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On the Abstract Čech Cohomology

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Abstract. The present Note is a survey of the authors' papers [11,12,14,16–18], concerning the introduction and study of the notion of the abstract Čech cohomology, as well as its applications. Here we have investigated: projective systems, injective systems, covering of a directed partially ordered set, abstract Čech cohomology, abstract Čech homology, Čech cohomology space, simplicial projective systems, de Rham cohomology space of projective systems, J-resolution of a projective system and acyclic resolution.

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1 Introduction

The Čech cohomology is a cohomology theory based on the intersection properties of open covers of a topological space. It is named for the mathematician Eduard Čech who in 1932 introduced it [2]. Let X be a topological space, and let $\mathcal{U}$ be an open cover of X. Define a simplicial complex $N(\mathcal{U})$ called the nerve of the covering, as follows: the vertices of $N(\mathcal{U})$ are all elements of $\mathcal{U}$, each pair $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$ determines one edge, in general, there is one q-simplex for each $(q+1)$-element subset $\{U_0, ..., U_q\}$ for which $U_0 \cap ... \cap U_q \neq \emptyset$. Geometrically, the nerve $N(\mathcal{U})$ is essentially a "dual complex" (in the sense of a dual graph, or Poincaré duality) for the covering $\mathcal{U}$. The idea of Čech cohomology is that, if we choose a cover $\mathcal{U}$ consisting of sufficiently small, connected open sets, the resulting simplicial complex should be a good combinatorial model for the space X. For such a cover, the Čech cohomology of X is defined to be the simplicial cohomology of the nerve. This idea can be formalized by the notion of a *good cover*, for which every open set and every finite intersection of open sets is contractible. However, a more general approach is to take the direct limit of the cohomology groups of the nerve over the system of all possible open covers of X, ordered by refinement. For a more precise description see [20], Chap. 6, Sec. 7. Let X be a topological space, and let $\mathcal{F}$ be a presheaf of abelian groups on X. Let $\mathcal{U}$ be an open cover of X. A q-simplex σ of $\mathcal{U}$ is an ordered collection of $q+1$ sets chosen from $\mathcal{U}$ such that the intersection of all these sets is non-empty. This intersection is called the *support* of σ and is denoted $|\sigma|$. Now let $\sigma = (U_0, ..., U_q)$ be such a q-simplex. The j-th partial boundary of σ is defined to be the $(q-1)$-simplex obtained by removing the j-th set from σ, that is: $\partial_j\sigma := (U_i)_{i\in\{0,\ldots,q\},i\neq j}$. The *boundary* of σ is defined as the alternating sum of the partial boundaries: $\partial\sigma := \sum\limits_{j=0}^{q}(-1)^{j+1}\partial_j\sigma$. A q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a map which associates to each q-simplex σ an element of $\mathcal{F}(|\sigma|)$ and we denote the set of all q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ by $C^q(\mathcal{U},\mathcal{F})$. $C^q(\mathcal{U},\mathcal{F})$ is an abelian group by pointwise addition. The cochain groups can be made into a cochain complex $(C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)$ by defining a coboundary operator (also called codifferential)

$$\delta_q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(U,F) : \omega \to \delta_q\omega, (\delta_q\omega)(\sigma) := \sum_{j=0}^{q+1}(-1)^j res^{|\partial_j\sigma|}_{|\sigma|}\omega(\partial_j\sigma),$$

(where $res^{|\partial_j\sigma|}_{|\sigma|}$ is the restriction morphism from $\mathcal{F}(|\partial_j\sigma|)$ to $\mathcal{F}(|\sigma|)$), and showing that $\delta^2 = 0$ (i.e., $\delta^{q+1}\circ\delta^q = 0$). A q-cochain is called a *q-cocycle* if it is in the kernel of δ_q and $Z^q(\mathcal{U},\mathcal{F}) := ker(\delta_q)$ is the set of all q-cocycles. Thus a q-cochain ω is a cocycle if for any $(q+1)$-simplex σ the cocycle condition $\sum\limits_{j=0}^{q+1}(-1)^j res^{|\partial_j\sigma|}_{|\sigma|}\omega(\partial_j\sigma) = 0$ holds. For example, ω is a 1-cocycle if $\forall A,B,C\in\mathcal{U}$, $\omega(B\cap C)|_{A\cap B\cap C} - \omega(A\cap C)|_{A\cap B\cap C} + \omega(A\cap B)|_{A\cap B\cap C} = 0$, where, for $U'\subset U''$, $\omega(U'')|_{U'}$ denotes $res^{U''}_{U'}$.

A q-cochain is called a *q-coboundary* if it is in the image of δ_{q-1} and $B^q(\mathcal{U},\mathcal{F})$ is the set of all q-coboundaries. For example, a 1-cochain ω is a 1-coboundary if there exists a 0-cochain ϖ such that $\forall A,B\in\mathcal{U}$, $\omega(A\cap B) = \delta_0(\varpi)(A\cap B) = \varpi(A)|_{A\cap B} - \varpi(B)|_{A\cap B}$.

The Čech cohomology of $\mathcal{U}$ with values in $\mathcal{F}$ is defined to be the cohomology of the cochain complex $(C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)$. Thus the q-th Čech cohomology is given by

$$\check{H}^q(\mathcal{U},\mathcal{F}) := H^q((C^{\cdot}(\mathcal{U},\mathcal{F}),\delta)) = Z^q(\mathcal{U},\mathcal{F})/B^q(\mathcal{U},\mathcal{F}).$$

The Čech cohomology of X is defined by considering refinements of open covers. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ then there is a map in cohomology $\check{H}^*(\mathcal{U},\mathcal{F}) \to \check{H}^*(\mathcal{V},\mathcal{F})$. The open covers of X form a directed set under refinement, so the above map leads to a direct system of abelian groups. The Čech cohomology of X with values in $\mathcal{F}$ is defined as the direct limit $\check{H}^*(X,\mathcal{F}) = \varinjlim\limits_{\mathcal{U}} \check{H}^*(\mathcal{U},\mathcal{F})$ of this system. Actually, the original Čech cohomology of X with coefficients in a fixed abelian group A, denoted $\check{H}^*(X;A)$, is defined as $\check{H}^*(X,\mathcal{F}_A)$ where $\mathcal{F}_A$ is the constant sheaf on X determined by A.

An excellent presentation of Čech cohomology was made by Kostake Teleman in [21] (Chp. II, Sect. 18). Probably it was one of the reasons why his book was translated in German and Russian, shortly after appearing in Romanian (see [Zbl 018953902], [Zbl 018953904]). In particular, K. Teleman proved that if X is homotopy equivalent to a CW-complex, then the Čech cohomology $\check{H}^*(X;A)$ is naturally isomorphic to the singular cohomology $H^*(X;A)$. (For an arbitrary space X this fact is false: if X is the closed topologist's sine curve, then $\check{H}^0(X;\mathbb{Z}) = \mathbb{Z}$, whereas $H^0(X;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$). Also, K. Teleman proved that if $\mathcal{S}$ is a locally finite simplicial polyhedron, then the singular cohomology $H^*(\mathcal{S};\mathbb{Z})$ and Čech cohomology $\check{H}^*(\mathcal{S};\mathbb{Z})$ are isomorphic with the cohomology $\check{H}^*(\Sigma,\mathbb{Z})$, associated to the cover Σ of $\mathcal{S}$ with stellar neighborhoods.

If X is a differentiable manifold and the cover $\mathcal{U}$ of X is a good "cover" (i.e., all the sets $U \in \mathcal{U}$ are contractible to a point, and all finite intersections of sets in $\mathcal{U}$ are either empty or contractible to a point), then $\check{H}^*(X;\mathbb{R})$ is isomorphic to the de Rham cohomology.

If X is compact Hausdorff, then Čech cohomology (with coefficients in a discrete group) is isomorphic to Alexander-Spanier cohomology.

