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Estimation of the number of one-point expansions of a topology* which is given on a finite set

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Abstract. Let X be a finite set and τ be a topology on X which has precisely m open sets. If $t(\tau)$ is the number of possible one-point expansions of the topology τ on $Y = X\bigcup\{y\}$, then $\frac{m\cdot(m+3)}{2}-1 \geq t(\tau) \geq 2\cdot m+log_2 m-1$ and $\frac{m\cdot(m+3)}{2}-1 = t(\tau)$ if and only if τ is a chain (i.e. it is a linearly ordered set) and $t(\tau) = 2\cdot m + log_2 m - 1$ if and only if τ is an atomistic lattice.

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1 Introduction

The present article is a continuation of the article [1].

The basic result of this article is Theorem 2, in which for any topology given on a finite set, estimations of the number of one-point expansions are obtained.

To the proof of this theorem we applied the following algorithm, which is proved in the article [1] and which allows to obtain any topology τ_1 that is a one-point expansion of the topology τ_0 given on a finite set.

Let τ_0 be some topology given on a finite set X_0 and $Y = X_0\bigcup\{y\}$.

1. We choose arbitrarily $V_0 \in \tau_0$.
2. We choose arbitrarily $U_0 \in \tau_0$ such that $U_0 \subseteq \bigcap\limits_{V\in\tau_0, V\not\subseteq V_0} V$ (consider that $\bigcap\limits_{V\in\emptyset} V = X_0$).
3. We determine the topology

$$\tilde{\tau_1}(V_0, U_0) = \{V \in \tau \mid V \subseteq V_0\} \cup \{U \cup \{y\} \mid U \in \tau, U \supseteq U_0\}.$$

2 Main results

Assume that (X, τ) is a topological space.

*If $Y = X\bigcup\{y\}$ then a topology $\widetilde{\tau}$ on the set Y is called a one-point expansion of the topology $\tau = \widetilde{\tau}|_X$.

Definition 1. A subset $\mathbf{B} \subseteq \tau$ is called *a base of the topological space* (X, τ) if any open set is a union of some sets from $\mathbf{B}$.

Definition 2. *A weight of the topological space* (X, τ) is the minimal cardinal number m for which there exists a base of the topological space (X, τ) of cardinality m.

Definition 3. *The minimal base* of the topological space (X, τ) is any base which has cardinality equal to the weight of the space (X, τ).

Theorem 1. *If X is a finite set and τ is a topology on X, then the topological space (X, τ) has the unique minimal base.*

Proof. For each element $x \in X$ we consider the set $V(x) = \bigcap\limits_{U \in \tau, x \in U} U$ and let $\mathcal{B} = \{V(x) \mid x \in X\}$.

Let's show that $\mathcal{B}$ is a base in the topological space (X, τ).

From the finiteness of the set τ it follows that $V(x) \in \tau$ for any $x \in X$, and hence, $\mathcal{B} \subseteq \tau$.

If now $U \in \tau$, from the definition of the set $V(x)$ it follows that $V(x) \subseteq U$ for any $x \in U$. Then $U = \bigcup\limits_{x \in U} \{x\} \subseteq \bigcup\limits_{x \in U} V(x) \subseteq U$, and hence, $\bigcup\limits_{x \in U} V(x) = U$.

From the randomness of U it follows that $\mathcal{B}$ is a base in the topological space (X, τ).

Let's show that $\mathcal{B}$ is a minimal base in the topological space (X, τ), i.e. that its cardinality is equal to the weight of the topological space (X, τ).

Let $\mathcal{B}'$ be some minimal base of the topological space (X, τ) and $x \in X$. As $V(x) \in \tau$ and $x \in V(x)$ then there exists $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V(x)$. From the definition of the set $V(x)$ it follows that $V(x) \subseteq V'$, and hence, $V(x) = V' \in \mathcal{B}'$. From the randomness of the element $x \in X$ it follows that $\mathcal{B} = \{V(x) \mid x \in X\} \subseteq \mathcal{B}'$.

Then $\mathcal{B} \subset \mathcal{B}'$ or $\mathcal{B} = \mathcal{B}'$.

