BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(66), 2011, Pages 5–16
ISSN 1024–7696

On some operations in the lattice of submodules determined by preradicals

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Abstract. In the lattice $\boldsymbol{L}({}_RM)$ of all submodules of a module ${}_RM$ four operations are defined using the standard preradicals: α-product, ω-product, α-coproduct and ω-coproduct. Some properties of these operations, as well as some connections with the lattice operations of $\boldsymbol{L}({}_RM)$ are indicated. For characteristic submodules these operations were studied in the work [5].

Mathematics subject classification: 16D90, 16S90, 06B23.
Keywords and phrases: Preradical, lattice, submodule, product (coproduct) of preradicals.

1 Definitions and preliminary facts

Let R be an associative ring with unity and R-Mod be the category of unitary left R-modules. For an arbitrary module ${}_RM \in R$-Mod we denote by $\boldsymbol{L}({}_RM)$ the lattice of all submodules of ${}_RM$. A submodule $N \in \boldsymbol{L}({}_RM)$ is called *characteristic* (fully invariant) in M if $f(N) \subseteq N$ for every R-endomorphism $f : {}_RM \to {}_RM$. The lattice of all characteristic submodules of ${}_RM$ will be denoted by $\boldsymbol{L}^{ch}({}_RM)$.

A *preradical* r of R-Mod by definition is a subfunctor of identity functor of R-Mod (i.e. $r(M) \subseteq M$ and $f\big(r(M)\big) \subseteq r(M')$ for every module $M \in R$-Mod and every R-morphism $f : M \to M'$). Obviously, $r(M)$ is a characteristic submodule of ${}_RM$. Moreover, the submodule $N \in \boldsymbol{L}({}_RM)$ is characteristic in ${}_RM$ if and only if there exists a preradical r of R-Mod such that $N = r(M)$.

If $r\big(r(M)\big) = r(M)$ for every $M \in R$-Mod, then r is called *idempotent* preradical; if $r\big(M / {}_RM)\big) = 0$ for every $M \in R$-Mod, then r is called a *radical*.

We denote by R-pr the family of all preradicals of the category R-Mod. Two operations ,,$\wedge$" and ,,$\vee$" are defined in R-pr by the following rules:

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(X), \qquad \big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big)(X) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(X)$$

for every $X \in R$-Mod and every family of preradicals $\{r_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$-pr. Then R-pr $(\wedge, \vee)$ possesses all properties of a complete lattice with the exception that it is not necessarily a set (in general case R-pr is a class), so it is called the "big lattice" of preradicals of R-Mod. In this lattice a special role is played by the following two types of preradicals. For every pair $N \subseteq M$, where $N \in \boldsymbol{L}({}_RM)$, we define the functions α_N^M and ω_N^M by the rules:

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$$\alpha_N^M(X) = \sum_{f\,:\,M\to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f\,:\,X\to M} f^{-1}(N),$$

for every $X \in R$-Mod. The following facts are well known ([1, 2]).

Proposition 1.1. 1) *α_N^M and ω_N^M are preradicals of R-Mod;*

2) *$\alpha_N^M(M)$ is the least characteristic submodule of ${}_RM$ containing N;*

3) *$\omega_N^M(M)$ is the largest characteristic submodule of ${}_RM$ contained in N.* □

Proposition 1.2. *If $N \in \boldsymbol{L}^{ch}({}_RM)$, then $\alpha_N^M(M) = N$ and $\omega_N^M(M) = N$. Moreover, for a preradical $r \in R$-pr we have:*

$$r(M) = N \quad \Leftrightarrow \quad \alpha_N^M \ \le\ r\ \le\ \omega_N^M. \qquad \square$$

So for a submodule $N \in \boldsymbol{L}^{ch}({}_RM)$ the preradical α_N^M is the least among preradicals $r \in R$-pr with the property $r(M) = N$. Dually, ω_N^M is the largest among preradicals $r \in R$-pr with $r(M) = N$.

Now we mention two particular cases:

a) *the idempotent preradical* r^M, defined by module ${}_RM$

$$r^M(X) = \sum_{f\,:\,M\to X} Im\, f \text{ – the } \textit{trace} \text{ of } M \text{ in } X \text{ (i.e. } r^M = \alpha_M^M);$$

b) *the radical* r_M defined by ${}_RM$

$$r_M(X) = \bigcap_{f\,:\,M\to X} Ker\, f \text{ – the } \textit{reject} \text{ of } M \text{ in } X \text{ (i.e. } r_M = \omega_0^M).$$

The following two operations in R-pr are very important in the theory of preradicals:

1) *the product* of preradicals $r,\, s \in R$-pr:

$$(r \cdot s)(X) = r\big(s(X)\big);$$

2) *the coproduct* of preradicals $r,\, s \in R$-pr:

$$(r : s)(X) \,/\, r(X) = s\big(X \,/\, r(X)\big)$$

for every $X \in R$-Mod.