In the articles [5] and [6], René Deheuvels develops a theory of homology of ordered sets which is a generalization of the Čech homology and cohomology. This author starts with an Abelian category C, with products and enough injectives, and with an ordered set $\mathcal{E}$. He considers $\mathcal{E}$, as a category in the usual way, having the objects all elements of $\mathcal{E}$ and a morphism $a_1 \to a_2$ being a relation $a_1 < a_2$. He denotes by $C(\mathcal{E})$ the category of covariant functors from the category $\mathcal{E}$ to the category C. Let $C^{\cdot}$ be the category of cochain complexes in the category C and let the functor $C^{\cdot}(\mathcal{E},-) : C(\mathcal{E}) \to C^{\cdot}$ be defined by: $C^n(\mathcal{E},A) = \prod\limits_{a_0>\ldots>a_n, a_i\in\mathcal{E}} A(a_0)$, $d : C^n(\mathcal{E},A) \to C^{n+1}(\mathcal{E},A)$, where if $f = (f_{a_0>\ldots>a_n}) \in C^n(\mathcal{E},A)$, $d(f)_{a_0>\ldots>a_{n+1}} = \eta^{a_1}_{a_0}(f_{a_1>\ldots>a_{n+1}}) + \sum\limits_{i=1}^{n+1} f_{a_0>\ldots>\hat{a}_i>\ldots>a_{n+1}}$, $\eta^{a_1}_{a_0}$ being the morphism $A(a_1) \to A(a_0)$ corresponding to the relation $a_1 < a_0$. The author proves that this functor is a resolving functor, i.e., for every n there is a canonical isomorphism $R^n\Gamma_{\mathcal{E}} \simeq H^n(C^{\cdot}(\mathcal{E},-))$, where $\Gamma_{\mathcal{E}}$ is the inverse limit functor $\Gamma_{\mathcal{E}} := \varprojlim\limits_{\mathcal{E}}$. The construction and this result have obvious duals. Let M be a "schéma simplicial" and let E be the set of simplices of M ordered by inclusion. If A is a constant functor then the author proves that $R^n\Gamma_{\mathcal{E}}(A)$ is isomorphic to the usual simplicial cohomology of M with coefficients in A. To define a generalized Čech cohomology and homology theory, Deheuvels introduced the notion of "order" of the ordered set $\mathcal{E}$ in the ordered set $\mathcal{E}'$. This is a function ρ in $\mathcal{E}$ with values in the set of subsets of $\mathcal{E}'$ such that if $a_2 \leq a_1$ then $\rho(a_2) \supseteq \rho(a_1)$ and if $a'_2 < a'_1, a'_2 \in \rho(a)$, then $a'_1 \in \rho(a)$. Given an "order" ρ of $\mathcal{E}$ in $\mathcal{E}'$, the author defines a functor $\rho^{-1} : C(\mathcal{E}') \to C(\mathcal{E}^*)$, where $\mathcal{E}^*$ is the dual of $\mathcal{E}$, by $\rho^{-1}(A')(a) = \Gamma_{\rho(a)}(A')$. Modulo technicalities the generalized Čech cohomology in the sense of Deheuvels is now the hyperderived of the composed functor $L_{\mathcal{E}^*}\rho^{-1}$ with $L_{\mathcal{E}} = \varinjlim\limits_{\mathcal{E}}$. The dual construction yields a generalized Čech homology theory. Let X be a topological space, let $\mathcal{E}$ be the dual of the set of non-empty open sets of X ordered by inclusion, and let $\mathcal{D}$ be the ordered set of all open coverings R of X. It is supposed that if $O \subseteq O'$ and $O' \in R$ then $O \in R$. The category $C(\mathcal{E})$ is then the category of presheaves on X. The "order" ρ of $\mathcal{D}$ in $\mathcal{E}$ is defined by $\rho(R) = R$. The corresponding generalized Čech cohomology in the Deheuvels sense is then shown to coincide with the usual Čech cohomology, at least when C has exact inductive limits. If X is a compact metric space and C is the category of abelian groups the author shows that the generalized Čech homology coincides with Steenrod homology theory.

We can see from this summary of the work [6] of René Deheuvels that this theory is indeed a very consistent generalization of the homology and cohomology Čech theories. But at the same time, it is clear that Deheuvels's theory is very sophisticated and difficult to apply and to find other examples. In addition, even the construction of this theory is very little similar to the construction of the Čech theory. This is the reason for which in 1974 the first and the third author proposed a theory of *abstract Čech cohomology* in [11] and [12], and not a generalization of the Čech theory as constructed Deheuvels, but simply following the Čech's construction. It is more easily applicable in other important situations. In the third chapter of the book [12], entitled "Simplicial complexes. Abstract Čech cohomology", this theory is developed in detail as follows: §3 Cohomology groups associated with a projective system or Čech abstract cohomology groups, §4 The exact cohomology sequence associated with a pair of projective systems, §5 Canonical projective systems, §6 Resolutions of a projective system, §7. Homology groups associated with an injective system. And in Chap. VII, the Čech homology and cohomology for a topological space are obtained as a "concretization" of the abstract Čech homology and cohomology. Then, in the thesis of the second author [15] and in his papers [14, 16, 17], as well as in the paper [18] of the third author, a number of examples and applications are given.

The present article is a synthesis paper including the results of the three authors about the abstract Čech homology and cohomology.

Finally, the authors wish to emphasize that they are impressed by recent research concerning multy-ary relations homology, studied by Academician Petru Soltan in [19] and [1]. They believe that this subject can be expanded by using abstract Čech (co)homology, as well as the theory of abstract Čech (co)homology can find one new, interesting and important application in the above mentioned field investigated by Academician Petru Soltan.

2 The authors' construction of the abstract Čech cohomology

Let $\mathcal{P} = (H_i, \alpha^i_j)_{i,j\in I}$ be a projective system of abelian groups. For our purpose we suppose that the partially ordered set $(I, \leq)$ of the indices over which the projective system $\mathcal{P}$ (or an inductive system $\mathcal{I}$) is given fulfills the following conditions:

(1) For every pair $i, j \in I$ there exists infimum $\inf(i, j)$, which is denoted by $i \wedge j$;

(2) For every subset J of I there exists $\sup J$, the supremum with respect to the relation $\leq$;

(3) There exists a minimal element $\theta \in I$, i.e., $\theta \leq i$ for every $i \in I$, (but this condition is not essential).

Definition 1. A subset J of I is called a *covering* of $(I, \leq)$ if for every $i \in I$ there exists $J_i \subset J$ such that $i \leq \sup J_i$.

In the set of coverings of $(I, \leq)$ a partial ordering can be introduced, namely, if J, J' are coverings of $(I, \leq)$, then $J' \prec J$ if for every $i' \in J'$ there exists an $i \in J$

such that $i' \leq i$.

In order to define the cohomology groups of a projective system $\mathcal{P} = (H_i, \alpha_j^i)_{i,j \in I}$, also we assume the condition

(4) The set of coverings of $(I, \leq)$ is directed with respect to the relation $\prec$, i.e., if J, J' are two arbitrary coverings of $(I, \leq)$, then there exists a covering J'' of $(I, \leq)$ such that $J'' \prec J$ and $J'' \prec J'$.

In these conditions ((1)-(4)) on the ordered set $(I, \leq)$, for a covering J of $(I, \leq)$ we can consider the cochain complex

$$C^*(J, \mathcal{P}) : ... \to C^q(\mathcal{J}, P) \xrightarrow{d^q} C^{q+1}(J, \mathcal{P}) \to ...,$$

where $C^q(J, \mathcal{P}) := \prod\limits_{i_0,...,i_q \in J; i_0 \wedge ... \wedge i_q \neq \theta} H_{i_0 \wedge .... \wedge i_q}$, and the boundary homomorphism d^q is defined by

$$(d^q t)_{i_0 ... i_{q+1}} = \sum_{p=0}^{q+1} (-1)^p \alpha_{i_0 \wedge .... \wedge i_{q+1}}^{i_0 \wedge ... \wedge \widehat{i}_p \wedge ... \wedge i_{q+1}} t_{i_0 \wedge ... \wedge \widehat{i}_p \wedge ... \wedge i_{q+1}}$$

for $t \in C^q(J, \mathcal{P})$.

