I^{\omega_1}$ and the tightness $t(B) = \aleph_1$. Thus the respective cardinal assumptions in Theorems 2, 3, 4 and Corollary 6 are essential.

We could also modify the definitions of $B$, $X$, and $Y$ above so that each of the spaces $B$, $X$, $Y$ would admit a structure of a topological group.
4 On Mal’cev spaces. Some questions

A Mal’cev operation on a space $X$ is a continuous mapping $\mu : X^3 \to X$ such that $\mu(x, x, z) = z$ and $\mu(x, y, y) = x$, for all $x, y, z \in X$. A space is called a Mal’cev space if it admits a Mal’cev operation (see [9–11, 16, 19]).

A homogeneous algebra on a space $G$ is a pair of binary continuous operations $p, q : G \times G \to G$ such that $p(x, x) = p(y, y)$, and $p(x, q(x, y)) = y$, $q(x, p(x, y)) = y$ for all $x, y \in G$. If the above conditions are satisfied, then the ternary operation $\mu(x, x, z) = q(x, p(y, z))$ is a Mal’cev operation (see [9, 10]).

A biternary algebra on a space $G$ is a pair of ternary continuous operations $\alpha, \beta : G \times G \times G \to G$ such that $\alpha(x, x, y) = y$, $\alpha(\beta(x, y, z), y, z)) = x$, and $\beta(\alpha(x, y, z), y, z)) = x$, for all $x, y, z \in G$ (see [16]).

In [9, 10] (see also [19]) it was proved that for an arbitrary space $G$ the following conditions are equivalent:

1) $G$ is a rectifiable space;
2) $G$ is homeomorphic to a homogeneous algebra;
3) There exists a structure of a biternary algebra on $G$.

A structure of a topological quasigroup on a space $G$ is a triplet of binary continuous operations $p, l, r : G \times G \times G \to G$ such that $p(x, l(x, y)) = p(r(y, x), x) = l(x, p(x, y)) = l(r(x, y), x) = r(p(y, x), x) = r(x, l(y, x)) = y$, for all $x, y \in G$. If there exists an element $e \in G$ such that $p(e, x) = p(x, e) = x$ for any $x \in G$, then we say that $G$ is a topological loop and $e$ is the identity of $G$. Any topological quasigroup admits the structure of a topological loop (see [16]). If $e \in G$ and $p(e, x) = x$ for any $x \in G$, then $x + y = p(y, x)$ and $x \cdot y = r(y, x)$ is a structure of a homogeneous algebra.

If $(G, \cdot)$ is a topological group with the neutral element $e$, then the mapping $\varphi(x, y) = (x, x^{-1} \cdot y)$ is a rectification on the space $G$ with the neutral element $e$, and the mappings $p(x, y) = x^{-1} \cdot y$ and $q(x, y) = x \cdot y$ form a structure of homogeneous algebra on $G$. Therefore, every topological quasigroup is a rectifiable space.

A space $X$ is called $\kappa$-perfect if the closure of any open subset of $X$ is a $G_\delta$-set in $X$.

**Proposition 3.** Let $X$ and $Y$ be any pseudocompact $\kappa$-perfect subspaces of a Tychonoff space $Z$ such that $Z = X \cup Y$, and $X, Y$ are dense in $Z$. Suppose further that $X$ and $Y$ are Mal’cev spaces. Then the space $Z$ is also $\kappa$-perfect.

**Proof.** Let $U$ be an open subset of the space $Z$. We put $V = X \cap U$ and $W = Y \cap U$. There exist two sequences $\{V_n : n \in \omega\}$ and $\{W_n : n \in \omega\}$ of open subsets of the space $Z$ such that:

- $cl_X V = \cap \{V_n \cap X : n \in \omega\}$ and $cl_Y W = \cap \{W_n \cap Y : n \in \omega\}$;
- $V_{n+1} \subseteq V_n$ and $W_{n+1} \subseteq W_n$ for any $n \in \omega$.

Obviously, $cl_Z U = cl_Z V = cl_Z W$. We put $F + cl_Z U$. 

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\[ \text{Some Addition Theorems for Rectifiable Spaces} \]
We affirm that \( \cap \{ V_n : n \in \omega \} \subseteq F \). Assume that \( H = (\cap \{ V_n : n \in \omega \}) \setminus F \neq \emptyset \).