2 $\boldsymbol{\alpha}$-product of submodules

Using preradicals of the form α_N^M the following operation is introduced in the lattice $\boldsymbol{L}({}_RM)$ of all submodules of an arbitrary module $M \in R$-Mod.

Definition 2.1. Let $M \in R$-Mod and $K, N \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$K \cdot N = \alpha_K^M(N) = \sum_{f\,:\,M\to N} f(K)$$

will be called the *α-product* in M of submodules K and N.

This operation was considered in [3] for the investigation of prime modules. The continuation of these studies can be found in [4]. For characteristic submodules $K, N \in \boldsymbol{L}^{ch}({}_RM)$ this operation coincides with the α-product defined in [5] by the rule: $K \cdot N = \alpha_K^M\, \alpha_N^M(M)$.

Some simple properties of α-product are indicated in the following statement.

Proposition 2.1. 1) *$K \cdot N \subseteq N$ and $K \cdot N$ is a characteristic submodule in N;*

2) *If $N \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \in \boldsymbol{L}^{ch}({}_RM)$ for every $K \in \boldsymbol{L}({}_RM)$;*

3) *If $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \subseteq K$, therefore $K \cdot N \subseteq K \cap N$;*

4) *If $K = 0$, then $0 \cdot N = 0$ for every $N \in \boldsymbol{L}({}_RM)$; if $N = 0$, then $K \cdot 0 = 0$ for every $K \in \boldsymbol{L}({}_RM)$;*

5) *If $N = M$, then for every $K \in \boldsymbol{L}({}_RM)$ the submodule $K \cdot M = \sum\limits_{f\,:\,M \to M} f(K)$ is the least characteristic submodule of M containing K;*

6) *If $K = M$, then for every $N \in \boldsymbol{L}({}_RM)$ we have $M \cdot N = \sum\limits_{f\,:\,M \to N} f(M) =$ $= r^M(N)$.* □

Proposition 2.2. *The operation of α-product is monotone in both variables:*

$$K_1 \subseteq K_2 \;\Rightarrow\; K_1 \cdot N \subseteq K_2 \cdot N \qquad \forall\, N \in \boldsymbol{L}({}_RM);$$
$$N_1 \subseteq N_2 \;\Rightarrow\; K \cdot N_1 \subseteq K \cdot N_2 \qquad \forall\, K \in \boldsymbol{L}({}_RM).$$ □

The following two results explore the associativity of α-product and are indicated in [3] (Lemma 2.1). For convenience we give also the sketch of proofs.

Proposition 2.3. *The following relation is true:*

$$(K \cdot N) \cdot L \subseteq K \cdot (N \cdot L)$$

for every $K, N, L \in \boldsymbol{L}({}_RM)$.

Proof. Every pair of morphisms $f : M \to N$, $g : M \to L$ determines a morphism $h = gf : M \to L$ and since by definition $N \cdot L = \sum\limits_{g\,:\,M \to L} g(N)$ we have $g\big(f(m)\big) \in N \cdot L$. So we can consider that $h \in Hom_R(M, N \cdot L)$. For every $a \in K$ we have $f(a) \in K \cdot N$ and $g\big(f(a)\big) \in (K \cdot N) \cdot L$. Therefore we obtain $g\big(f(a)\big) = h(a) \in K \cdot (N \cdot L) = \sum\limits_{h\,:\,M \to N \cdot L} h(K)$, proving the statement. □

Proposition 2.4. *If M is a projective module, then the operation of α-product in $\boldsymbol{L}({}_RM)$ is associative:*

$$(K \cdot N) \cdot L = K \cdot (N \cdot L)$$

for every $K, N, L \in \boldsymbol{L}({}_RM)$.

Proof. We consider the module $U = \dot{\sum\limits_{g\,:\,M \to L}} N_g$, $N_g = N$, with canonical projections $p_g : U \to U_g$. We can define the mapping:

$$h : U \to N \cdot L, \quad h(x) = \sum_{g\,:\,M \to L} g\big(p_g(x)\big) \in N \cdot L = \sum_{g\,:\,M \to L} g(N),\ x \in U.$$

Then h is an epimorphism, since every element of $N \cdot L$ by definition has the form $\sum\limits_{i=1}^{t} g_i(n_{g_i})$. By projectivity of ${}_RM$ for every $f : M \to N \cdot L$ there exists a morphism