The cohomology groups of this cochain complex are denoted by $\{(H^q(J, P)\}_q$. If J' is another covering of $(I, \leq)$ such that $J' \prec J$, one obtains, for every $q \geq 0$, a well defined homomorphism $\alpha^q_{J' \prec J} : H^q(J, \mathcal{P}) \to H^q(J', \mathcal{P})$ such that $\{H^q(J, P), \alpha^q_{J', J}\}_{J, J' \in \mathcal{A}_{(I, \leq)}}$ is an inductive system over the set $(\mathcal{A}_I, \prec)$ of all coverings of $(I, \leq)$.

In the imposed conditions there exists $\varinjlim\limits_{J \in \mathcal{A}_{(I, \leq)}} H^q(J, \mathcal{P})$, and this group, denoted by $\check{H}^q(\mathcal{P})$ or $\check{H}^q((I, \leq), \mathcal{P})$, is called the *q-th cohomology group of the projective system* $\mathcal{P}$.

If $(I, \leq)$ is an ordered set satisfying the conditions (1)-(4), denote by $\underleftarrow{(I, \leq)}\ (Ab)$ the category of projective systems of abelian groups indexed over $(I, \leq)$ and of morphisms of projective systems.

Proposition 1. *([12], Cor. 3.3, p. 100). For every integer $q \geq 0$ we can define a covariant functor*

$$\check{H}^q((I, \leq), -) : \underleftarrow{(I, \leq)}\ (Ab) \longrightarrow Ab,$$

which assigns to every projective system $\mathcal{P}$ indexed over $(I, \leq)$ the q-th abstract Čech cohomology group $\check{H}^q(\mathcal{P})$.

Proposition 2. *([12], Prop. 4.1, p. 101). For any exact sequence $0 \to P' \xrightarrow{\varphi} P \xrightarrow{\psi} P'' \to 0$ in the category $\underleftarrow{(I, \leq)}\ (Ab)$ we get the following exact sequence of abstract Čech cohomology*

$$0 \to \check{H}^0((I, \leq), \mathcal{P}') \xrightarrow{\varphi^{*0}} \check{H}^0((I, \leq), \mathcal{P}) \xrightarrow{\psi^{*0}} H^0((I, \leq), \mathcal{P}'') \to \check{H}^1((I, \leq), \mathcal{P}')....$$

Let $\mathcal{P} = (H_i, \alpha_{ij})$ be an object in the category $(\underleftarrow{I, \leq})(\mathrm{Ab})$. An element $t_i \in H_i$ is called a *section* of $\mathcal{P}$ over the index i. A system of sections $\{t_i\}_{i \in J \subset I}$ is called *coherent* if $\alpha_{i \wedge j, i} t_i = \alpha_{i \wedge j, j} t_j$, for any $i, j \in I$.

Definition 2. A projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *complete* if for any coherent system of sections $\{t_i\}_{i \in J}$, there exists a section $t_k \in H_k$, $k = \sup J$, such that $\alpha_{ik} t_k = t_i$, for all $i \in J$.

The projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *essential* if the following property is satisfied: for $k \in I$ and $t_k \in H_k$, there exists $J \subset I$ such that $k = \sup J$ and $\alpha_{ik} t_k = 0$ for every $i \in J$ implies t_k=0.

The projective system $\mathcal{P} = (H_i, \alpha_{ij})$ is called *canonical* if it is complete and essential.

Proposition 3. *([12], Prop. 5.2, p. 103). If $\mathcal{P} = (H_i, \alpha_{ij})$ is a canonical projective system over $(I, \leq)$ for which there exists $k = \sup I$, then $\check{H}^0(P) \simeq H_k$.*

Let $\mathcal{P} = (H_i, \alpha_{ij})$ be a projective system over $(I, \leq)$ satisfying the conditions (1)-(4).

Definition 3. A *cohomological resolution* of $\mathcal{P} = (H_i, \alpha_{ij})$ is an exact sequence in the category $(\underleftarrow{I, \leq})(\mathrm{Ab})$,

(RP):...$\to \mathcal{P}^{n-1} \xrightarrow{D^{n-1}} \mathcal{P}^n \xrightarrow{D^n} \mathcal{P}^{n+1} \to$ ...
such that:
1. $\mathcal{P}^{-1} = \mathcal{P}$, 2. $\mathcal{P}^n = 0$ for $n < -1$, 3. $\check{H}^q((I, \leq), \mathcal{P}^n) = 0$, for all $q \geq 1$ and $n \geq 0$.

A resolution (RP) of a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ over $(I, \leq)$ induces a superior semiexact sequence (i.e., a cochain complex)

$$(*) \ldots \to \check{H}^0(\mathcal{P}^0) \xrightarrow{(D^0)^*} \check{H}^0(\mathcal{P}^1) \xrightarrow{(D^1)^*} \check{H}^0(\mathcal{P}^2) \xrightarrow{(D^2)^*} \ldots$$

Theorem 1. *([12], Th. 6.1, p. 104) The abstract Čech cohomology groups of a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ are isomorphic to the cohomology groups of the cochain complex (*) associated to a cohomological resolution (RP) of $\mathcal{P}$.*

Remark 1. If a projective system $\mathcal{P} = (H_i, \alpha_{ij})$ admits a cohomological resolution whose terms are canonical projective systems, then, by Proposition 3 and Theorem 1 the abstract Čech cohomology groups of $\mathcal{P}$ can be immediately determined.

The abstract Čech homology is defined by categorical duality. Let $\mathcal{I} = \{G_i, \alpha_{ij}\}_{(I, \leq)}$ be an inductive system of abelian groups over a directed partially ordered set $(I, \leq)$ satisfying conditions (1)-(4). For a covering $J \subset I$ a chain complex $C_*(J, \mathcal{I})$ is defined by taking

$$C_q(J, \mathcal{I}) := \bigoplus_{(i_0, \ldots, i_q) \in \Sigma_q} G_{i_0 \wedge \ldots \wedge i_q}$$

and $d_q : C_q(J,\mathcal{I}) \to C_{q-1}(J,\mathcal{I})$ given by

$$d_q t_{i_0\wedge\ldots\wedge i_q} = \sum_{p=0}^{q} (-1)^p \alpha_{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_q, i_0\wedge\ldots\wedge i_q} t_{i_0\wedge\ldots\wedge i_q}.$$

The groups $H_q(C_*(J,\mathcal{I}))$ are denoted by $H_q(J,\mathcal{I})$. If we consider two coverings J, J' with $J' \prec J$, then for every $q \geq 0$ there exist natural homomorphisms $\alpha_{J'\prec J} : H_q(J',\mathcal{I}) \to H_q(J,\mathcal{I})$ such that a projective system $(H_q(J,\mathcal{I})\alpha_{J'\prec J}$ is obtained. The projective limit $\varprojlim_{\mathcal{J}\in\mathcal{A}_\mathcal{I}} H_q(J,\mathcal{I})$ is called the *q-th abstract Čech homology of the inductive system* $\mathcal{I}$ and it is denoted by $\check{H}_q(\mathcal{I})$ or by $\check{H}_q(|I|,\mathcal{I})$. The properties of abstract Čech homology are dual to those of the abstract Čech cohomology.

