On Wallman compactifications of $T_0$-spaces and related questions

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Abstract. We study the compactification of the Wallman-Shanin type of $T_0$-spaces. We have introduced the notion of compressed compactification and proved that any compressed compactification is of the Wallman-Shanin type. The problem of the validity of the equality $\omega(X \times Y) = \omega X \times \omega Y$ is examined. Two open questions have arisen.

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1 Introduction. Preliminaries

Any space is considered to be a $T_0$-space. We use the terminology from [6, 9]. By $|A|$ we denote the cardinality of a set $A$, $w(X)$ be the weight of a space $X$, $\mathbb{N} = \{1, 2, ...\}$. The intersection of $\tau$ open sets is called a $G_{\tau}$-set. For any space $X$ denote by $P_{\tau}X$ the set $X$ with the topology generated by the $G_{\tau}$-sets of the space $X$.

Let $\tau$ be an infinite cardinal. A space $X$ is called $\tau$-subtle if on $X$ the closed $G_{\tau}$-sets form a closed base.

Let $X$ be a dense subspace of a space $Y$. The space $Y$ is called a compressed extension of the space $X$ if for some infinite cardinal $\tau$ the set $X$ is dense in the space $P_{\tau}Y$ and $Y$ is $\tau$-subtle. The cardinal $\tau$ is called the index of compressing of the extension $Y$ of $X$ and put $ic(X \subset Y) \leq \tau$.

Any completely regular space is $\aleph_0$-subtle, i.e. is $\tau$-subtle for any infinite cardinal $\tau$.

Example 1.1. Let $\tau$ be an uncountable cardinal, $I = [0, 1]$ and $L$ be a dense subset of $I^\tau$ of the cardinality $\leq \tau$. Denote by $T_1$ the topology of the Tychonoff cube $I^\tau$ and $T$ is the topology generated by the open base $T_1 \cup \{U \setminus L : U \in T_1\}$. Denote by $X$ the set $I^\tau$ with the topology $T$. The set $L$ is closed in $X$. If $m < \tau$, $H$ is a $G_m$-set of $X$ and $L \subseteq H$, then the set $H$ is dense in $X$. Thus $X$ is a Hausdorff space which is $m$-subtle for any $m < \tau$.

Example 1.2. A space $X$ is called feebly compact if any locally finite family of open non-empty sets is finite. Let $Y$ be an $\aleph_0$-subtle extension of the feebly compact space $X$. Then $Y$ is a compressed extension of the space $X$ and $ic(X \subset Y) \leq \tau$. 
Example 1.3. A completely regular space $X$ is feebly compact if and only if it is pseudocompact. Thus any completely regular extension $Y$ of a pseudocompact space $X$ is compactified and $ic(X \subseteq Y) = R_0$.

Example 1.4. Let $Y$ be the one-point Alexandroff compactification of an uncountable discrete space $X$. Then $ic(X \subseteq Y) = R_0$.

Definition 1.5. A family $L$ of subsets of a space $X$ is called a WS-ring if $L$ is a family of closed subsets of $X$ and $F \cap H, F \cup H \in L$ for any $F, H \in L$.

Definition 1.6. A family $L$ of subsets of a space $X$ is called a WF-ring if $L$ is a WS-ring and $X \setminus F = \cup \{H \in L : H \cap F = \emptyset\}$ for any $F \in L$.

The family $F(X)$ of closed subsets of a space $X$ is a WS-ring. The family $F(X)$ is a WF-ring if and only if $X$ is a $T_1$-space.

Definition 1.7. A $g$-compactification of a space $X$ is a pair $(Y, f)$, where $Y$ is a compact $T_0$-space, $f : X \to Y$ is a continuous mapping, the set $f(X)$ is dense in $Y$ and for any point $y \in Y \setminus f(X)$ the set $\{y\}$ is closed in $Y$. If $f$ is an embedding, then we say that $Y$ is a compactification of $X$ and consider that $X \subseteq Y$, where $f(x) = x$ for any $x \in X$.

Fix a WS-ring $L$ of a space $X$. For any $x \in X$ we put $\xi(x, L) = \{F \in L : x \in F\}$. Denote by $M(L, X)$ the family of all ultrafilters $\xi \in L$. Let $\omega_L X = M(L, X) \cup \{\xi \subseteq L : \xi = \xi(x, y) \text{ for some } x \in X\}$. Consider the mapping $\omega_L : X \to \omega_L X$, where $\omega_L X = \xi(x, L)$ for any $x \in X$. On $\omega_L X$ consider the topology generated by the closed base $< L > = \{< H > = \{\xi \in \omega_L X : H \in \xi\} : H \in L\}$.

Theorem 1.8 (M. Choban, L. Calmuţchi [5]). If $L$ is a WS-ring of a space $X$, then:

1. $(\omega_L X, \omega_L)$ is a $g$-compactification of the space $X$.
2. $< H > = cl_{\omega_L X} \omega_L(H), H = \omega_L^{-1}(< H >)$ and $< H > \cap \omega_L(X) = \omega_L(H)$ for any $H \in L$.
3. $L$ is a WF-ring if and only if $\omega_L X$ is a $T_1$-space.

Definition 1.9. A $g$-compactification $(Y, f)$ of a space $X$ is called a Wallman-Shanin $g$-compactification of the space $X$ if $(X, f) = (\omega_L X, \omega_L)$ for some WS-ring $L$.