$\overline{f} : M \to U$ such that $f = h\overline{f}$. For every $a \in K$ we have $f(a) \in K \cdot (N \cdot L)$ and $f(a) = h\overline{f}(a) = \sum\limits_{g\,:\,M \to L} g\big(p_g\,\overline{f}(a)\big)$. Since $p_g\,\overline{f} \in Hom_R(M, N)$, we obtain $p_g\,\overline{f}(a) \in K \cdot N$ and then by definition $\sum\limits_{g\,:\,M \to L} g\big(p_g\,\overline{f}(a)\big) \in (K \cdot N) \cdot L$, therefore $f(a) \in (K \cdot N) \cdot L$. This proves that $K \cdot (N \cdot L) \subseteq (K \cdot N) \cdot L$, and the inverse inclusion follows from Proposition 2.3. □

In continuation we will study some relations between the operation of α-product and the lattice operations of $\boldsymbol{L}({}_RM)$. For that we need the following fact on the operation ,,$\vee$" (join) in the lattice R-pr.

Lemma 2.5. *For every submodules $N, K \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\alpha^M_{N+K} = \alpha^M_N \ \vee \ \alpha^M_K.$$

Proof. For every $X \in R$-Mod by definitions it follows:

$$(\alpha^M_N \ \vee \ \alpha^M_K)\,(X) = \alpha^M_N(X) + \alpha^M_K(X) =$$

$$= \Big(\sum_{f\,:\,M \to X} f(N)\Big) + \Big(\sum_{f\,:\,M \to X} f(K)\Big) = \sum_{f\,:\,M \to X} f(N + K) = \alpha^M_{N+K}(X).$$ □

Proposition 2.6. *For every module $M \in R$-Mod the operation of α-product is left distributive with respect to the sum of submodules:*

$$(K_1 + K_2) \cdot N = (K_1 \cdot N) + (K_2 \cdot N)$$

for every K_1, K_2, $N \in \boldsymbol{L}({}_RM)$.

Proof. Using Lemma 2.5 and definitions, we obtain:

$$(K_1 + K_2) \cdot N = \alpha^M_{K_1+K_2}(N) = \big(\alpha^M_{K_1} \ \vee \ \alpha^M_{K_2}\big)(N) =$$

$$= \alpha^M_{K_1}(N) + \alpha^M_{K_2}(N) = (K_1 \cdot N) + (K_2 \cdot N).$$ □

Proposition 2.7. *For every submodules K, N_1, $N_2 \in \boldsymbol{L}({}_RM)$ the following relation is true: $K \cdot (N_1 + N_2) \ \supseteq \ (K \cdot N_1) + (K \cdot N_2)$. If $N_1 \cap N_2 = 0$, then we have: $K \cdot (N_1 \oplus N_2) \ = \ (K \cdot N_1) \ \oplus \ (K \cdot N_2)$.*

Proof. The first relation follows from the monotony of α-product (Proposition 2.2). In the second relation it is sufficient to verify the inclusion $(\subseteq)$. Let i_1, $i_2\,(p_1, p_2)$ be the canonical injections (projections) of a direct sum $N_1 \oplus N_2$. Every morphism $f : M \to N_1 \oplus N_2$ can by uniquely represented as $i_1 g + i_2 h$, where $g = p_1 f : M \to N_1$, $h = p_2 f : M \to N_2$. For every $a \in K$ we have $f(a) \in K \cdot (N_1 \oplus N_2)$. But at the same time

$$f(a) = p_1 f(a) + p_2 f(a) = g(a) + h(a) \in (K \cdot N_1) \ \oplus \ (K \cdot N_2),$$

proving the needed inclusion. □

We conclude this section with the remark on the particular case when ${}_RM = {}_RR$, i.e. $\boldsymbol{L}({}_RR)$ is the lattice of left ideals of the ring R. For every $I,\ J \in \boldsymbol{L}({}_RR)$ we have:

$$I \cdot J = \alpha_I^R(J) = \sum_{f\,:\,R\,\to\,J} f(I) = \sum_{j\in J} I \cdot j = IJ,$$

so the α-product of left ideals coincides with the ordinary product of left ideals in R.

3 $\boldsymbol{\omega}$-product of submodules

In a similar mode as in the previous case we will now define another operation in the lattice $\boldsymbol{L}({}_RM)$ with the help of preradicals of the forme ω_N^M (Section 1).

Definition 3.1. Let $M \in R$-Mod and $K,\ N \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$K \odot N = \omega_K^M(N) = \bigcap_{f:N\,\to\,M} f^{-1}(K)$$

will be called the *ω-product* in M of submodules K and N, i.e. $K \odot N = \{n \in N \,|\, f(n) \in K \ \text{ for every } \ f : N \to M\}$.