3 Examples

Example 1. Let (X, A) be a pair of topological spaces, with A a closed subspace of X, and let Γ be a presheaf over X. Consider the set $I := \{U|U$ open subset of X and $U \supset A\}$. This ordered set (by the inclusion relation) satisfies the conditions (1), (2) and (4). Then we consider the restriction Γ/I, and denote the q-th cohomology group of this projective system by $\check{H}^q([X,A];\Gamma)$. One can prove that, for every $q \geq 0$, there exists a commutative diagram

$$\begin{tikzcd}
 & \check{H}^q((X,A);\Gamma) \arrow[dd] \\
\check{H}^q([X,A];\Gamma) \arrow[ur] \arrow[dr] & \\
 & \check{H}^q(X;\Gamma_A)
\end{tikzcd}$$

where $\check{H}^q((X,A);\Gamma)$ and $\check{H}^q(X;\Gamma_A)$ are the Čech cohomology groups with

$$\Gamma_A(U) = \begin{cases} \Gamma(U) & \text{if } U \cap A \neq \emptyset, \\ 0 & \text{if } U \subset X - A \end{cases}$$

for every open subset u of X.

Moreover, for these cohomology groups an excision theorem can be proved too:

$$\check{H}^q([X-V, A-V];\Gamma) \simeq \check{H}^q([X,A];\Gamma).$$

Example 2. Let

$$G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 .... \leftarrow G_n \xleftarrow{\varphi_n} G_{n+1} \leftarrow ....$$

be a sequence in the category of abelian groups. We obtain a projective system (sequence) $\mathcal{P} := \{G_n, \alpha_{n,m}\}_{n,m\in\mathbb{N}}$ by taking $\alpha_{n,m} = \varphi_n \circ \varphi_{n+1} \circ ... \circ \varphi_{m-1}$.

Let us denote $I = \{J_m = \{m, m+1, ...\}\}$, with $\mathbb{N} \supset J_1 \supset J_2 \supset ....$

For every $J_n \in I$ we define $Q_{J_n} := \varprojlim\limits_{k \in J_n} G_k$, and if $J_n \subseteq J_m$, then we obtain a homomorphism $\alpha_{J_n J_m} : Q_{J_m} \longrightarrow Q_{J_n}$. For the projective system $\mathcal{Q} := \{Q_{J_n}, \alpha_{J_n J_m}\}$, we have:

$$\check{H}^0(\mathcal{Q}) = \varprojlim_{\mathbb{N}} \mathcal{P} = \varprojlim_{\mathbb{N}} G_n$$

and

$$\check{H}^q(\mathcal{Q}) = 0$$

for $q \geq 1$.

Example 3. If the sequence of abelian groups considered above is semiexact, we consider the projective system $\mathcal{P} = \{G_k, \alpha_{kh}\}_{h,k \in \mathbb{N}}$ with $\alpha_{kh} = 0$ for $h > k+1$.

A covering of $\mathbb{N}$ (in the sense of our definition) has the form $J = \{n_1 \leq ... \leq n_k \leq ...\}$, and we can prove that any two such coverings are cohomologically equivalent. In this case we obtain $\check{H}^0(\mathcal{P}) = 0$, and $\check{H}^1(\mathcal{P})$ is the factor group of the group of infinite dimensional matrices of the form

$$A = \begin{pmatrix} 0 & y_{12} & y_{13} & y_{13} & \cdots & \cdots \\ -y_{12} & y'_{12} & y_{23} & y_{24} & y_{24} & \cdots \\ -y_{13} & -y_{23} + y'_{13} & y'_{23} & y_{34} & y_{35} & y_{35} \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

by the group of infinite dimensional matrices of the form

$$B = \begin{pmatrix} 0 & y_{12} & y_{12} + \varphi_1(y_{23}) & y_{12} + \varphi_1(y_{23}) & \cdots & \cdots \\ -y_{12} & 0 & y_{23} & y_{23} + \varphi_2(y_{34}) & \cdots & \cdots \\ -y_{12} - \varphi_1(y_{23}) - y_{23} & 0 & y_{24} & y_{34} + \varphi_3(y_{45}) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $y_{ih} \in G_i$ and $y'_{ih} \in Ker\varphi_i$.

Example 4. Let X be a Hausdorff topological space, and $\mathcal{C}\tau$ the set of the closed subsets of X. This set satisfies the conditions (1)-(4), with $A \wedge B = A \cap B$ and $\sup A_\alpha := \overline{\bigcup\limits_\alpha A_\alpha}$, $\theta = \emptyset$, for $A, B, A_\alpha \in \mathcal{C}\tau$. If $\mathcal{I}$ is an inductive system over the set $\mathcal{C}\tau$, then $\check{H}_q(\mathcal{I}) = \varprojlim\limits_{\mathcal{M} \in \mathcal{D}} \check{H}_q(M, I)$, where $\mathcal{D}$ is the set of all dense parts M in X. In particular,

$$\check{H}_0(I) = \varprojlim_{M \in \mathcal{D}} \big(\bigoplus_{x \in M} \mathcal{I}(x)\big).$$

In general, we cannot consider the cohomology since the coverings in the above sense do not form a directed set. However, the problem is possible in the case of some topological spaces which are not Hausdorff. For example, let X be a set with the "excluded point topology", i.e., the topology which is obtained by declaring open, in addition to X itself, all sets which do not include a given point $p \in X$. In this case the Čech cohomology and homology are less interesting than the cohomology and homology with coefficients in projective and inductive systems over the closed subsets of X.

If $\mathcal{P}$ is a projective system and $\mathcal{I}$ is an inductive system over the set of closed parts of X, then:

$\check{H}^0(\mathcal{P}) = \prod_{x\in X} \mathcal{P}(\{x,p\})$, $\check{H}^q(\mathcal{P}) = 0$ for $q \geq 1$, and

$\check{H}_0(\mathcal{I}) = \bigoplus_{x\in X} \mathcal{I}(\{x,p\})$, $\check{H}_q(\mathcal{I}) = 0$ for $q \geq 1$.

Finally, we mention that if the topological space (X,τ) has the property that for every open covering $\mathcal{U}$ of X there exists an open covering $\mathcal{U}'$ of X such that $\mathcal{U}'$ is a refinement of $\mathcal{U}$ and $\mathcal{CU}$ is a closed covering of X, then for every presheaf Γ on X there exists an isomorphism of the group $\check{H}^q(X,\Gamma)$ with the group $\check{H}_q(\mathcal{I})$, where $\mathcal{I}$ is the inductive system defined by $\mathcal{I}(A) := \Gamma(\mathcal{C}A)$ for every $A \in \mathcal{C}\tau$.

4 Application [18]

In this section, as an application of our abstract Čech cohomology, we consider some projective systems associated with a standard simplex Δ^n by taking as the indices the faces of Δ^n. As examples of such projective systems are the *simple cellular sheaves* considered in [3,4,7] and [13], and whose cohomology groups appear in the calculation of the K_G-groups of some particular G-spaces, by using the Atiyah-Hirzebruch spectral sequences. The cohomology groups of a linear simplex X with coefficients in a simple cellular sheaf are computed in [3] and [4] by means of a finite closed covering of X and using a Corollary of Leary's theorem ([9], p. 209).

We replace these coverings (which are rather complicated and which require the verification of some difficult acyclicity conditions) by a covering in the sense considered in [11] and [12], and which in fact consists only of the vertices of the standard simplex Δ^n. In this way we can calculate the cohomology groups with coefficients in a simple cellular sheaf for every linear simplex.

Let Δ^n be the n-dimensional standard simplex and let Σ be the set of the (closed) faces of Δ^n to which we add the empty set $\emptyset$. If $A_0, A_1, ..., A_n$ are the vertices of Δ^n, denote by $\Delta^n_{i_0 i_1 ... i_p}$ the face of Δ^n spanned by the vertices $A_{i_0}, A_{i_1}, ..., A_{i_p}$.

If $\sigma = \Delta^n_{i_0...i_p}, \sigma' = \Delta^n_{j_0...j_q} \in \Sigma$, we define the relation

$\sigma \leq \sigma'$ if and only if $j_0,...,j_q \subseteq i_0,...,i_p$, i.e., σ' is a face of σ.