By construction, \( H \) is a \( G_\delta \)-subset of \( Z \) and \( H \subseteq Y \). Fix a point \( b \in H \). There exists a continuous function \( f : Z \to [0,1] \) such that \( b \in f^{-1}(0) \subseteq H \). Then the function \( g(x) = 1/f(x) \) is a continuous unbounded function on the space \( X \), a contradiction. Thus \( \cap \{ V_n : n \in \omega \} \subseteq F \cap \{ W_n : n \in \omega \} \subseteq F \). If \( U_n = V_n \cup W_n \), then \( \cap \{ W_n : n \in \omega \} = F \).

**Proposition 4.** Let \( X \) and \( Y \) be pseudocompact subspaces of a compact Hausdorff space \( B \) such that \( X \) and \( Y \) are dense in \( B \) and \( Z = X \cup Y \). Suppose further that \( X \) and \( Y \) are Mal’cev spaces. Then:

1) The space \( B \) is a \( \kappa \)-perfect Mal’cev space.
2) There exist Mal’cev operations \( \mu, \eta : B^3 \to B \) on \( B \) such that \( \mu(X^3) = X \) and \( \eta(Y^3) = Y \).
3) If \( X \) is rectifiable, then \( B \) is also rectifiable, and there exists a structure of homogeneous algebra \( \{ +, \cdot \} \) on \( B \) such that \( X \) is a subagebra of \( B \).
4) If \( X \) is a topological quasigroup, then there exists a structure of a topological loop on \( B \) such that \( X \) is a subloop of \( B \).
5) If the space \( X \) is a topological group, then there exists a structure of a topological group on \( B \) such that \( X \) is a subgroup of \( B \).
6) \( B = \beta X = \beta Y \).

**Proof.** Since \( X \) is a pseudocompact Mal’cev space, the Stone-Čech compactification \( \beta X \) of \( X \) is a compact Mal’cev space [17]. Any compact Mal’cev space is \( \kappa \)-perfect [10]. Thus, \( X \) and \( Y \) are \( \kappa \)-perfect spaces, since they are dense subspaces of \( \kappa \)-perfect spaces. By Proposition 3, the space \( B \) is \( \kappa \)-perfect. Now we need the following known fact:

**Fact 1:** If \( Z \) is a pseudocompact subspace of a \( \kappa \)-perfect compact Hausdorff space \( B \) such that \( Z \) is dense in \( B \), then \( B = \beta Z \).

Really, let \( F_1 \) and \( F_2 \) be two closed subsets of \( Z \) and \( f : Z \to \mathbb{R} \) be a continuous function such that \( F_1 \subseteq f^{-1}(-2) \) and \( F_2 \subseteq f^{-1}(2) \). There exist two open subsets \( U \) and \( V \) of \( B \) such that \( U \cap Z \subseteq f^{-1}(-3,-1) \) and \( V \cap Z \subseteq f^{-1}(1,3) \). Then \( H = cl_B U \cap cl_B V \) is a \( G_\delta \)-subset of \( B \) and, by construction, \( H \subseteq B \setminus Z \). Since \( Z \) is pseudocompact, we have \( H = \emptyset \). Since \( F_1 \subseteq U \) and \( F_2 \subseteq V \), we have \( cl_B F_1 \cap cl_B F_2 = \emptyset \). Therefore \( B = \beta Z \).

Thus, \( B = \beta X = \beta Y \). Statements 1 and 6 are proved. The space \( X^n \) is pseudocompact for any \( n \in \omega \). Hence, by virtue of Glicksberg’s Theorem ([12], Problem 3.12.20(d)), any continuous binary operation on \( X \) admits continuous extension on \( B \). Statements 2, 3 and 4 are established.

**Proposition 5.** Let \( X \) be a subalgebra of a homogeneous algebra \( G \). If the space \( G \) is regular and Lindelöf, and the space \( X \) is of pointwise countable type and is dense in \( G \), then there exist a separable metrizable homogeneous algebra \( G' \) and a homomorphism \( g : G \to G' \) such that \( X = g^{-1}(g(X)) \) and the mapping \( g \) is open and perfect. In particular, it follows that \( X \) is a Lindelöf p-space.
**Proof.** By the assumptions, there is a pair of binary continuous operations \( p, q : G \times G \to G \) on the space \( G \) such that:
- \( p(x, x) = p(y, y) \), and \( p(x, q(x, y)) = y, q(x, p(x, y)) = y \) for all \( x, y \in G \);
- \( p(x, y) \in X \) and \( q(x, y) \in X \) for all \( x, y \in X \).