Definition 1.10. A $g$-compactification $(Y, f)$ of a space $X$ is called a Wallman-Frink $g$-compactification of the space $X$ if $(X, f) = (\omega_L X, \omega_L)$ for some WF-ring $L$.

The compactifications of the Wallman-Shanin type were introduced by N. A. Shanin [10] and studied by many authors (see [1, 5, 7, 11–14] and the references in these articles). Any Wallman-Frink $g$-compactification is a Wallman-Shanin $g$-compactification. The Wallman compactification $\omega X = \omega_{F(X)} X$ is a Wallman-Shanin compactification of $X$. The compactification $\omega X$ is a Wallman-Frink compactification if and only if $X$ is a $T_1$-space. There exists Hausdorff compactifications of discrete spaces which are not Wallman-Shanin compactifications [11, 13].
2 Comparison of the WS-rings

Following [7] and [14] on the family $\mathcal{F}(X)$ of closed subsets of a space $X$ consider the binary relation $\sim: A \sim B$ if and only if the set $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is relatively compact in $X$, i.e. its closure in $X$ is compact.

For any family $\mathcal{L} \subseteq \mathcal{F}(X)$ we put $m\mathcal{L} = \{ F \in \mathcal{F}(X) : F \sim A \text{ for some } A \in \mathcal{L} \}$. A family $\mathcal{L}$ is called maximal if $\mathcal{L} = m\mathcal{L}$.

**Lemma 2.1.** If $\mathcal{L}$ is a WS-ring of subsets of a space $X$, then $m\mathcal{L}$ is a WS-ring too.

**Proof.** Follows from the relations $(A \cap B) \Delta (F \cap H) \subseteq (A \Delta F) \cup (B \Delta H)$ and $(A \cup B) \Delta (F \cup H) \subseteq (A \Delta F) \cup (B \Delta H)$.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space $X$. We put $\mathcal{L} \leq \mathcal{M}$ if $\mathcal{L} \subseteq \mathcal{M}$ and for each $\xi \in \omega_\mathcal{M}X$ we have $\xi \cap \mathcal{L} \in \omega_\mathcal{L}X$.

**Lemma 2.2.** Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space $X$ and $\mathcal{L} \leq \mathcal{M}$. Then there exists a unique continuous mapping $\varphi : \omega_\mathcal{M}X \to \omega_\mathcal{L}X$ such that $\omega_\mathcal{L}(x) = \varphi(\omega_\mathcal{M}(x))$ for any $x \in X$, i.e. $\omega_\mathcal{L} = \varphi \circ \omega_\mathcal{M}$.

**Proof.** By definition for any $\xi \in \omega_\mathcal{M}X$ we have $\xi \cap \mathcal{L} \in \omega_\mathcal{L}X$. We put $\varphi(\xi) = \xi \cap \mathcal{L}$. Thus $\varphi$ is a mapping of $\omega_\mathcal{M}X$ into $\omega_\mathcal{L}X$. Obviously, $\varphi(\omega_\mathcal{M}X) = \omega_\mathcal{L}X$.

If $x \in X$, then $\xi(x, \mathcal{L}) = \xi(x, \mathcal{M}) \cap \mathcal{L}$. Hence $\omega_\mathcal{L}(x) = \varphi(\omega_\mathcal{M}(x))$. For any $F \in \mathcal{L}$ we have $\varphi^{-1}(\{ \xi \in \omega_\mathcal{L}X : F \cap \xi \}) = \{ \eta \in \omega_\mathcal{L}X : F \cap \eta \}$ for any $F \in \mathcal{L}$. Thus the mapping $\varphi$ is continuous.

If $f : \omega_\mathcal{M}X \to \omega_\mathcal{L}X$ is a continuous mapping and $\omega_\mathcal{L} = f \circ \omega_\mathcal{M}$, then $f^{-1}(\{ \xi \in \omega_\mathcal{L}X : F \cap \xi \}) = \{ \eta \in \omega_\mathcal{L}X : F \cap \eta \}$ for any $F \in \mathcal{L}$. Thus $f = \varphi$. The proof is complete.

**Theorem 2.3.** Let $\mathcal{L}$ be a WS-ring and a closed base of a space $X$ and $F \in \mathcal{L}$ for any closed compact subset $F$ of $X$. Then $(\omega_\mathcal{L}X, \omega_\mathcal{L}) = (\omega_{m\mathcal{L}}X, \omega_{m\mathcal{L}})$. Moreover, $\omega_\mathcal{L}X$ is a compactification of the space $X$.

**Proof.** For any $\xi \in \omega_\mathcal{L}X$ we put $\varphi(\xi) = \xi \cap \mathcal{L}$.

1. $\varphi(\xi) \in \omega_\mathcal{L}X$.

Let $F \in \mathcal{L}$ and $F \notin \xi$. Then there exists $H \in \xi$ such that $F \cap H = \emptyset$. Since $H \in m\mathcal{L}$, we have $H \sim \Phi$ for some $\Phi \in \mathcal{L}$. Hence, there exists a closed compact subset $\Phi_1 \in \mathcal{L}$ such that $H \Delta \Phi_1 = H \Delta \Phi_1$.

Case 1. $\Phi_1 \notin \xi$.

In this case $\cap \xi \neq \emptyset$ and there exists a point $x \in \Phi_1 \subseteq X$ such that $\xi = \xi(x, m\mathcal{L})$.