This is a partial ordering on the set Σ.

Definition 4. A projective system $\mathcal{P} = (H_\sigma, \alpha^{\sigma'}_\sigma)_{\sigma,\sigma'\in\Sigma}$ over the above partially ordered set $(\Sigma, \leq)$ is called a simplicial projective system.

Lemma 1. *The partially ordered set* $(\Sigma, \leq)$ *satisfies the conditions (1)-(4).*

Proof. (1) If $\sigma' = \Delta^n_{i_0...i_p}$, $\sigma'' = \Delta^n_{j_0...j_q}$, then for $\sigma = \Delta^n_{k_0,...,k_r}$, with $\{k_0,...,k_r\} = \{i_0,...,i_p\} \cup \{j_0,...,j_q\}$, we have $\sigma = \sigma' \wedge \sigma''$ (σ is sometimes the joint $\sigma' * \sigma''$).

(2) If $\Sigma' \subset \Sigma$, we have $\sup \Sigma' = \bigcap_{\sigma'\in\Sigma'} \sigma'$. Here it is necessary to suppose that Σ contains the empty set $\emptyset$.

Now we say that a subset Σ' of the set Σ is a *covering* if for every face $\sigma \in \Sigma$ there exists a subset $\Sigma'_\sigma \subset \Sigma'$ such that $\sup \Sigma'_\sigma \neq \emptyset$ and $\sigma \leq \sup \Sigma'_\sigma$.

If Σ', Σ'' are two coverings, then we have $\Sigma' \prec \Sigma''$ if and only if for every face $\sigma' \in \Sigma'$ there exists a face $\sigma'' \in \Sigma''$ such that $\sigma' \leq \sigma''$, i.e., σ'' is a face of σ'. Now the condition (4) is verified because if Σ', Σ'' are two coverings, then $\Sigma' \bigcap \Sigma''$ satisfies the relations $\Sigma' \bigcap \Sigma'' \prec \Sigma'$ and $\Sigma' \bigcap \Sigma'' \prec \Sigma''$. $\square$

By Lemma 1 we can consider the abstract Čech cohomology groups $\check{H}^q(\mathcal{P}_n)$ of a simplicial projective system $\mathcal{P}_n = (H_\sigma, \alpha_\sigma^{\sigma'})_{\sigma,\sigma' \in \Sigma}$.

Now we recall from [7] and [4] that a *cellular sheaf* over Δ^n is a sheaf $\mathcal{F}$ on the topological space Δ^n with the property that for every open face $\overset{\circ}{\sigma}$ of Δ^n the restriction $\mathcal{F}/\overset{\circ}{\sigma}$ is a simple sheaf. For such a sheaf, if $\overset{\circ}{\sigma}$ and $\overset{\circ}{\sigma}{}'$ are two open faces of Δ^n with $\overset{\circ}{\sigma} \cap \sigma' \neq \emptyset$ then there exists a homomorphism $\varphi_{\sigma\sigma'} : \mathcal{F}/\overset{\circ}{\sigma} \to \mathcal{F}/\overset{\circ}{\sigma}{}'$ satisfying the condition that if $\overset{\circ}{\sigma} \cap \sigma' \cap \sigma'' \neq \emptyset$ and $\overset{\circ}{\sigma}{}' \cap \sigma'' \neq \emptyset$, then $\varphi_{\sigma'\sigma''} \circ \varphi_{\sigma\sigma'} = \varphi_{\sigma\sigma''}$ ([7], Prop.1). Also, from the definition of the above homomorphisms $\varphi_{\sigma\sigma'}$ one deduces that $\varphi_{\sigma\sigma}$ is the identity. Together with the Prop. 3 of [7] and remarking that if σ, σ' are two faces of Δ^n then $\overset{\circ}{\sigma} \cap \sigma' \neq \emptyset$ if and only if σ is a face of σ', i.e., if and only if $\sigma' \leq \sigma$, we obtain the following theorem.

Theorem 2. *Every simple cellular sheaf F over the standard simplex Δ^n defines a simplicial projective system $\mathcal{P}(\mathcal{F}) = \mathcal{P}_n$, and conversely, any simplicial projective system $\mathcal{P}_n$ induces a cellular simple sheaf over Δ^n.*

Theorem 3. *The abstract Čech cohomology groups of a simplicial projective system*

$$\mathcal{P}_n := \{H_{i_0\wedge\ldots\wedge i_q}, \alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i}_p\wedge\ldots\wedge i_q}\}$$

are given by :

$$\check{H}^q(\mathcal{P}_n) \cong \frac{\bigcap\limits_{0\leq i_0<i_1<\ldots<i_{q+1}\leq n} Z_{i_0 i_1\ldots i_{q+1}}}{\bigcap\limits_{0\leq i_0<i_1<\ldots<i_{q+1}\leq n} B_{i_0 i_1\ldots i_{q+1}}}$$

for $q = 0, 1, ..., n$ (and 0 otherwise), with

$$Z_{i_0\ldots i_{q+1}} = H_{i_1\wedge\ldots\wedge i_{q+1}} \prod\nolimits_{H_{i_0\wedge\ldots\wedge i_{q+1}}} \bigoplus_{p=1}^{q+1} H_{i_0\wedge\ldots\wedge\widehat{i}_p\wedge\ldots\wedge i_{q+1}}$$

,

$$B_{i_0\ldots i_{q+1}} = Im\alpha_{i_0\wedge\ldots\wedge i_{q+1}}^{i_0\wedge\ldots\wedge i_q} \prod\nolimits_{H_{i_0\wedge\ldots\wedge i_{q+1}}} \bigoplus_{p=1}^{q} Im\alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i}_p\wedge\ldots\wedge i_q}$$

where $A\prod_S B$ denotes a fibered product of A and B over S.

Remark 2. If the homomorphisms $\alpha_{i_0\wedge\ldots\wedge i_q}^{i_0\wedge\ldots\wedge\widehat{i_p}\wedge\ldots\wedge i_q}$ are all injective, then the formulas from Theorem 3 become more simple. For example, in this case we have

$$\check{H}^0(\mathcal{P}_n) \cong \bigcap_{i=0}^{n} H_i, \check{H}^n(\mathcal{P}_n) \cong \frac{H_{012\ldots n}}{H_0' + H_1' + \ldots + H_n'}$$

where $H_p' = H_{0\ldots\widehat{p}\ldots n}$.

Remark 3. For $n = 1, 2, 3$ we find some results of [3],[4]. Thus, if $n = 1$ we obtain

$$\check{H}^0(\mathcal{P}_1) \cong H_0 \prod\nolimits_{H_{01}} H_1$$

and

$$\check{H}^1(\mathcal{P}_1) \cong \frac{H_{01}}{Im\alpha_{01}^0 + Im\alpha_{01}^1},$$

which coincide respectively with the cohomology groups $H^q(\Delta^1, \mathcal{F})$, $q = 0, 1$.

We can establish a general result. Let $\mathcal{F}$ be a simple cellular sheaf over Δ^n. The calculation of the groups $H^q(\Delta^n, \mathcal{F})$ in [4] uses a closed covering $\mathcal{U} = \{U_0, \ldots, U_n\}$ with the acyclicity property $H^q(U_{i_0} \cap \ldots \cap U_{i_q}; F) = 0, q \geq 1$. Then, by Cor. 1 of [9], p. 209, the natural homomorphism $H^q(\mathcal{U}, \mathcal{F}) \to H^q(\Delta^n, \mathcal{F})$ is an isomorphism. Then the cochain complex $C^*(\mathcal{U}, \mathcal{F})$ is given by

$$C^q(\mathcal{U}, \mathcal{F}) = \prod \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q})$$

and

$$(d^q t)_{U_{i_0} \cap \ldots \cap U_{i_q}} = \sum_{p=0}^{q+1} (-1)^p \mathcal{F}_{U_{i_0} \cap \ldots \cap U_{i_{q+1}}}^{U_{i_0} \cap \ldots \cap \widehat{U}_{i_p} \cap \ldots \cap U_{i_{q+1}}} t_{U_{i_0} \cap \ldots \cap \widehat{U}_{i_p} \cap \ldots \cap U_{i_{q+1}}}.$$

By the choice of the covering $\mathcal{U}$ and because $\mathcal{F}$ is a simple cellular sheaf one verifies easily that $C^*(\mathcal{U}, \mathcal{F})$ is equivalent to the cochain complex which appears in the proof of Theorem 3. Thus we have the following result.