In the case when $K,\ N \in \boldsymbol{L}^{ch}({}_RM)$ this operation coincides with the ω-product of characteristic submodules, defined in [5] by the rule: $K \odot N = \omega_K^M\,\omega_N^M(M)$.

Now we formulate some elementary properties of ω-product in $\boldsymbol{L}({}_RM)$.

Proposition 3.1. 1) *$K \odot N \subseteq N$ and $K \odot N$ is a characteristic submodule in N;*

2) *If $N \in \boldsymbol{L}^{ch}({}_RM)$, then $K \odot N \in \boldsymbol{L}^{ch}({}_RM)$ for every $K \in \boldsymbol{L}({}_RM)$;*

3) *$K \odot N \subseteq K$, therefore $K \odot N \subseteq K \cap N$;*

4) $0 \odot N = 0, \quad K \odot 0 = 0$;

5) *$K \odot M = \omega_K^M(M) = \bigcap_{f:M\,\to\,M} f^{-1}(K)$ is the largest characteristic submodule of M contained in K; therefore if $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \odot M = K$;*

6) $M \odot N = \omega_M^M(N) = \bigcap_{f:N\,\to\,M} f^{-1}(M) = N.$ □

Proposition 3.2. *The operation of ω-product is monotone in both variables:*

$$K_1 \subseteq K_2 \ \Rightarrow\ K_1 \odot N \subseteq K_2 \odot N \qquad \forall\, N \in \boldsymbol{L}({}_RM);$$

$$N_1 \subseteq N_2 \ \Rightarrow\ K \odot N_1 \subseteq K \odot N_2 \qquad \forall\, K \in \boldsymbol{L}({}_RM).$$ □

We remark that if $K \in \boldsymbol{L}^{ch}({}_RM)$, then $K \cdot N \subseteq K \odot N$ for every $N \in \boldsymbol{L}({}_RM)$, since $\alpha_K^M \le \omega_K^M$ and $\alpha_K^M(N) \subseteq \omega_K^M(N)$, so we have:

$$K \cdot N \subseteq K \odot N \subseteq K \cap N.$$

Proposition 3.3. *For every submodules $K,\ N,\ L \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$(K \odot N) \odot L \supseteq K \odot (N \odot L).$$

Proof. Let $l \in K \odot (N \odot L)$. By definition this means that:

1) $l \in (N \odot L)$, i.e. $g(l) \in N$ for every $g : L \to M$;

2) $h(l) \in K$ for every $h : N \odot L \to M$.

We must verify that

$$l \in (K \odot N) \odot L \;=\; \{x \in L \,|\, f\big(g(x)\big) \in K \quad \forall\, f : N \to M, \;\; \forall\, g : L \to M\}.$$

For every pair of morphisms $g : L \to M$ and $f : N \to M$ we can define the morphism $h : N \odot L \to M$ by the rule:

$$h(m) = f\big(g(m)\big) \quad \forall\, m \in N \odot L,$$

using the fact that $g(m) \in N$ by the definition of $N \odot L$.

From $l \in (K \odot N) \odot L$ it follows $h(l) \in K$, therefore $h(l) = f\big(g(l)\big) \in K$ for every $f : N \to M$ and $g : L \to M$, but this means that $l \in (K \odot N) \odot L$. □

For the study of relation between ω-product and intersection in $\boldsymbol{L}({}_RM)$ the following remark is useful.

Lemma 3.4. *For every submodules $N, K \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\omega^M_{N\cap K} \;=\; \omega^M_N \;\wedge\; \omega^M_K.$$

Proof. By definitions, for every module $X \in R$-Mod we have:

$$(\omega^M_N \;\wedge\; \omega^M_K)\,(X) \;=\; \omega^M_N(X) \;\bigcap\; \omega^M_K(X) =$$

$$= \big\{x \in X \mid f(x) \in N \;\; \forall\, f : X \to M\big\} \;\bigcap\; \big\{x \in X \mid f(x) \in K \;\; \forall\, f : X \to M\big\} =$$

$$= \big\{x \in X \mid f(x) \in N \;\bigcap\; K \;\; \forall\, f : X \to M\big\} = \omega^M_{N\cap K}(X).$$ □

Proposition 3.5. *For every module $M \in R$-Mod the operation of ω-product is left distributive with respect to the intersection of submodules:*

$$(K_1 \;\bigcap\; K_2) \odot N \;=\; (K_1 \odot N) \;\bigcap\; (K_2 \odot N).$$

Proof. Applying Lemma 3.4 we obtain:

$$(K_1 \;\bigcap\; K_2) \odot N \;=\; \omega^M_{K_1 \bigcap K_2}(N) \;=\; (\omega^M_{K_1} \;\wedge\; \omega^M_{K_2})\,(N) \;=$$

$$= \omega^M_{K_1}(N) \;\bigcap\; \omega^M_{K_2}(N) \;=\; (K_1 \odot N) \;\bigcap\; (K_2 \odot N).$$ □