In this case $\varphi(\xi) = \xi(x, \mathcal{L}) = \omega_\mathcal{L}X$.

Case 2. $\Phi_1 \notin \xi$.

In this case there exists $H_1 \in \xi$ such that $H_1 \subseteq H$ and $H_1 \cap \Phi_1 = \emptyset$. Since $\mathcal{L}$ is a base, there exists $H_2 \in \mathcal{L}$ such that $H_1 \subseteq H_2$ and $H_2 \cap \Phi_1 = \emptyset$. Then $H_2 \in \xi$ and $H_2 \cap F = \emptyset$. Thus $H_2 \in \varphi(\xi)$ and $H_2 \cap F = \emptyset$. Hence $\varphi(\xi)$ is a maximal filter in $\mathcal{L}$, i.e. $\varphi(\xi) \in \omega_\mathcal{L}X$. Claim 1 is proved.

By virtue of Lemma 2.2, $\varphi : \omega_{m\mathcal{L}}X \to \omega_\mathcal{L}X$ is the unique continuous mapping for which $\omega_\mathcal{L} = \varphi \circ \omega_{m\mathcal{L}}$.

Claim 2. If $\xi \in \omega_\mathcal{L}X$, $F \in \xi$, $H \in \mathcal{L}$, $\cap \xi = \emptyset$ and $F \sim H$, then $H \in \xi$. 

There exists a compact subset $\Phi \in \mathcal{L}$ such that $F \triangle H \subseteq \Phi$. Let $H \not\subseteq \xi$. Then there exists $L \in \xi$ such that $L \subseteq F, L \cap H = \emptyset$ and $L \cap \Phi = \emptyset$. Then $F \subseteq H \cup \Phi$ and $L \cap (H \cup \Phi) = \emptyset$, a contradiction.

Claim 3. $\varphi : \omega_{m\mathcal{L}}X \rightarrow \omega_{\mathcal{L}}X$ is a homeomorphism.

Let $\xi_1, \xi_2 \in \omega_{m\mathcal{L}}X$, $\xi_1 \neq \xi_2$ and $\eta = \varphi(\xi_1) = \varphi(\xi_2)$. In this case $\cap \eta = \emptyset$. Thus there exist $H_1 \in \xi_1 \setminus \xi_2$ and $H_2 \in \xi_2 \setminus \xi_1$ such that $H_1 \cap H_2 = \emptyset$. Since $H_1 \cap H_2 \in m\mathcal{L}$, there exist $F_1, F_2 \in \mathcal{L}$ and a compact subset $\Phi \in \mathcal{L}$ such that $F_1 \sim H_1, F_2 \sim H_2, F_1 \triangle H_1 \subseteq \Phi, F_2 \triangle H_2 \subseteq \Phi$. By virtue of Claim 2, we have $F_1 \in \xi_1$ and $F_2 \in \xi_2$. Then $F_1, F_2 \in \eta$ and $F_1 \cap F_2 \subseteq \Phi$, i.e. $\Phi \in \eta$, a contradiction. Therefore $\varphi$ is a one-to-one mapping. Let $H \in m\mathcal{L}$ and $H_1 = \{\xi \in \omega_{m\mathcal{L}}X : H \subseteq \xi\}$. Assume that $\eta \in \omega_{m\mathcal{L}}X$ and $H \not\subseteq \eta$.

Case 1. $\cap \eta \neq \emptyset$.

In this case $\eta = \xi(x, m\mathcal{L})$ for some $x \in X$ and $x \not\subseteq H$. Since $\mathcal{L}$ is a base of $X$, there exists $F \in \mathcal{L}$ such that $x \not\subseteq F$ and $H \subseteq F$. Thus $\varphi(\eta) \not\subseteq c_{\omega_{m\mathcal{L}}X}\varphi(H_1)$.

Case 2. $\cap \eta = \emptyset$.

In this case there exists $F \in \mathcal{L}$ such that $H \sim F$. We can assume that $H \subseteq F$. Then $F \not\subseteq \eta$ and $\varphi(\eta) \not\subseteq c_{\omega_{m\mathcal{L}}X}\varphi(H_1) \subseteq \{\xi \in \omega_{m\mathcal{L}}X : F \in \xi\}$. Therefore the set $\varphi(H_1)$ is closed for any $H \in m\mathcal{M}$. Since $\{H_1 : H \in m\mathcal{M}\}$ is a closed base of $\omega_{\mathcal{L}}X$, the mapping $\varphi$ is closed. Hence $\varphi$ is a homeomorphism. The proof is complete.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space $X$. We put $\mathcal{L} << \mathcal{M}$ if for any two sets $F_1, F_2 \in \mathcal{L}$, with the empty intersection $F_1 \cap F_2 = \emptyset$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of $X$, $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$. If $\mathcal{L} << \mathcal{M}$ and $\mathcal{M} << \mathcal{L}$, then we put $\mathcal{L} \approx \mathcal{M}$.