Theorem 4. *If $\mathcal{F}$ is a simple cellular sheaf over the standard simplex Δ^n and if $\mathcal{P}(\mathcal{F})$ is its associated projective system by Theorem 2, then there exists a natural isomorphism*

$$H^q(\Delta^n, \mathcal{F}) \cong \check{H}^q(\mathcal{P}(\mathcal{F}))$$

for every integer q.

Remark 4. If we replace the standard simplex Δ^n by an arbitrary CW-complex, then a simple cellular sheaf also defines a projective system having as the set of indices the set of cells, but unfortunately this set does not satisfy the condition (1). But in [8] a method for the calculation of the cohomology groups of an arbitrary polyhedron with coefficients in a simple cellular sheaf by using the simplicial cohomology with local coefficients was given. This leads us to believe that it is possible to extend our method, that of abstract Čech cohomology, from the case of standard simplex Δ^n to the general case of an arbitrary polyhedron.

5 De Rham type theorems

5.1 J–resolutions of a projective system [14]

In this section we assume that $(I, \leq)$ satisfies (1)–(4) (see Section 2) and for $J \in I$ we put $\Sigma_s(J) = \times_s J$. Let $P = (H_i, \alpha_{ij})_{i,j \in I}$ be a projective system (of abelian groups) over I and consider the cochain complex of projective systems over I

$$0 \to P \xrightarrow{j} P^0 \xrightarrow{D^0} P^1 \to \ldots \to P^q \xrightarrow{D^q} P^{q+1} \to \ldots \tag{1}$$

where $j = (j_i)_{I \in I}$ and $D^q = (D^q_i)_{i \in I}$.

Definition 5. A *J–resolution* of the projective system P over I is a cochain complex (1) satisfying the following conditions:

a. there exists $J \in A_I$ such that for any $s \geq 0$ and $(i_0, i_1, \ldots, i_s) \in \Sigma_s(\mathrm{J})$,

$$\begin{aligned} 0 \to H_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} &\xrightarrow{j_{i_0 \wedge i_1 \wedge \ldots \wedge i_s}} H^0_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} \to \ldots \\ \to H^q_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} &\xrightarrow{D^q_{i_0 \wedge i_1 \wedge \ldots \wedge i_s}} H^{q+1}_{i_0 \wedge i_1 \wedge \ldots \wedge i_s} \to \ldots \end{aligned}$$

is an exact sequence;

b. the sequence (1) is exact with respect to P and P^0.

Remark 5. 1. Let $f : P \to Q$ be a projective systems isomorphism. If $J \in \mathcal{A}_I$, then f induces an isomorphism from $H^q(J, P)$ to $H^q(J, Q)$, hence

$$\check{H}^q(|I|, P) \simeq \check{H}^q(|I|, Q)$$

for any $q \geq 0$.

2. If the sequence (1) is a J–resolution of the projective system P then its projective limit

$$0 \longrightarrow \varprojlim_{i \in I} H_i \xrightarrow{\varprojlim_{i \in I} j_i} \varprojlim_{i \in I} H^0_i \longrightarrow \ldots \longrightarrow \varprojlim_{i \in I} H^q_i \xrightarrow{\varprojlim_{i \in I} D^q_i} \varprojlim_{i \in I} H^{q+1}_i \longrightarrow \ldots \tag{2}$$

is a cochain complex and it is exact with respect to the terms $\varprojlim_{i \in I} H_i$ and $\varprojlim_{i \in I} H^0_i$.

Let (1) be a J–resolution of the projective system P and denote by $C^s(J, P^q)$ and $\mathcal{C}(J, P)$ the s–dimensional cochain groups associate to the systems P^q and P, respectively. By Remark 5 we can assume that $\mathcal{C}^s(J, P) \leq C^{s,0}(J, P)$ by the inclusion morphism j^s. Then the sequence

$$0 \to \mathcal{C}^s(J, P) \xrightarrow{j^s} C^{s,0}(J, P) \to \ldots \to C^{s,q}(J, P) \xrightarrow{d^{s,q}} C^{s,q+1}(J, P) \to \ldots \tag{3}$$

is exact for each $s \geq 0$, $q \geq 1$, and we have the following

Proposition 4. *If the projective system P has a J–resolution (1) then (3) is a resolution for its p–dimensional cochain group.*

Definition 6. A J–resolution of the projective system P is *acyclic* if $H^s(J, P^q) = 0$ for any $q \leq 0$ and $s \geq 1$.

We remark that if for any $J \in A_I$, (1) is an acyclic J–resolution of the projective system P then it is a resolution of P. Conversely, if there exists $J \in A_I$ such that $J \prec J'$ for each $J' \in A_I$ then any resolution of P is a J–resolution.

Now, we can state

Theorem 5. *If the projective system P admits an acyclic J–resolution of canonical projective systems then*

$$H^0(J, P) \simeq \ker \varprojlim_{i \in I} D_i^0,$$
$$H^q(J, P) \simeq \ker \varprojlim_{i \in I} D_i^q / \mathrm{Im} \varprojlim_{i \in I} D_i^{q-1} \quad for \quad q \geq 1.$$

We notice that in [15], the above isomorphism is effectively exhibited.

Denote by $P^* = \oplus_{q \geq 0} P^q$ the differential projective system associate to (1), with the codifferential $d = \oplus_{q \geq 0} D^q$. We have the following generalization of Theorem 5.

Theorem 6. *[17] Let P be a projective system over the $\wedge$–semilattice $(I, \leq)$, $J \in \mathcal{A}_I$ and (1) be a J–resolution of P by canonical projective systems. If $H^p(H^q(J, P^*)) = 0$ for all $p \geq 0$ and $q \geq 1$ then*

$$H^p(J, P) \simeq H^p(\varprojlim_{i \in I} H_i^0 P^*).$$

Now, we present some applications of this result. The first one is the following de Rham type theorem:

Theorem 7. *If for any $J \in \mathcal{A}_I$, the sequence (1) is an acyclic J–resolution of canonical projective systems for P, then the following isomorphisms occur:*

$$\check{H}^0(|I|, P) \simeq \ker \varprojlim_{i \in I} H_i^0 D_i^0,$$
$$\check{H}^q(|I|, P) \simeq \ker \varprojlim_{i \in I} D_i^q / \mathrm{Im} \varprojlim_{i \in I} D_i^{q-1} \quad for \quad q \geq 1.$$

From Remark 5 we deduce that the conclusion of Theorem 7 is still valid if there exists $J \in \mathcal{A}_I$ such that $J \prec J'$ for any $J' \in A_I$.

On the other hand, if the set I has a supremum k then from Theorem 5 we obtain

$$H^0(J, P) \simeq H_k, \qquad H^q(J, P) \simeq \ker d_k^q / \mathrm{Im} d_k^{q-1} \quad \text{for} \quad q \geq 1 \tag{4}$$

and, moreover, if $k \in J$ then $H^q(J, P) = 0$ for any $q \geq 1$.