If $\mathcal{B} \subset \mathcal{B}'$, then from the finiteness of the set $\mathcal{B}'$ it follows that cardinality of the set $\mathcal{B}$ will be less than cardinality of the set $\mathcal{B}'$. We have received a contradiction with the choice of the base $\mathcal{B}'$.

Hence $\mathcal{B} = \mathcal{B}'$.

From the randomness of the base $\mathcal{B}'$ it follows that the minimal base of the topological space (X, τ) is unique, and moreover, this minimal base can be received by the method which is specified in the beginning of the proof of the theorem. $\square$

Proposition 1. *Let X be a finite set and $\tau = \{\emptyset = W_1, \ldots, W_n = X\}$ be a topology on the set X. If the topology τ is a chain (i.e. it is a linearly ordered set), then τ has precisely $\dfrac{n \cdot (n+3)}{2} - 1$ one-point expansions.*

Proof. As τ is a chain we can consider that $W_i \subset W_{i+1}$ for all $1 \leq i < n$.

If $1 \leq i \leq n-1$ and $V_0 = W_i$ (designations for V_0 and U_0 are given in Algorithm 1), then $\bigcap\limits_{V \in \tau_0, V \not\subseteq W_i} V = \bigcap\limits_{j=i+1}^{n} W_j = W_{i+1}$. Then U_0 can take $i+1$ values, namely, it can be any W_j for $1 \leq j \leq i+1$.

If $V_0 = X$ then $\bigcap\limits_{V\in\tau_0, V\nsubseteq X} V = \bigcap\limits_{V\in\emptyset} V = X$. Then U_0 can take n values. As

$$\sum_{i=1}^{n-1}(i+1) + n = \frac{(n+2)\cdot(n-1)}{2} + n =$$

$$\frac{n^2+n-2}{2} + n = \frac{n^2+n+2n-2}{2} = \frac{n\cdot(3+n)}{2} - 1$$

then we have $\frac{n\cdot(n+3)}{2} - 1$ various pairs (V_0, U_0) and hence the topology τ has precisely $\frac{n\cdot(n+3)}{2} - 1$ various one-point expansions.

The proposition is completely proved. □

Definition 4. As it is usual (see, for example, [3]), a lattice $(L, \leq)$ is called *a distributive lattice* if $\inf\{a, \sup\{b, c\}\} = \sup\{\inf\{a, b\}, \inf\{a, c\}\}$ for any $a, b, c \in L$.

Definition 5. As it is usual, a nonzero element a of a lattice $(L, \leq)$ with zero is called *an atom* if between 0 and a there are no other elements of the lattice $(L, \leq)$.

Definition 6. As it is usual, a lattice $(L, \leq)$ with zero is called (see, for example, [2]) *an atomistic lattice* if for any nonzero element $a \in L$ there exists a finite set $S \subseteq L$ of atoms of the lattice L such that $a = sup\, S$.

Remark 1. From ([3, VIII, §2, Lemma 2] it follows that in any distributive, atomistic lattice $(L, \leq)$ for any element $a \in L$ there exists the unique set $S \subseteq L$ of atoms of the lattice L for which $a = inf S$.

Remark 2. It is known that if (X, τ) is a topological space then $(\tau, \leq)$ is a distributive lattice with zero $0 = \emptyset$, in which $\sup\{U, V\} = U \bigcup V$ and $\inf\{U, V\} = U \bigcap V$.

Proposition 2. *Let X be a finite set and τ be a topology on the set X. If τ is an atomistic lattice and $\{W_1, \ldots, W_n\}$ is the set of all atoms of the lattice τ, then the topology τ has precisely $2^{(n+1)} + n - 1$ one-point expansions.*

Proof. Let $Y = \{y_1, \ldots, y_n\}$ and $\tau' = \{M \mid M \subseteq Y\}$ be the discrete topology on the set Y.

If we map each subset $M = \{y_{i_1}, \ldots, y_{i_k}\} \in \tau'$ of the set Y on the subset $\bigcup\limits_{j=1}^{k} W_{i_j} \in \tau$ of the set X, then we define a mapping $\psi : \tau' \to \tau$.