**Proposition 2.4.** Let $\mathcal{L}, \mathcal{M}$ be two WS-rings and closed bases of a space $X$ and $F \in \mathcal{L} \cap \mathcal{M}$ for any closed compact subset $F$ of $X$. The next assertions are equivalent:

1. $\mathcal{L} \ll \mathcal{M}$,
2. $m\mathcal{L} \ll m\mathcal{M}$,
3. $m\mathcal{L} \ll \mathcal{M}$

**Proof.** Let $\mathcal{L} \ll \mathcal{M}$. Assume that $F_1, F_2 \in m\mathcal{L}$ and $F_1 \cap F_2 = \emptyset$. By virtue of Theorem 2.3, we have $c_{\omega_{\mathcal{L}}X}F_1 \cap c_{\omega_{\mathcal{L}}X}F_1 = \emptyset$. Then there exist two sets $L_1, L_2 \in \mathcal{L}$ such that $L_1 \cap L_2 = \emptyset, F_1 \subseteq L_1$ and $F_2 \subseteq L_2$. Since $\mathcal{L} \ll \mathcal{M}$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of $X$, $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$. Therefore $m\mathcal{L} \ll \mathcal{M}$ and $m\mathcal{L} \ll m\mathcal{M}$. The implications $1 \rightarrow 3 \rightarrow 2 \rightarrow 3$ are proved. Theorem 2.3 completes the proof.

**Proposition 2.5.** Let $\omega_{\mathcal{L}}X$ and $\omega_{\mathcal{M}}X$ be Hausdorff compactifications of a space $X$ and $F \in \mathcal{M}$ for any compact subset $F$ of $X$. The next assertions are equivalent:

1. There exists a continuous mapping $\varphi : \omega_{\mathcal{M}}X \rightarrow \omega_{\mathcal{L}}X$ such that $\varphi(x) = x$ for any $x \in X$.
2. $\mathcal{L} \ll \mathcal{M}$,
3. $m\mathcal{L} \ll \mathcal{M}$.
4. $m\mathcal{L} \ll m\mathcal{M}$.

**Proof.** Let $\varphi : \omega_{\mathcal{M}}X \rightarrow \omega_{\mathcal{L}}X$ be a continuous mapping and $\varphi(x) = x$ for any $x \in X$. Fix $F_1, F_2 \in \mathcal{L}$ such that $F_1 \cap F_2 = \emptyset$. Then $c_{\omega_{\mathcal{L}}X}F_1 \cap c_{\omega_{\mathcal{L}}X}F_2 = \emptyset$. Since $\varphi$ is a
continuous mapping, then \(cl_{\omega_X} F_1 \cap cl_{\omega_X} F_2 = \emptyset\). The family \(\{cl_{\omega_M} X H : H \in \mathcal{M}\}\) is a closed base of a compact space \(\omega_M X\). Thus there exists \(H_1, H_2 \in \mathcal{M}\) such that \(H_1 \cap H_2 = \emptyset, F_1 \subseteq H_1, F_2 \subseteq H_2\). Implication 1 \(\rightarrow\) 2 is proved.

Assume that \(\mathcal{L} \ll \mathcal{M}\). There exist two continuous mappings \(f : \beta X \rightarrow \omega_X\) and \(g : \beta X \rightarrow \omega_M X\) of the Stone-Čech compactification \(\beta X\) of \(X\) such that \(f(x) = g(x) = x\) for any \(x \in X\). It is sufficient to prove that \(\varphi(x) = f(g^{-1}(x))\) is a singleton for any \(x \in \omega_M X\). Let \(y \in \omega_M X\) and \(x_1, x_2 \in \varphi(y)\) be two distinct points of \(\omega_X\). Obviously, \(y \in \omega_M X \setminus X\) and \(\varphi(y) \subseteq \omega_X\). There exists \(F_1, F_2 \in \mathcal{L}\) such that \(x_1 \in cl_{\omega_X} F_1, x_2 \in cl_{\omega_X} F_2\) and \(F_1 \cap F_2 = \emptyset\). Let \(H_1, H_2 \in \mathcal{M}\), \(H = H_1 \cap H_2\) be a compact subset of \(X\), \(F_1 \subseteq H_1, F_2 \subseteq H_2\). Then \(H = cl_{\omega_M} X H_1 \cap cl_{\omega_M} X H_2\). Let \(\Phi_1 = f^{-1}(x_1)\) and \(\Phi_2 = f^{-1}(x_2)\). Then \(y \in g(\Phi_1) \cap g(\Phi_2)\). Since \(\Phi_1 \subseteq cl_{\beta X} F_1\) and \(g(cl_{\beta X} F_1) = cl_{\omega_M} X F_1\) we have \(g(\Phi_1) \subseteq cl_{\omega_M} X F_1 \subseteq cl_{\omega_M} X H_1\) and \(g(\Phi_2) \subseteq cl_{\omega_M} X F_2 \subseteq cl_{\omega_M} X H_2\). Hence \(Y \in H \subseteq X\), a contradiction. Implication 2 \(\rightarrow\) 1 is proved. Proposition 2.4 completes the proof.

**Corollary 2.6.** Let \(\omega_X\) and \(\omega_M X\) be Hausdorff compactifications of a space \(X\). Then \(\omega_X = \omega_M X\) if and only if \(\mathcal{L} \approx \mathcal{M}\).

3. **On compressed compactification**

**Theorem 3.1.** If \((Y, f)\) is a compressed \(g\)-compactification of a space \(X\), then \((Y, f)\) is a Wallman-Shanin \(g\)-compactification of the space \(X\).

**Proof.** Let \(\tau\) be a cardinal number for which:
- \(f(X)\) is dense in \(P_\tau Y\);
- the closed \(G_\tau\)-subsets of \(Y\) form a closed base of the space \(Y\).