Proposition 5. *Under the hypotheses from Theorem 7, if there exists* $\sup I = k$ *then*

$$\check{H}^0(|I|, P) \simeq H_k, \qquad \check{H}^q(|I|, P) = 0 \quad for \quad q \geq 1.$$

5.2 Existence of J–resolutions

We remark that, in order to obtain de Rham type theorems for projective systems, the existence of a J–resolution is essential. This problem is solved in [16], at least partially. First, we assume $(I, \leq)$ to be a $\wedge$–semilattice and for $i \in I$ and $J \subseteq I$ we put $i \wedge J = \{i \wedge j\}_{j \in J}$. Then the group of q–dimensional cochains of P relative to $i \wedge J$, is

$$K^q(i \wedge J) = \prod_{\Sigma_q} H_{i \wedge i_o \wedge \ldots \wedge i_q}.$$

If $i' \in I$, $i' \leq i$, then for $(i_0, \ldots, i_q) \in \Sigma_q$ and $t \in K^q(i \wedge J)$ we put

$$(\alpha^{*q}_{i'i} t)_{i' \wedge i_o \wedge \ldots \wedge i_q} = \alpha_{i' \wedge i_o \wedge \ldots \wedge i_q, i \wedge i_o \wedge \ldots \wedge i_q} t_{i \wedge i_o \wedge \ldots \wedge i_q}.$$

Thus we obtain the projective system $K^q(J, P) = \{K^q(i \wedge J), \alpha^{*q}_{i'i}\}_{i,i' \in I}$ and denote by $\partial^q : K^q(J, P) \to K^{q+1}(J, P)$ the coboundary operator. Moreover it induces a morphism of projective systems. Another such morphism is $f = (f_i)_{i \in I} : P \to K^0(i \wedge J, P)$, where $[f_i(h_i)]_{i \wedge j} = \alpha_{i \wedge j, i}(h_i)$ for $h_i \in H_i$ and $j \in J$. Then we have

Proposition 6. *Assume that there exists* $\sup I$ *and let* P *be a canonical projective system on* I. *Then the sequence*

$$0 \longrightarrow P \xrightarrow{f} K^0(J, P) \xrightarrow{(\partial^0)} \ldots \longrightarrow K^q(J, P) \xrightarrow{(\partial^q)} K^{q+1}(J.P) \longrightarrow \ldots$$

is an acyclic J*–resolution of* P *for any cofinal subset* J *of* I.

5.3 Canonical system associate to a projective system [15]

The determination of these groups is a hard and subtle problem and we know them explicitly in very few cases.

It is well-known that under some additional restrictions on the set $(I, \leq)$, to any projective system we can associate a canonical projective system. More precisely, we have the following

Proposition 7. *([10], Prop. 5.1, p. 102) If* $(I, \leq)$ *is a filter at left then to any projective system* P *over* $(I, \leq)$*, a canonical projective system* P^* *can be associated.*

Under some stronger conditions on $(I, \leq)$ we can get a simple relation between Čech cohomologies of P and P^*.

Theorem 8. *Let* $(I, \leq)$ *be a* $\wedge$*–semilattice such that any* $J \in \mathcal{A}_I$ *admits a refinement* $\overline{J} \in \mathcal{A}_I$ *with the property that for every* $(i_0, \ldots, i_q) \in \Sigma_q$ *there exist* $\overline{i}_0, \ldots, \overline{i}_q \in \overline{J}$ *such that* $\overline{i}_0, \ldots, \overline{i}_q \in J_{i_0 \wedge \ldots \wedge i_q}$.
Then the Čech cohomology groups of P *and* P^* *are isomorphic in each dimension, that is*

$$\check{H}^q(|I|, P) \simeq \check{H}^q(|I|, P^*)$$

for all $q \geq 0$.

5.4 Examples

Now, we use the above results to recover some classical theorems (see Introduction).

Example 9. Let M be a smooth n–dimensional manifold and denote by $\widetilde{\Lambda}^q$ its q–forms sheaf and by $\widetilde{\mathbb{R}}$ the sheaf associated to the constant presheaf on M. If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of M then the sheaves sequence

$$0 \longrightarrow \widetilde{\mathbb{R}} \xrightarrow{j} \widetilde{\Lambda}^0 \xrightarrow{d} \widetilde{\Lambda}^1 \longrightarrow \ldots \longrightarrow \widetilde{\Lambda}^q \xrightarrow{d} \widetilde{\Lambda}^{q+1} \longrightarrow \ldots \tag{5}$$

is a cochain complex with respect to the exterior derivative d. (5) is exact for the terms $\widetilde{\mathbb{R}}$, $\widetilde{\Lambda}^0$ and $H^s(\mathcal{U}, \widetilde{\Lambda}^q) = 0$ for $s \geq 1$ and $q \geq 0$. If, moreover, the cover $\mathcal{U}$ is contractible then (5) is an acyclic $\mathcal{U}$–resolution of the sheaf $\widetilde{\mathbb{R}}$ and we can apply the isomorphisms (4). On the other hand, any open cover of M can be refined by a contractible one, so the set $\mathcal{A}^c$ of contractible covers of M is cofinal in the set of all open covers and then

$$\check{H}^q(M, \widetilde{\mathbb{R}}) \simeq \varinjlim_{\mathcal{U}\in\mathcal{A}^c} H^q(\mathcal{U}, \widetilde{\mathbb{R}})$$

and therefore from the isomorphism $\check{H}^q(M, \widetilde{\mathbb{R}}) \simeq \check{H}^q(M, \mathbb{R})$ we deduce the classical de Rham theorem.

Example 10. Let (X, τ) be a topological space and

$$\Delta(U): \quad \ldots \longrightarrow \Delta_q(U) \xrightarrow{d_{q,U}} \Delta_{q-1}(U) \longrightarrow \ldots \longrightarrow \Delta_0(U) \longrightarrow 0$$

the singular chain sequence associate to $U \in \tau$. Denote by $\Delta^q(U)$ the group of q–dimensional singular cochains with coefficients in an abelian group G and for $l \in \Delta^q(U)$ we put $d^q_U(l) = l d_{q+1,U}$. For $V \in \tau$, $V \supset U$, we define the morphism $\alpha^q_{UV} : \Delta^q(V)$, which, to each cochain on V, associates its restriction to U. Then for each $q \geq 0$, $\Delta^q = \{\Delta^q(U), \alpha^q_{UV}\}_{U,V\in\tau}$ is a complete projective system and we obtain the cochain complex

$$\Delta^*: \quad 0 \longrightarrow \Delta^0 \longrightarrow \ldots \longrightarrow \Delta^q \xrightarrow{d^q} \Delta^{q+1} \longrightarrow 0$$

where $d^q = (d^q_U)_{U\in\tau} : \Delta^q \to \Delta^{q+1}$ and $d^q_U(l) = l d_{q+1,U}$.

Now, if $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is an open cover for X and $U \cap \mathcal{U}$ is its trace on U then we consider the group $\Delta_q(U \cap \mathcal{U})$ spanned by all singular simplexes whose images belong to $U_\alpha \in \mathcal{U}$ for some $\alpha \in I$. Denoting by $\Delta^q(U \cap \mathcal{U})$ the group of all morphisms from $\Delta_q(U \cap \mathcal{U})$ to G, as above, we can construct a complete projective system $\overline{\Delta}^q = \{\Delta^q(U \cap \mathcal{U}), \overline{\alpha}^q_{UV}\}_{U,V\in\tau}$ and the associate cochain complex $\overline{\Delta}^*$, obtained from Δ^*.