We put \( e = p(x, x) \). If \( a \in G \), then \( p_a(x) = p(a, x) \) and \( q_a(x) = q(a, x) \) for any \( x \in G \). We have \( q_a^{-1} = p_a \) and \( q_a(e) = a \). Thus, \( p_a \) and \( q_a \) are homeomorphisms. Moreover, \( p_a(X) = q_a(X) = X \) for each \( a \in X \).

Let \( F \) be a non-empty compact subspace of \( X \) with a countable base of open neighbourhoods in \( X \). We can assume that \( e \in F \). Since \( X \) is dense in \( G \), the set \( F \) also has a countable base of open neighbourhoods in the space \( G \). Therefore, \( X \) and \( G \) are \( p \)-spaces (see [6], Proposition 2.1).

Since \( F \) is a compact \( G_δ \)-subset of the Lindelöf algebra \( G \), there exist a separable metrizable homogeneous algebra \( G' \) and a homomorphism \( g : G \to G' \) such that \( F = g^{-1}(g(F)) \) and \( g \) is a perfect mapping [10]. The quotient homomorphism of a Mal’cev algebra is an open mapping [10]. Thus, the mapping \( g \) is open. We can assume that \( e' = g(e) \) and \( p(z, z) = e' \) for any \( z \in G' \).

Let \( b \in g(X) \subseteq G' \). Fix \( a \in X \cap g^{-1}(b) \). If \( H = g^{-1}(e') \), then \( H \subseteq F \subseteq X \) and \( g_a(H) = g^{-1}(g_a(e')) \subseteq X \). Thus, \( g^{-1}(b) = g^{-1}(g_a(e')) \subseteq X \). Therefore, \( g^{-1}(g(X)) = X \). The proof is complete. \( \square \)

Since a pseudocompact rectifiable space \( X \) can be considered as a subalgebra of the compact homogeneous algebra \( G = \beta X \), Proposition 5 yields

**Corollary 8.** If \( G \) is a pseudocompact rectifiable space of pointwise countable type, then \( G \) is compact.

**Theorem 9.** Suppose that \( B \) is a compact Hausdorff space, \( B = X \cup Y \) and \( X \cap Y = \emptyset \), where \( X \) and \( Y \) are non-locally compact rectifiable spaces. Then \( X \) and \( Y \) either are both pseudocompact, or are both Lindelöf \( p \)-spaces.

**Proof.** Clearly, \( X \) is the remainder of \( Y \) in the Hausdorff compactification \( B \) of \( Y \); similarly, \( Y \) is the remainder of \( X \) in the Hausdorff compactification \( B \) of \( X \).

By the Dichotomy Theorem for remainders of rectifiable spaces (see [6]), each of the spaces \( X \) and \( Y \) is either pseudocompact, or Lindelöf. If both of them are pseudocompact, then we are done.

Assume now that at least one of the subspaces \( X \) and \( Y \), say \( X \), is Lindelöf.

By the Dichotomy Theorem for remainders in [6], the remainder \( Y \) of \( X \) in \( B \) is either pseudocompact or Lindelöf.

**Case 1:** \( Y \) is pseudocompact.

Lindelöfness of \( X \) implies that \( Y \) is a space of countable type, by the theorem of Henriksen and Isbell [14]. Then Corollary 8 implies that \( Y \) is compact. Therefore, \( Y \) is closed in \( B \). Hence, \( X \) is open in \( B \), which implies that \( X \) is locally compact, a contradiction. Thus, Case 1 is impossible.

**Case 2:** \( Y \) is Lindelöf.
Lindelöfness of $X$ and $Y$ implies that $X$ and $Y$ are spaces of countable type, by the theorem of Henriksen and Isbell [14]. Therefore, by Proposition 2.1 from [6], $X$ and $Y$ are Lindelöf $p$-spaces. The proof is complete.

Observe that Theorem 9 doesn’t generalize to homogeneous Mal’cev spaces. Indeed, a non-metrizable compactum can be represented as the union of two disjoint dense copies of Sorgenfrey line (take the ”double arrow” space). It was shown in [17] that Sorgenfrey line is a Mal’cev space. It is well-known that Sorgenfrey line is not a $p$-space (see [5]). It is also clear that Sorgenfrey line is not pseudocompact.

**Question 1.** Is every rectifiable space of countable type paracompact? Normal?

**Question 2.** Does every rectifiable space of countable type admit a perfect mapping onto a metrizable (rectifiable) space?

**References**


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