We put \(Z = f(X)\). Denote by \(\mathcal{F}_\tau(Y)\) the family of all closed \(G_\tau\)-subsets of \(Y\). By construction, \(\mathcal{L} = \{f^{-1}(H) : H \in \mathcal{F}_\tau(Y)\}\) is a \(WS\)-ring of closed subsets of the space \(X\).

**Claim 1.** If \(H \in \mathcal{F}_\tau(Y)\), then \(H = cl_Y (H \cap Z)\).

Obiously, \(cl_Y (H \cap Z) \subseteq H\). Let \(y \in H, U\) be an open subset of \(Y\) and \(y \in U\). Then \(Y = U \cap H\) is a \(G_\tau\)-subset of \(Y\). Since \(Z\) is \(G_\tau\)-dense in \(Y\), we have \(V \cap Z \neq \emptyset\). Hence \(U \cap (H \cap Z) \supseteq V \cap Z \neq \emptyset\) and \(y \in cl_Y (H \cap Z)\). Claim is proved.

**Claim 2.** \((Y, f) = (\omega_X, \omega_X)\).

Let \(\xi \in \omega_X\).

Case 1. \(\cap \xi \neq \emptyset\).

There exists \(x \in X\) such that \(\xi = \xi(x, \mathcal{L})\). In this case we put \(\varphi(\xi) = f(x)\)

Case 2. \(\cap \xi = \emptyset\).

In this case \(\xi\) is an \(\mathcal{L}\)-ultrafilter. Let \(\xi = \{L_\alpha : \alpha \in A\}\). For each \(\alpha \in A\) there exists a unique \(H_\alpha \in \mathcal{F}_\tau(Y)\) such that \(L_\alpha = f^{-1}(H_\alpha)\). By construction, \(\eta = \{H_\alpha : \alpha \in A\}\) is an \(\mathcal{F}_\tau(Y)\)-ultrafilter and \(\cap \eta = \emptyset\). There exists a unique point \(y \in Y \setminus Z\) such that \(y \in \cap \eta\). We put \(\varphi(\xi) = y\).

The mapping \(\varphi : \omega_X \rightarrow Y\) of \(\omega_X\) onto \(Y\) is constructed. Obviously, the mapping \(\varphi\) is one-to-one. By construction, \(\varphi(\omega_X) = f(x)\) for any \(x \in X\) and \(\varphi(\{\xi \in \omega_X : f^{-1}(H) \in \xi\}) = H\) for each \(H \in \mathcal{F}_\tau(Y)\). Hence the mapping \(\varphi\) is a homeomorphism. The proof is complete.
Corollary 3.2. Let $Y$ be a Hausdorff compactification of a space $X$ and the space $X$ is $G_\tau$-dense in $Y$. Then $Y$ is a Wallman-Frink compactification of the space $X$.

Corollary 3.3 (R. A. Alo, H. L. Shapiro [1], E. Wajch [14]). Let $X$ be a pseudo-compact space. Then any Hausdorff compactification $Y$ of $X$ is a Wallman-Frink compactification.

Corollary 3.4. Let $(Y, f)$ be a Hausdorff $g$-compactification of a feebly compact space $X$. Then $(Y, f)$ is a Wallman-Frink $g$-compactification of the space $X$.

For any discrete uncountable space the family of compressed Hausdorff compactifications is large. Moreover, this fact is valid for Hausdorff paracompact locally compact non-Lindelöf spaces.

Theorem 3.5. Let $X$ be a Hausdorff locally compact space which contains an uncountable discrete family of open non-empty subsets. Assume that $\dim X = 0$. Then the family $\mathcal{B}$ of all compressed Hausdorff compactifications of $X$ is uncountable and $\beta X = \sup \mathcal{B}$.

Proof. Fix $n \geq 2$. There exists a family $\{X_\mu : \mu \in M\}$ of open-and-closed subsets of $X$ such that for any $\mu \in M$ the set $X_\mu$ is compact and there exist $n$ distinct points $b_1, b_2, ..., b_n \in X_\mu$. The sets $\{B_i = \{b_\mu : \mu \in M\} : i \leq n\}$ are closed and disjoint. Fix $n$ distinct points $b_1, b_2, ..., b_n \in \beta X \setminus X$. Since the sets $\{\text{cl}_{\beta X} B_i : i \leq n\}$ are disjoint we can assume that $b_i \notin \bigcup \{\text{cl}_{\beta X} B_j : j \leq n, j \neq i\}$ for any $i \leq n$. Fix $n$ open-and-closed subsets $\{H_i : i \leq n\}$ of $\beta X \setminus X$ such that $b_i \in H_i, H_i \cap H_j = \emptyset, H_i \cap \text{cl}_{\beta X} B_j = \emptyset$ for any $i \neq j$ and $\beta X \setminus X = \bigcup \{H_i : i \leq n\}$. Then there exists a compactification $Y$ of $X$ and a continuous mapping $f : \beta X \to Y$ such that $f(x) = x$ for any $x \in X$ and $f^{-1}(f(b_i)) = H_i$ for any $i \leq n$. The compactification $Y$ is compressed. By construction, the compressed compactifications $\mathcal{B}$ of $X$ separate the points of $\beta X$. Thus $\beta X = \sup \mathcal{B}$. The proof is complete.