Let L^q_U be the subgroup of all elements of $\Delta^q(U \cap \mathcal{U})$ vanishing on $\Delta_q(U' \cap \mathcal{U})$ for all U' in some cover of U and consider the quotient group $\widetilde{\Delta}^q(U \cap \mathcal{U}) = \Delta^q(U \cap \mathcal{U})/L^q_U$. Then $\overline{\alpha}^q_{UV}$ and the differential $\overline{d}^q_U$ naturally induce the quotient morphisms $\widetilde{\alpha}^q_{UV}$ and

$\widetilde{d}^q_U$, respectively. Thus, from $\overline{\Delta}^*$ we obtain the following cochain complex of canonical projective systems

$$\widetilde{\Delta}^* : \quad 0 \longrightarrow \widetilde{\Delta}^0 \longrightarrow \ldots \longrightarrow \widetilde{\Delta}^q \xrightarrow{\widetilde{d}^q} \widetilde{\Delta}^{q+1} \longrightarrow 0$$

Moreover, for any cover of X, the sequences $\overline{\Delta}^*$ and $\widetilde{\Delta}^*$ are homotopy equivalent. Also, the sequence

$$0 \longrightarrow \ker \widetilde{d}^0 \xrightarrow{\hookrightarrow} \widetilde{\Delta}^0 \longrightarrow \ldots \longrightarrow \widetilde{\Delta}^q \xrightarrow{\widetilde{d}^q} \widetilde{\Delta}^{q+1} \longrightarrow 0$$

is an acyclic $\mathcal{U}$–resolution of the projective system $\ker \widetilde{d}^0$, for any contractible cover $\mathcal{U}$ of X and from the isomorphisms (4) and since the projective systems $\widetilde{\Delta}^q$ are canonical, we deduce

$$\check{H}^q(\mathcal{U}, \ker \widetilde{d}^0) \simeq \check{H}^q(\mathcal{U}, G) \simeq H^q(X, G)$$

and we find a Leray's theorem,[10], asserting that the Čech cohomology groups relative to a contractible cover of the topological space X and with coefficients in the abelian group G, are isomorphic, in each dimension, with the singular cohomology groups of X, with coefficients in G.

6 Spectral sequence [17]

Let $P^* = \oplus_{q \geq 0} P^q$ be a differential projective system over I, with the codifferential $d = \oplus_{q \geq 0} D^q$. For $J \subset I$ we consider the coboundary homomorphism associated to the projective system P^q, denoted by $d_{10}^{pq} = (d^p)^q : C^p(J, P^q) \to C^{p+1}(J, P^q)$ (see Section 2) and the homomorphism $d_{01}^{pq} = (-1)^p (\widetilde{D^q})^p : C^p(J, P^q) \to C^p(J, P^{q+1})$, where $(\widetilde{D^q})^p$ is induced by the codifferential D^q. Then the following equalities hold

$$d_{10}^{p+1,q} d_{10}^{pq} = 0, \quad d_{01}^{p,q+1} d_{01}^{pq} = 0, \quad d_{01}^{p,q+1} d_{10}^{pq} + d_{10}^{p+1,q} d_{01}^{pq} = 0 \tag{6}$$

These equalities show that for any $J \subset I$, the codifferentials d_{10}^{pq} and d_{01}^{pq} define on the group

$$C(J, P^*) = \oplus_{p \geq 0, q \geq 0} C^p(J, P^q)$$

a double complex structure and then we consider its first and second filtration given by

$$'C_p(J, P^*) = \oplus_{\overline{p} \geq p}^{q \geq 0} C^{\overline{p}}(J, P^q), \qquad ''C_q(J, P^*) = \oplus_{\overline{q} \geq q}^{p \geq 0} C^p(J, P^{\overline{q}}).$$

We remark that the first filtration $'C_p(J, P^*)$ is regular if $P^q = 0$ for $q < q_0$ for some q_0, while the second filtration $''C^p(J, P^*)$ is always regular.

Now, assume that $J \in \mathcal{A}_I$ and denote by $'E_r^{pq}(J, P^*)$ and $''E_r^{pq}(J, P^*)$ the terms of the spectral sequences corresponding to the first filtration and the second filtration of the complex $C(J, P^*)$, respectively. Then we have

Proposition 8. *If $(I, \leq)$ is a $\wedge$–semilattice then*

$$
\begin{array}{ll}
{}'E_1^{pq}(J, P^*) \simeq C^p(J, \mathcal{H}^q(P^*)), & {}'E_2^{pq}(J, P^*) \simeq H^p(J, \mathcal{H}^q(P^*)) \\
{}''E_1^{pq}(J, P^*) \simeq H^p(J, P^p), & {}''E_2^{pq}(J, P^*) \simeq H^p(H^q(J, P^*))
\end{array}
$$

where $\mathcal{H}^q(P^*) = \ker D^q / D^{q-1}(P^{q-1})$ and $H^q(J, P^*) = \oplus_{p \geq 0} H^q(J, P^p)$ is endowed with the differential induced by d_{01}.

One more interesting result can be obtained for the term ${}''E_1^{p0}(J, P^*)$, namely:

Theorem 11. *If the projective systems P^p are canonical for all $p \geq 0$ and $(I, \leq)$ is a $\wedge$–semilattice then for $J \in \mathcal{A}_I$ we have*

$$
{}''E_1^{p0}(J, P^*) \simeq H^p(\varprojlim_{i \in I} (P^*)).
$$

Assuming that $(I, \leq)$ is a $\wedge$–semilattice, for the term ${}''E_2^{p0}(J, P^*)$ we can prove the existence of a homomorphism ${}''E_2^{p0}(J, P^*) \longrightarrow H^p(C(J, P^*)$ if $(I, \leq)$. But if, moreover, all projective systems are canonical then we also get the homomorphisms

$$
\mu^p : H^p(\varprojlim_{i \in I} (P^*)) \longrightarrow H^p(C(J, P^*)
$$

and we can state the following

Proposition 9. *Let P and Q be two projective systems over the $\wedge$–semilattice $(I, \leq)$ and $J \in \mathcal{A}_I$. If P and Q have J–resolutions of the form (1) by canonical projective systems then for any differential projective systems morphism $f : (P^*) \longrightarrow (Q^*)$ the following diagrams are commutative*

$$
\begin{array}{ccc}
H^p(\varprojlim_{i \in I} (P^*)) & \xrightarrow{\mu^p} & H^p(J, P) \\
{\scriptstyle H^p(\varprojlim_{i \in I} f)} \big\downarrow & & \big\downarrow {\scriptstyle g^p} \\
H^p(\varprojlim_{i \in I} (Q^*)) & \xrightarrow[\mu^p]{} & H^p(J, Q)
\end{array}
$$

where g^p are induced by the morphism j corresponding to the J–resolution of Q and f^0 is the component $P^0 \longrightarrow Q^0$ of f.

The homomorphisms μ^p are also used to prove the following

Theorem 12. *If the projective systems P^p are canonical for all $p \geq 0$ and the sequences $H^q(J, P^*)$ are acyclic for all $q \geq 1$ then there exists a spectral sequence with the term E_2 given by*

$$
E_2^{pq} \simeq H^p(J, \mathcal{H}^q(P^*))
$$

and whose term E_∞ is the graded group associated to some filtration of the sequence $\varprojlim_{i \in I} (P^*)$.

Finally, we notice some related results concerning projective systems admitting a J–resolution. The first one is concerning the term ${}'E_2^{pq}(J, P^*)$.

Proposition 10. *Let P be a projective system over the $\wedge$–semilattice $(I, \leq)$ and $J \in \mathcal{A}_I$. If P has a J–resolution (1) then*

$$ {}'E_2^{pq}(J, P^*) = 0 $$

for $q \geq 1$.

Now, using Proposition 10, we obtain the following characterization of the groups $H^p(C(J, P^*))$.

Proposition 11. *If the projective system P over the $\wedge$–semilattice $(I, \leq)$ has a J–resolution (1) then*

$$ H^p(C(J, P^*) \simeq H^p(J, P) $$

for $p \geq 0$ and $J \in \mathcal{A}_I$, where $P^* = \oplus_{q \geq 0} P^q$ is the differential projective system associated to (1).

Also, we have a result similar to the Theorem 12.

Proposition 12. *Let P be a projective system over the $\wedge$–semilattice $(I, \leq)$ and $J \in \mathcal{A}_I$. For any J-resolution (1) of P with canonical projective systems there exists a spectral sequence with the term E_2 given by*

$$ E_2^{pq} \simeq H^p(H^q(J, P^*)) $$

and whose term E_∞ is the graded group associated to the abstract Čech cohomology of P.

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