In the particular case when ${}_RM = {}_RR$ we have the specification of ω-product in the lattice $\boldsymbol{L}({}_RR)$ of left ideals of the ring R. For every left ideals $J, I \in \boldsymbol{L}({}_RR)$ we obtain:

$$J \;\odot\; I = \omega^R_J(I) = \bigcap_{f\,:\,I \to R} f^{-1}(J) = \{i \in I \mid f(i) \in J \;\; \forall\, f : {}_RI \to {}_RR\} \;\subseteq\; J \;\bigcap\; I.$$

4 $\boldsymbol{\alpha}$-coproduct of submodules

The following two operations which will be introduced in continuation are in some sense dual to the previous operations (α-product and ω-product) and are obtained by replacing the product of preradicals with its coproduct (Section 1).

Definition 4.1. Let $M \in R$-Mod and $N, K \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$(N : K) = \pi_N^{-1}\big(\alpha_K^M(M \,/\, N)\big) = \{m \in M \mid m + N \in \sum_{f\,:\,M \to M/N} f(K)\}$$

will be called the *α-coproduct* in M of submodules N and K, where $\pi_N : M \to M \,/\, N$ is the natural morphism. In other form:

$$(N : K) \,/\, N = \alpha_K^M(M \,/\, N).$$

Some properties of α-coproduct are collected in

Proposition 4.1. *Let $M \in R$-Mod and $N, K \in \boldsymbol{L}({}_RM)$. Then:*

1) $(N : K) \supseteq N + K$;
2) $(N : K) \,/\, N$ *is a characteristic submodule in* $M \,/\, N$; *if* $N \in \boldsymbol{L}^{ch}({}_RM)$, *then* $(N : K) \in \boldsymbol{L}^{ch}({}_RM)$;
3) *If* $N + K = M$ *(in particular, if* $N = M$ *or* $K = M$*), then* $(N : K) = M$;
4) *If* $N = 0$, *then for every* $K \in \boldsymbol{L}({}_RM)$ *the submodule* $(0 : K)$ *is the least characteristic submodule of* M *containing* K; *therefore if* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $(0 : K) = K$;
5) *If* $K = 0$, *then* $(N : 0) = N$ *for every* $N \in \boldsymbol{L}({}_RM)$. □

Proposition 4.2. *The operation of α-coproduct is monotone in both variables:*

$$N_1 \subseteq N_2 \;\Rightarrow\; (N_1 : K) \subseteq (N_2 : K) \qquad \forall\, K \in \boldsymbol{L}({}_RM);$$
$$K_1 \subseteq K_2 \;\Rightarrow\; (N : K_1) \subseteq (N : K_2) \qquad \forall\, N \in \boldsymbol{L}({}_RM).$$ □

Proposition 4.3. *If the module $M \in R$-Mod is projective, then for every submodules $N, K, L \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\big((N : K) : L\big) \subseteq \big(N : (K : L)\big).$$

Proof. Let $m \in \big((N : K) : L\big)$. Then by definition we have $m + (N : K) \in \alpha_L^M\big(M \,/\, (N : K)\big)$, i.e. $m + (N : K) = \sum_{g_i : M \to M/(N:K)} g_i(l_i)$, where $l_i \in L$. Since ${}_RM$ is projective, for every morphism $g_i : M \to M \,/\, (N : K)$ there exists a morphism $f_i : M \to M \,/\, N$ such that $\varphi f_i = g_i$, where $\varphi : M \,/\, N \to M \,/\, (N : K)$ is the epimorphism determined by the inclusion $N \subseteq (N : K)$ $\big($i.e. $\varphi(m + N) = m + (N : K)\big)$. Therefore:

$$m + (N : K) = \sum_{g_i : M \to M/(N:K)} g_i(l_i) \ =$$

$$= \sum_{f_i : M \to M/N} (\varphi f_i)\,(l_i) \ = \ \varphi\Big(\sum_{f_i : M \to N} f_i(l_i)\Big) \in M \,/\, (N : K).$$