4 Cartesian products of compactifications

Let $A$ be a non-empty set, $\{X_\alpha : \alpha \in A\}$ be a family of non-empty spaces, $X = \prod \{X_\alpha : \alpha \in A\}$, $(b_\alpha X_\alpha, \varphi_\alpha)$ be a family of $g$-compactifications of given spaces $X_\alpha$. Then $bX = \prod \{b_\alpha X_\alpha : \alpha \in A\}$ and the mapping $\varphi : X \to bX$, where $\varphi((x_\alpha : \alpha \in A)) = (\varphi_\alpha(x_\alpha) : \alpha \in A)$ for any $(x_\alpha : \alpha \in A) \in X$, is a $g$-compactification of the space $X$. If each $b_\alpha X_\alpha$ is a compactification of the space $X_\alpha$, then $bX$ is a compactification of the space $X$. Let $\mathcal{L}_\alpha$ be a $WS$-ring of closed sets of the space $X_\alpha$. We put $\mathcal{L}' = \{\Pi \{H_\alpha : \alpha \in A\} : H_\alpha \in \mathcal{L}_\alpha, \alpha \in A\}, \mathcal{L} = \{H_1 \cup H_2 \cup \ldots \cup H_n : H_1, H_2, ..., H_n \in \mathcal{L}', n \in \mathbb{N}\}$.

Now we put $\mathcal{L} = \otimes \{\mathcal{L}_\alpha : \alpha \in A\}$.

Theorem 4.1. The family $\mathcal{L}$ of closed subsets of the space $X$ is a $WS$-ring and $\omega_\mathcal{L} X = \Pi \{\omega_{\mathcal{L}_\alpha} X_\alpha : \alpha \in A\}$.

Proof. Obviously, $\mathcal{L}$ is a $WS$-ring of closed subsets of the space $X$.

Let $\xi$ be an $\mathcal{L}'$-filter. Obviously $e(\xi) = \{H \cup F : H \in \xi, F \in \mathcal{L}\}$ is a $\mathcal{L}$-filter. Moreover, $\xi$ is an $\mathcal{L}'$-ultrafilter if and only if $e(\xi)$ is an $\mathcal{L}$-ultrafilter.
Let \( x = (x_\alpha : \alpha \in A) \), \( \xi_\alpha \subseteq \mathcal{L}_\alpha \) and \( \xi = \{ \Pi \{ H_\alpha : \alpha A \} : H_\alpha \in \xi_\alpha, \alpha \in A \} \). Then:
- if any \( \xi_\alpha \) is an \( \mathcal{L}_\alpha \)-filter, then \( \xi \) is an \( \mathcal{L}' \)-filter;
- \( \xi_\alpha \) is an \( \mathcal{L}_\alpha \)-ultrafilter if and only if \( \xi \) is an \( \mathcal{L}' \)-ultrafilter;
- \( \xi_\alpha = \xi(x_\alpha, \mathcal{L}_\alpha) \) for any \( \alpha \in A \) if and only if \( \xi = \xi(x, \mathcal{L}') \) and \( e(\xi) = \xi(x, \mathcal{L}). \)

These facts complete the proof.

Theorem 4.2. If \( |A| \geq 2 \), \( \omega X = \Pi \{ \omega X_\alpha : \alpha \in A \} \) and any \( X_\alpha \) is an infinite \( T_1 \)-space, then any space \( X_\alpha \) is countably compact.

Proof. Fix \( \beta \in A \). Assume that the space \( X_\beta \) is not countably compact. Then \( X_\beta \) contains an infinite, discrete and closed subset \( F = \{ b_n : n \in \mathbb{N} \} \).

Since \( \omega Z_1 = cl_{\omega Z} Z_1 \) for any closed subspace \( Z_1 \) of a \( T_0 \)-space \( Z \), we can assume that \( X_\beta F \).

We put \( Y_\beta = \Pi \{ \omega X_\alpha : \alpha \in A \setminus \{ \beta \} \} \). Obviously, \( X = X_\beta \times Y_\beta \) and \( \omega X = \omega X_\beta \times \omega Y_\beta \).

If the space \( Y_\beta \) is not countably compact, then \( Y_\beta \) contains a discrete infinite space \( Z \) and \( \omega (X_\beta \times Z) = cl_{\omega X} (X_\beta \times Z) = (\omega X_\beta \times \omega Z) \), a contradiction with the Glicksberg’s theorem ([6], Problem 3.12.20(d)). Thus we can assume that the space \( Y_\beta \) is countably compact.

In the space \( Y_\beta \) fix a set \( L = \{ c_n : n \in \mathbb{N} \} \), where \( c_n \neq c_m \) for distinct \( n, m \in \mathbb{N} \). The set \( \Phi = \{ (b_n, c_n) : n \in \mathbb{N} \} \) is closed and discrete in \( X \). Projection \( p : X_\beta \times Y_\beta \to X_\beta \) is a continuous closed mapping. Fix an ultrafilter \( \xi \) of closed subsets of the space \( X \) for which \( \Phi \in \xi \) and \( \cap \xi = \emptyset \). Then \( p(\xi) = \{ p(H) : H \in \xi \} \) is an ultrafilter of closed subsets of the space \( X_\beta \). If \( \cap p(\xi) = \emptyset \), then there exists a unique point \( b \in X_\beta \) for which \( \{ b \} = \cap p(\xi) \). In this case \( \{ b \} \times Y_\beta \in \xi \) and \( \cap \xi = \emptyset \). Since \( \Phi \in \xi \), there exists a unique \( n \in \mathbb{N} \) such that \( b = b_n \) and \( (b_n, c_n) \in H \cap (\{ b \} \times Y_\beta) \) for each \( H \in \xi \), a contradiction with \( \cap \xi = \emptyset \). Thus \( \cap p(\xi) = \emptyset \). Hence there exists a unique \( b \in \omega X_\beta \setminus X_\beta \) for which \( \{ b \} = \cap \{ cl_{X_\beta} H : H \in p(\xi) \} \).