As the lattices τ' and τ are distributive and atomistic lattices, then (see Remark 1) in each of them we shall present any element uniquely as the supremum of some set of atoms. And as the sets $\{W_1, \ldots, W_n\}$ and $\{\{y_1\}, \ldots, \{y_n\}\}$ are sets of all atoms in the lattices τ and τ', accordingly, then the mapping $\psi : \tau' \to \tau$ is a lattice isomorphism. Then (see [1], Theorem 2.6) the topologies τ and τ' have the same number of one-point expansions and hence (see [1], Theorem 2.7) the topology τ has precisely $2^{(n+1)} + n - 1$ one-point expansions.

The proposition is completely proved. □

Theorem 2. *Let X be a finite set and τ be a topology on X which has precisely m open sets. If $t(\tau)$ is the number of possible one-point expansions of the topology τ on the set $Y = X \bigcup \{y\}$, then the following statements are true:*

$\mathcal{A}$) $\dfrac{m \cdot (m+3)}{2} - 1 \geq t(\tau) \geq 2 \cdot m + log_2 m - 1$;

$\mathcal{B}$) $\dfrac{m \cdot (m+3)}{2} - 1 = t(\tau)$ *if and only if* τ *is a chain (i.e. it is a linearly ordered set);*

$\mathcal{C}$) $t(\tau) = 2 \cdot m + log_2 m - 1$ *if and only if* τ *is an atomistic lattice.*

Proof. $\mathcal{A}$) Let (see Theorem 1) $\{V_1, \ldots, V_k)\}$ be the minimal base in the topological space (X, τ).

As any $U \in \tau$ can be presented as the union of some sets from $\{V_1, \ldots, V_k\}$, then the number of all open sets in the topological space (X, τ) does not exceed the number 2^k of all subsets of the set $\{V_1, \ldots, V_k\}$, and hence, $m \leq 2^k$.

For every $1 \leq i \leq k$ we consider the set $U_i = \bigcup\limits_{U \in \tau, V_i \not\subseteq U} U$.

From the construction of minimal base (see the proof of Theorem 1) it follows that there exists a subset $\{x_1, \ldots, x_k\}$ of the set X such that $V_i = V(x_i) = \bigcap\limits_{U \in \tau, x_i \in U} U$ for $1 \leq i \leq k$. Then for any $1 \leq i \leq k$ it follows that $x_i \in U$ if and only if $V_i \subseteq U$ for any $U \in \tau$, and hence, $U_i = \bigcup\limits_{U \in \tau, x_i \notin U} U$.

Prove first that $U_i \neq U_j$ for $i \neq j$.

We assume the contrary, i.e. that $U_s = U_l$ for some $s \neq l$.

Then from the minimality of the base $\{V_1, \ldots, V_k)\}$ in the topological space (X, τ) it follows that $V_s \neq V_l$. Let (for definiteness) $V_s \not\subseteq V_l$. Then $x_s \notin V_l$ (otherwise $V_s = \bigcap\limits_{U \in \tau, x_s \in U} U \subseteq V_l$), and hence, $V_s \subseteq \bigcup\limits_{U \in \tau, x_s \notin U} U = U_s = U_l$. Then $x_l \in V_l \subseteq U_l$.

We obtain a contradiction with the construction of the sets U_l, and hence, $U_i \neq U_j$ for $i \neq j$.

Now let's apply Algorithm 1 for calculation of the number $t(\tau)$ of one-point expansions of the topology τ.

The following 3 cases are possible:

1. $V_0 = X$;

2. $V_0 = U_i$ for some $1 \leq i \leq k$;

3. $V_0 \notin \{X\} \bigcup \{U_i \mid 1 \leq i \leq k\}$.

Consider each of these cases separately.

1. Let $V_0 = X$. Then $\bigcap\limits_{V \not\subseteq V_0} V = \bigcap\limits_{V \in \emptyset} V = X$, and hence, U_0 can take n values. Then the number of all pairs (X, U_0), where $U_0 \subseteq X$, is equal to m.

2. Now let $V_0 = U_i$ for some $1 \leq i \leq k$. Then

$$\bigcup_{U \in \tau, U \not\subseteq V_0} U = \bigcup_{U \in \tau, U \not\subseteq U_i} U = \bigcup_{U \in \tau, x_i \notin U_i} U = V_i \neq \emptyset,$$

and hence, in this case the set U_0 can take not less than two values.

Then the number of all pairs (U_i, U_0) for $1 \leq i \leq k$ and $U_0 \subseteq \bigcap\limits_{V\in\tau, V\nsubseteq U_i} V$ is not less than $2 \cdot k$.