Considering the inverse image in $M \,/\, N$ we have:

$$(m + N) \ - \sum_{f_i : M \to M/N} f_i(l_i) \ \in \ Ker\,\varphi \ = \ (N : K) \,/\, N \ = \ \alpha_K^M(M \,/\, N),$$

and so $(m+N) - \sum_{f_i : M \to M/N} f_i(l_i) \ = \sum_{f_j : M \to M/N} f_j(k_j)$, where $k_j \in K$. Therefore:

$$m + N \ = \sum_{f_i : M \to M/N} f_i(l_i) \ + \sum_{f_j : M \to M/N} f_j(k_j) \ \in \ \alpha_{(K:L)}^M(M \,/\, N),$$

since $l_i \in L \subseteq (K : L)$ and $k_j \in K \subseteq (K : L)$. By definition this means that $m \in \big(N : (K : L)\big)$. □

Proposition 4.4. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\big(N : (K_1 + K_2)\big) = (N : K_1) + (N : K_2),$$

i.e. the α-coproduct is right distributive with respect to the sum of submodules.

Proof. By Lemma 2.5 we have $\alpha_{K_1+K_2}^M = \alpha_{K_1}^M \ \vee \ \alpha_{K_2}^M$, therefore:

$$\big(N : (K_1 + K_2)\big) \,/\, N \ = \ \alpha_{K_1+K_2}^M(M \,/\, N) \ = \ \alpha_{K_1}^M(M \,/\, N) \ + \ \alpha_{K_2}^M(M \,/\, N) =$$

$$= \ \big[(N : K_1) \,/\, N\big] \ + \ \big[(N : K_2) \,/\, N\big] \ = \ \big[(N : K_1) + (N : K_2)\big] \,/\, N,$$

which implies the statement. □

Now we concretize the operation of α-coproduct for the lattice of left ideals $\boldsymbol{L}({}_RR)$ of the ring R.

Proposition 4.5. *For every left ideals N, $K \in \boldsymbol{L}({}_RR)$ the following relation is true:*

$$(N : K) \ = \ KR + N.$$

Proof. By definition $(N : K) = \pi_N^{-1}\big(\alpha_K^R(R \,/\, N)\big)$, where $\pi_N : R \to R \,/\, N$ is the natural morphism. Since $Hom_R(R, R \,/\, N) \ \cong \ R \,/\, N$, we have

$$\alpha_K^R(R \,/\, N) \ = \sum_{f : R \to R/N} f(K) \ = \ \sum_{r \in R} K(r + N) \ = \ K\big(\sum_{r \in R}(r + N)\big) \ =$$

$$= \ K(R \,/\, N) = (KR + N) \,/\, N,$$

therefore $(N : K) = KR + N$. □

If K is an ideal, then $(N : K) = N + K$ for every $N \in \boldsymbol{L}({}_RR)$. So in the lattice $\boldsymbol{L}^{ch}({}_RR)$ of two-sided ideals of R the α-coproduct coincides with the ordinary sum of ideals.

In particular from Proposition 4.5 it follows also that

$$(N : K)L = (KR + N)L = KRL + NL = KL + NL =$$
$$= (K + N)L = (N + K)L = (K : N)L$$

for every N, K, $L \in \boldsymbol{L}({}_RR)$.

5 $\boldsymbol{\omega}$-coproduct of submodules

In this section we consider an operation in $\boldsymbol{L}({}_RM)$ similar to the α-coproduct replacing α_N^M by ω_N^M.

Definition 5.1. Let $M \in R$-Mod and N, $K \in \boldsymbol{L}({}_RM)$. The following submodule of M:

$$(N \odot K) = \pi_N^{-1}\big(\omega_K^M(M/N)\big) = \{m \in M \mid m + N \in \bigcap_{f\,:\,M/N \to M} f^{-1}(K)\} =$$

$$= \{m \in M \mid f(m + N) \in K \ \ \forall f : M/N \to M\}$$

will be called the *ω-coproduct* in M of submodules N and K, where $\pi_N : M \to M/N$ is the natural morphism. Therefore:

$$(N \odot K)/N = \omega_K^M(M/N) = \bigcap_{f\,:\,M/N \to M} f^{-1}(K).$$

The ω-coproduct $(N \odot K)$ can be expressed in other form ([3]), using the fact that there exists a bijection between the morphisms $g : M \to M$ with the condition $g(N) = 0$, and all morphisms $f : M/N \to M$. Taking this into account, we can present the ω-coproduct as follows:

$$(N \odot K) = \big\{m \in M \mid g(m) \in K \ \ \forall g : M \to M, \ \ g(N) = 0\big\}.$$

If N, $K \in \boldsymbol{L}^{ch}({}_RM)$ then this operation coincides with the ω-coproduct of characteristic submodules defined in [5] by the rule:

$$(N \odot K) = (\omega_N^M : \omega_K^M)(M).$$

This operation (in other notations and other order of terms) was used in [3] for the study of coprime modules. The continuation of these studies is in [6], where coprime preradicals and coprime modules are investigated. As in the previous cases we start with some elementary properties of this operation.