Since \( \omega X = \omega X_\beta \times \omega Y_\beta \), there exists a unique \( c \in \omega Y_\beta \setminus Y_\beta \) such that \( (b, c) \in \cap \{ cl_{\omega X} H : H \in \xi \} \). In this case \( X_\beta \times \{ c \} \in \xi \). There exists a unique \( n \in \mathbb{N} \) and some \( H \in \xi \) such that \( (b_n, c_n) \in \Phi \cap (X_\beta \times \{ c \}) \) and \( H \cap (X_\beta \times \{ c \}) = \emptyset \). Then \( (b, c) \in cl_{\omega X} H \cap cl_{\omega X} (X_\beta \times \{ c \}) \) and \( cl_{\omega X} H \cap cl_{\omega X} (X_\beta \times \{ c \}) \), a contradiction. The proof is complete.

Theorem 4.3. Let \( f : X \to Y \) be a continuous closed mapping of a space \( X \) onto a space \( Y \). Then there exists a unique continuous mapping \( \omega f : \omega X \to \omega Y \) such that \( f = \omega f | X \). Moreover, the mapping \( \omega f \) is closed.

Proof. If \( \xi \) is an ultrafilter of closed subsets of \( X \), then \( \omega f(\xi) = \{ f(H) : H \in \xi \} \) is an ultrafilter of closed subsets of \( Y \). The mapping \( \omega f \) is constructed.

Let \( \tau \) be an infinite cardinal number. A space \( X \) is called initial \( \tau \)-compact if any open cover \( \gamma \) of \( X \) of the cardinality \( \leq \tau \) contains a finite subcover.

We say that the sequential character \( s_\chi(X) < \tau \) if for any non-closed subset \( H \) of \( X \) there exist a subset \( Y \subseteq X \) and a point \( x \in X \setminus H \) such that \( x \in Y \), \( x \in cl_X (H \cap Y) \) and \( \chi(Y, x) < \tau \). A space \( X \) is sequential if and only if \( s_\chi(X) \leq \aleph_0 \).
Theorem 4.4. Let $\tau$ be an infinite cardinal number, $X$ be an initial $\tau$-compact space, $Y$ be a compact space and $s\chi(Y) \leq \tau$. Then:

1. The projection $p : X \times Y \rightarrow Y$, where $p(x,y) = y$ for each $(x,y) \in X \times Y$, is a continuous closed-and-open mapping.

2. There exists a continuous bijection $\varphi : \omega(X \times Y) \rightarrow \omega X \times Y$ such that $\varphi(x,y) = (x,y)$ for all $(x,y) \in X \times Y$.

Proof. It is well known that the projection $p$ is continuous and open.

Let $y_0 \in Y, W$ be an open subset of $X \times Y$ and $p^{-1}(y_0) = X \times \{y_0\} \subseteq W$. We put $V = \{y \in Y : p^{-1}(y) \subseteq W\}$. Obviously, $y_0 \in V$. We affirm that the set $V$ is open in $Y$. Suppose that the set $V$ is not open in $Y$. Then the set $Y \setminus V$ is not closed in $Y$. Thus there exist a point $z \in V$ and a subspace $Z \subseteq Y$ such that $z \in Z$, $z \in \text{cl}_Z(Z \cap (Y \setminus V))$ and $\chi(Z,z) \leq \tau$. We fix an open base $\{V_\alpha : \alpha \in A\}$ of the space $Z$ at the point $z$ such that $|A| \leq \tau$. For any $\alpha \in A$ consider the set $U_\alpha = \cup\{U : U \text{ is open } X, U \times V_\alpha \subseteq W\}$. Obviously $X = \cup\{U_\alpha : \alpha \in A\}$.

Since $X$ is $\tau$-compact and $|A| \leq \tau$, there exists a finite set $B \subseteq A$ such that $X = \cup\{U_\alpha : \alpha \in B\}$. There exists an element $\beta \in A$ for which $V_\beta \subseteq \cap\{V_\alpha : \alpha \in B\}$. Then $U_\beta \supseteq \cup\{U_\alpha : \alpha \in B\} = X$. Hence $U_\beta = X$ and $X \times V_\beta = U_\beta \times V_\beta \subseteq W$. Therefore $z \in V_\beta \subseteq V$ and $z \notin \text{cl}_Y(Y \setminus V)$, a contradiction. Assertion 1 is proved.