3. Let $V_0 \notin \{X\} \bigcup \{U_1, \ldots, U_k\}$. Then $\emptyset \subseteq \bigcap\limits_{V\in\tau, V\nsubseteq V_0} V$, and hence, U_0 can take not less than one value. Then the number of all pairs (V_0, U_0) is not less than $1 \cdot (m-k-1) = m-k-1$.

Then the number of all pairs (V_0, U_0) will be not less than

$$m + 2\cdot k + m - k - 1 = 2m + k - 1 \geq 2m + \log_2 m - 1.$$

Then (see [1], Theorem 2.7) the topology τ has not less than $2m + \log_2 m - 1$ one-point expansions, i.e. $t(\tau) \geq 2m + \log_2 m - 1$.

Now let's show that $t(\tau) \leq \dfrac{m\cdot(m+3)}{2} - 1$.

Let $\tau = \{W_1, \ldots, W_m\}$ be such a numbering of the set τ that $W_i \nsubseteq W_j$ for $j < i$ (such a numbering of the set τ is possible as the set τ is finite). Then the set $\{W_j \in \tau \mid W_j \subseteq W_i\}$ has no more than i subsets for every $1 \leq i \leq m$.

If $V_0 = W_i$ for $1 \leq i \leq m-1$, then $U_0 \subseteq \bigcap\limits_{V\in\tau, V\nsubseteq W_i} V \subseteq W_{i+1}$, and hence it has no more than $i+1$ subsets. And as for $V_0 = X$ the set $U_0 \subseteq X$ has m subsets, then the number of all pairs (V_0, U_0) is no more than

$$(\sum_{i=1}^{m-1}(i+1)) + m = \frac{(m+2)\cdot(m-1)}{2} + m =$$

$$\frac{m^2+2m-m-2+2m}{2} = \frac{m\cdot(m+3)}{2} - 1.$$

Then the topology τ has no more than $\dfrac{m\cdot(3+m)}{2} - 1$ one-point expansions.

So, we have proved that $2\cdot m + log_2 m - 1 \leq t(\tau) \leq \dfrac{m\cdot(m+1)}{2} + m - 1$.

The statement $\mathcal{A}$) is proved.

$\mathcal{B}$) If τ is a chain then (see Proposition 1) $\dfrac{m\cdot(m+3)}{2} - 1 = t(\tau)$.

If τ is not a chain, then $W_k \nsubseteq W_{k+1}$ (the definition of W_i at the end of the proof of the statement $\mathcal{A}$)) for some $1 < k < m$. Then the number of possible values for the set U_0 if $V_0 = W_k$ is strictly less than $k+1$, and hence,

$$t(\tau) < (\sum_{i=1}^{m-1}(i+1)) + m = \frac{(m+2)\cdot(m-1)}{2} + m =$$

$$\frac{m^2+2m-m-2+2m}{2} = \frac{m\cdot(m+3)}{2} - 1.$$

The statement $\mathcal{B}$) is proved.

$\mathcal{C}$) If τ is an atomistic lattice and n is the number of atoms, then (see Proposition 2) $t(\tau) = 2^{(n+1)} + n - 1$. So in this case $m = 2^n$, then $t(\tau) = 2 \cdot m + log_2 m - 1$.

If τ is not an atomistic lattice, then the set of all atoms is not a base of a topological space (X, τ), and hence, there exists $1 \leq i \leq k$ such that V_i (definition of sets V_j see in the beginning of the proof of statement $\mathcal{A}$)) is not an atom. Then there exists $\emptyset \neq V' \in \tau$ such that $V' \subset V_i$. As $\bigcup\limits_{U \in \tau, x_i \notin U_i} U = V_i$ (see the beginning of the proof of the case 2 of the statement $\mathcal{A}$)), then the set $\{U \in \tau \mid U \not\subseteq \bigcup\limits_{U \in \tau, x_i \notin U_i} U\}$ contains not less than three subsets from τ, instead of two as we considered in the proof of the statement $\mathcal{A}$) (see the case 2). Hence, in this case we have that $t(\tau) > 2m + \log_2 m - 1$.

The statement $\mathcal{C}$) is proved, and hence, the theorem is proved completely. $\square$

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