Proposition 5.1. 1) *$(N \odot K) \supseteq N$ and $(N \odot K)/N$ is a characteristic submodule of M/N;*

2) *If $N = M$, then $(M \odot K) = M$ for every $K \in \boldsymbol{L}({}_RM)$;*

3) *If* $K = M$, *then* $(N \odot M) = M$ *for every* $N \in \boldsymbol{L}({}_RM)$;

4) *If* $N = 0$, *then* $(0 \odot K)$ *is the largest characteristic submodule contained in* K *for every* $K \in \boldsymbol{L}({}_RM)$; *so if* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $(0 \odot K) = K$;

5) *If* $K = 0$, *then* $(N \odot 0) = \pi_N^{-1}\Big(\bigcap\limits_{f\,:\,M/N \to M} Ker f\Big)$ *for every* $N \in \boldsymbol{L}({}_RM)$, *where* $\pi_N : M \to M\big/N$ *is the natural morphism;*

6) *If* $N \in \boldsymbol{L}^{ch}({}_RM)$, *then* $(N \odot K) \in \boldsymbol{L}^{ch}({}_RM)$ *for every* $K \in \boldsymbol{L}({}_RM)$;

7) *If* $N, K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $(N \odot K) \supseteq K$, *therefore* $(N \odot K) \supseteq N + K$. □

Proposition 5.2. *The operation of* ω*-coproduct is monotone in both variables:*

$$N_1 \subseteq N_2 \quad \Rightarrow \quad (N_1 \odot K) \subseteq (N_2 \odot K) \qquad \forall\, K \in \boldsymbol{L}({}_RM);$$

$$K_1 \subseteq K_2 \quad \Rightarrow \quad (N \odot K_1) \subseteq (N \odot K_2) \qquad \forall\, N \in \boldsymbol{L}({}_RM). \qquad \square$$

Two results on associativity of this operation are mentioned in [3] (Lemma 4.1). We remind these statements with short proofs.

Proposition 5.3. *For every* $M \in R$*-Mod the relation*

$$\big((N \odot K) \odot L\big) \subseteq \big(N \odot (K \odot L)\big)$$

is true, where $N, K, L \in \boldsymbol{L}({}_RM)$.

Proof. By definition we have:

$m \in \big((N \odot K) \odot L\big) \Leftrightarrow g(m) \in L \quad \forall\, g : M \to M, \quad g(N \odot K) = 0;$

$m \in \big(N \odot (K \odot L)\big) \Leftrightarrow f(m) \in (K \odot L) \quad \forall\, f : M \to M, \quad f(N) = 0 \Leftrightarrow$

$\Leftrightarrow hf(m) \in L \quad \forall\, h : M \to M, \quad h(K) = 0$ and $\forall\, f : M \to M, \quad f(N) = 0.$

If $m \in \big((N \odot K) \odot L\big)$ and we have a pair of morphisms $f, h : M \to M$ such that $f(N) = 0$ and $h(K) = 0$, then by definition $f(N \odot K) \subseteq K$ and so $hf(N \odot K) = 0$. By assumption, $hf(m) \in L$ for every such pair of morphisms, and by definition this means that $m \in \big(N \odot (K \odot L)\big)$. □

Proposition 5.4. *If* ${}_RM$ *is injective and artinian, then the operation of* ω*-coproduct in* $\boldsymbol{L}({}_RM)$ *is associative:*

$$\big((N \odot K) \odot L\big) = \big(N \odot (K \odot L)\big),$$

for every $N, K, L \in \boldsymbol{L}({}_RM)$.

Proof. Since ${}_RM$ is artinian there exists a finite number of endomorphisms $f_1, \ldots, f_n : M \to M$ with $f_j(N) = 0$ such that $(N \odot K) = \bigcap\limits_{j=1}^{n} f_j^{-1}(N)$. We define the morphism $t : M\big/(N \odot K) \to \prod\limits_{1}^{n}(M\big/K)$ by the rule: $t\big(m + (N \odot K)\big) = \big(f_1(m) + K, \ldots, f_n(m) + K\big)$ and observe that t is a monomorphism.

Let $m \in \big(N \odot (K \odot L)\big)$, i.e. $hf(m) \in L$ for every $f, h : M \to M$ with $f(N) = 0$ and $h(K) = 0$. Let $g : M \to M$ be an arbitrary morphism with $g(N \odot K) = 0$. Then g can be expressed in the form $g = g' \cdot \pi_{(N \odot K)}$, where $\pi_{(N \odot K)} : M \to M / (N \odot K)$ is natural and $g' \in Hom_R\big(M / (N \odot K), M\big)$. Since M is injective and t is mono, there exists a morphism $q : \prod_1^n (M / K) \to M$ such that $g' = qt$.