Consider the projection $f : X \times Y \rightarrow X$. The mappings $f$ and $p$ are continuous open-and-closed. Then there exist two continuous closed mappings $\omega f : \omega(X \times Y) \rightarrow \omega X$ and $\omega p : \omega(X \times Y) \rightarrow \omega X$ such that $f = \omega f|X \times Y$ and $p = \omega p|X \times Y$. Consider the continuous mapping $\varphi : \omega(X \times Y) \rightarrow \omega X \times Y$ for which $\varphi(z) = (\omega f(z), \omega p(z))$ for each $z \in \omega(X \times Y)$. By construction, we have $\varphi(z) = ((x,y), p(x,y)) = (x,y) = z$ for each $z = (x,y) \in X \times Y \subseteq \omega(X \times Y)$. Fix $z \in \omega(X \times Y) \setminus (X \times Y)$. Then there exists a unique ultrafilter $\xi$ of closed subsets of $X \times Y$ for which $\{z\} = \cap\{\text{cl}_{\omega(X \times Y)}H : H \in \xi\}$. The family $p(\xi) = \{g(H) : H \in \xi\}$ is an ultrafilter of closed subsets of the space $Y$. There exists a unique point $y(z) = \omega g(z) \in \cap\{\text{cl}_Yg(H) : H \in \xi\}$. In this case $X(\xi) = X \times \{y(z)\} \in \xi$. Thus $\xi = \{H \cap X(\xi) : H \in \xi\} \subseteq \xi$ is an ultrafilter of closed subsets of the subspace $X(\xi)$ of $X \times Y$.

Let $\xi, \eta$ be two ultrafilters of closed subsets of the space $X \times Y$, $z \in \cap\{\text{cl}_{\omega(X \times Y)}H : H \in \xi\}$ and $z' \in \cap\{\text{cl}_{\omega(X \times Y)}H : H \in \eta\}$. Assume that $y(z) = y(z')$. Then $X(\xi) = X(\eta)$ and there exist $H \in \xi$ and $L \in \eta$ such that $H \cap L = \emptyset$. Since $f|X(\xi) : X(\xi) \rightarrow X$ is a homeomorphism, $f(\xi) = f(\eta) = f(\overline{\eta})$ and $f(H) \cap f(L) = \emptyset$. Thus $f(\xi) \neq f(\eta)$ and $\omega f(z) = \cap\{\text{cl}_{\omega X}f(M) : M \in \xi\} \neq \cap\{\text{cl}_{\omega X}f(P) : P \in \xi \eta\} = \omega f(z')$. Therefore $\varphi$ is a bijection. The proof is complete.

Corollary 4.6. Let $\tau$ be an infinite cardinal number, $X$ be an initial $\tau$-compact normal space, $Y$ be a compact Hausdorff space and $s\chi(Y) \leq \tau$. Then:

1. $\omega(X \times Y) = \omega X \times Y$.

2. $X \times Y$ is an initial $\tau$-compact normal space.

Remark 4.7. Let $X$ be a first countable normal countably compact not paracompact space and $Y = \beta X$. By virtue of Tamano’s Theorem (see [6], Theorem 5.1.38), the space $X \times Y$ is not normal. Then $\omega X = \beta X$ and $\omega(X \times Y) \neq \beta(X \times Y) = \omega X \times Y$. Thus the restriction $s\chi(Y) \leq \tau$ in the above assertions is essential.
5 Remainders of compactifications

The main result of the section is the following theorem.

**Theorem 5.1** For any space $Y$ the following assertions are equivalent:

1. $Y$ is a $T_1$-space.

2. There exists a $T_0$-space $X$ such that the spaces $Y$ and $\omega X \setminus X$ are homeomorphic.

3. There exists a $T_1$-space $X$ such that the spaces $Y$ and $\omega X \setminus X$ are homeomorphic.

**Proof.** Let $X$ be a $T_0$-space and $Y = \omega X \setminus X$. Any ultrafilter of closed sets $\xi$ represents a point $\xi \in \omega X$ for which the set $\{\xi\}$ is closed in $\omega X$. Thus $Y$ is a $T_1$-space. Implication $2 \to 1$ is proved. Implication $3 \to 2$ is obvious.

Let $Y$ be a non-empty $T_1$-space. If $Y$ is compact, then we put $Z = Y$. Let $Y$ be a non-compact space. Consider a point $b \notin Y$. In this case $Y$ is an open subspace of the space $Z = Y \cup \{b\}$, where the base of the space $Z$ at the point $b$ is the family $\{Z \setminus \Phi : \Phi$ is a closed compact subset of $Y\}$. By construction $Z$ is a compact $T_1$-space. Fix an infinite cardinal number $\tau \geq w(Z)$. Denote by $W(\tau^+)$ the space of all ordinal numbers of the cardinality $\leq \tau$ in the topology generated by the linear order. Then $W(\tau^+)$ is a normal initial $\tau$-compact space and $\omega W(\tau^+) \setminus (\tau^+) = \{c\}$ is a singleton.

If the space $Y$ is compact, we consider the space $X = W(\tau^+) \times Y$ as a subspace of the compact space $\omega W(\tau^+) \times Z$. Further, if the space $Y$ is not compact, then we consider the space $X = (W(\tau^+) \times Y) \cup \{(c, b)\}$ as a subspace of the compact space $\omega W(\tau^+) \times Z$.

Since the space $X$ is initial $\tau$-compact and $s(x)(\tau) \leq \tau$, the mapping $g : X \to Z$, where $g(z, y) = y$ for any $(z, y) \in X$, is continuous and open-and-closed. Hence $\omega X = \omega W(\tau^+) \times Z$. By construction, the spaces $\omega X \setminus X = \{c\} \times Y$ and $Y$ are homeomorphic. The proof is complete.

Any Hausdorff locally compact space is a Wallman remainder of some normal space.

**Question 1.** Under which conditions a completely regular space is a Wallman remainder of some normal space?

**Question 2.** Under which conditions a $T_1$-space is a Wallman remainder of some completely regular (regular, Hausdorff) space?

Other problems about remainders of spaces have been examined recently in [2–4].

**References**


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