Now we consider the morphisms $u_j = i_j \pi_K : M \to \prod_1^n (M / K)$ $(j = 1, \ldots, n)$, where $\pi_K : M \to M / K$ is natural, and $i_j : M / K \to \prod_1^n (M / K)$ are the canonical injections. Then:

$$g(m) = q\, t\, \pi_{(N \odot K)}(m) = q\, t(m + (N \odot K)\big) = q(f_1(m) + K, \ldots, f_n(m) + K) =$$
$$= q\big(\pi_K f_1(m), \ldots, \pi_K f_n(m)\big) = q\big(i_1 \pi_K f_1(m) + \ldots + i_n \pi_K f_n(m)\big) =$$
$$= q\big(u_1 f_1(m), \ldots, u_n f_n(m)\big) = q\, u_1\, f_1(m) + \ldots + q\, u_n\, f_n(m),$$

where the morphism $h_j = q\, u_j : M \to M$ has the property $h_j(K) = 0$, and the morphisms f_j are given with $f_j(N) = 0$. From the assumption that $m \in \big(N \odot (K \odot L)\big)$ we obtain $q\, u_j\, f_j(m) \in L$ for every $j = 1, \ldots, n$, so $g(m) \in L$ for every $g : M \to M$ with $g(N \odot K) = 0$. By definition this means that $m \in \big((N \odot K) \odot L\big)$, proving the inclusion $(\supseteq)$, the inverse inclusion is true by Proposition 5.3. □

Now we will prove the right distributivity of ω-product in $\boldsymbol{L}({}_RM)$ with respect to the intersection of submodules.

Proposition 5.5. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}({}_RM)$ the following relation is true:*

$$\big(N \odot (K_1 \cap K_2)\big) = (N \odot K_1) \cap (N \odot K_2).$$

Proof. By Lemma 3.4 we have $\omega^M_{K_1 \cap K_2} = \omega^M_{K_1} \wedge \omega^M_{K_2}$, therefore:

$$\big(N \odot (K_1 \cap K_2)\big) / N = \omega^M_{K_1 \cap K_2}(M / N) = \omega^M_{K_1}(M / N) \cap \omega^M_{K_2}(M / N) =$$
$$= \big[(N \odot K_1) / N\big] \cap \big[(N \odot K_2) / N\big] = \big[(N \odot K_1) \cap (N \odot K_2)\big] / N,$$

which implies the statement. □

Remark. The distributivity relations from Propositions 2.6, 3.5, 4.4 and 5.5 can be generalized to infinite distributivity, i.e. the following relations are true:

$$\Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) \cdot N = \sum_{\alpha \in \mathfrak{A}} (K_\alpha \cdot N), \quad \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\Big) \odot N = \bigcap_{\alpha \in \mathfrak{A}} (K_\alpha \odot N),$$
$$\Big(N : \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big)\Big) = \sum_{\alpha \in \mathfrak{A}} (N : K_\alpha), \quad \Big(N \odot \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\Big)\Big) = \bigcap_{\alpha \in \mathfrak{A}} (N \odot K_\alpha).$$

Finally, we will specify the form of ω-coproduct in the lattice $\boldsymbol{L}({}_RR)$ of left ideals of R. Let N, $K \in \boldsymbol{L}({}_RR)$. By definition we have:

$$(N \odot K) \;=\; \big\{a \in R \mid g(a) \in K \quad \forall\, g \,:\, {}_RR \to {}_RR \;\text{ with }\; g(N) \;=\; 0\big\}.$$

If for $g \,:\, {}_RR \to {}_RR$ we denote $a_g = g(1_R)$, then $g(a) = a \cdot a_g$ for every $a \in R$ and $Ker\, g = \{a \in R \mid a \cdot a_g = 0\} = (0 : a_g)_l$ (left annihilator of a_g). The condition $g(N) = 0$ means that $N \cdot a_g = 0$, i.e. $a_g \in (0 : N)_r$ (right annihilator of N).

If $a \in (N \odot K)$, then $g(a) \in K$, i.e. $a \cdot a_g \in K$ or $a \in (K : a_g)_l$ for every $g \,:\, {}_RR \to {}_RR$ with $a_g \in (0 : N)_r$. So we obtain that $a \in \big(K : (0 : N)_r\big)_l$. Therefore:

$$(N \odot K) = \big(K : (0 : N)_r\big)_l.$$

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Received March 20, 2011