BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(66), 2011, Pages 102–111
ISSN 1024–7696

On Wallman compactifications of T_0-spaces and related questions

L. I. Calmuţchi, M. M. Choban

Abstract. We study the compactification of the Wallman-Shanin type of T_0-spaces. We have introduced the notion of compressed compactification and proved that any compressed compactification is of the Wallman-Shanin type. The problem of the validity of the equality $\omega(X \times Y) = \omega X \times \omega Y$ is examined. Two open questions have arisen.

Mathematics subject classification: 54D35, 54B10, 54C20, 54D40.
Keywords and phrases: Compactification, ring of sets, sequential character.

1 Introduction. Preliminaries

Any space is considered to be a T_0-space. We use the terminology from [6, 9]. By $|A|$ we denote the cardinality of a set A, $w(X)$ be the weight of a space X, $\mathbb{N} = \{1, 2, ...\}$. The intersection of τ open sets is called a G_τ-set. For any space X denote by $P_\tau X$ the set X with the topology generated by the G_τ-sets of the space X.

Let τ be an infinite cardinal. A space X is called *τ-subtle* if on X the closed G_τ-sets form a closed base.

Let X be a dense subspace of a space Y. The space Y is called a *compressed extension* of the space X if for some infinite cardinal τ the set X is dense in the space $P_\tau Y$ and Y is τ-subtle. The cardinal τ is called the *index of compressing* of the extension Y of X and put $ic(X \subset Y) \leq \tau$.

Any completely regular space is $\aleph_0$-subtle, i.e. is τ-subtle for any infinite cardinal τ.

Example 1.1. Let τ be an uncountable cardinal, $I = [0, 1]$ and L be a dense subset of I^τ of the cardinality $\leq \tau$. Denote by $\mathcal{T}_1$ the topology of the Tychonoff cube I^τ and $\mathcal{T}$ is the topology generated by the open base $\mathcal{T}_1 \cup \{U \setminus L : U \in \mathcal{T}_1\}$. Denote by X the set I^τ with the topology $\mathcal{T}$. The set L is closed in X. If $m < \tau$, H is a G_m-set of X and $L \subseteq H$, then the set H is dense in X. Thus X is a Hausdorff space which is not m-subtle for any $m < \tau$.

Example 1.2. A space X is called feebly compact if any locally finite family of open non-empty sets is finite. Let Y be an $\aleph_0$-subtle extension of the feebly compact space X. Then Y is a compressed extension of the space X and $ic(X \subset Y) \leq \tau$.

Example 1.3. A completely regular space X is feebly compact if and only if it is pseudocompact. Thus any completely regular extension Y of a pseudocompact space X is compressed and $ic(X \subseteq Y) = \aleph_0$.

Example 1.4. Let Y be the one-point Alexandroff compactification of an uncountable discrete space X. Then $ic(X \subseteq Y) = \aleph_0$.

Definition 1.5. *A family $\mathcal{L}$ of subsets of a space X is called a WS-ring if $\mathcal{L}$ is a family of closed subsets of X and $F \cap H, F \cup H \in \mathcal{L}$ for any $F, H \in \mathcal{L}$.*

Definition 1.6. A family $\mathcal{L}$ of subsets of a space X is called a WF-ring if $\mathcal{L}$ is a WS-ring and $X \setminus F = \cup\{H \in \mathcal{L} : H \cap F = \emptyset\}$ for any $F \in \mathcal{L}$.

The family $\mathcal{F}(X)$ of closed subsets of a space X is a WS-ring. The family $\mathcal{F}(X)$ is a WF-ring if and only if X is a T_1-space.

Definition 1.7. *A g-compactification of a space X is a pair (Y, f), where Y is a compact T_0-space, $f : X \to Y$ is a continuous mapping, the set $f(X)$ is dense in Y and for any point $y \in Y \setminus f(X)$ the set $\{y\}$ is closed in Y. If f is an embedding, then we say that Y is a compactification of X and consider that $X \subseteq Y$, where $f(x) = x$ for any $x \in X$.*

Fix a WS-ring $\mathcal{L}$ of a space X. For any $x \in X$ we put $\xi(x, \mathcal{L}) = \{F \in \mathcal{L} : x \in F\}$. Denote by $M(\mathcal{L}, X)$ the family of all ultrafilters $\xi \in \mathcal{L}$. Let $\omega_{\mathcal{L}}X = M(\mathcal{L}, X) \cup \{\xi \subseteq \mathcal{L} : \xi = \xi(x, y)$ for some $x \in X\}$. Consider the mapping $\omega_{\mathcal{L}} : X \to \omega_{\mathcal{L}}X$, where $\omega_{\mathcal{L}}X = \xi(x, \mathcal{L})$ for any $x \in X$. On $\omega_{\mathcal{L}}X$ consider the topology generated by the closed base $< \mathcal{L} >= \{< H >= \{\xi \in \omega_{\mathcal{L}}X : H \in \xi\} : H \in \mathcal{L}\}$.

Theorem 1.8 (M. Choban, L. Calmuţchi [5]). *If $\mathcal{L}$ is a WS-ring of a space X, then:*

1. *$(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ is a g-compactification of the space X.*
2. *$< H >= cl_{\omega_{\mathcal{L}}X}\omega_{\mathcal{L}}(H)$, $H = \omega_{\mathcal{L}}^{-1}(< H >)$ and $< H > \cap \omega_{\mathcal{L}}(X) = \omega_{\mathcal{L}}(H)$ for any $H \in \mathcal{L}$.*
3. *$\mathcal{L}$ is a WF-ring if and only if $\omega_{\mathcal{L}}X$ is a T_1-space.*

Definition 1.9. *A g-compactification (Y, f) of a space X is called a Wallman-Shanin g-compactification of the space X if $(X, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ for some WS-ring $\mathcal{L}$.*

Definition 1.10. *A g-compactification (Y, f) of a space X is called a Wallman-Frink g-compactification of the space X if $(X, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ for some WF-ring $\mathcal{L}$.*

The compactifications of the Wallman-Shanin type were introduced by N. A. Shanin [10] and studied by many authors (see [1, 5, 7, 11–14] and the references in these articles). Any Wallman-Frink g-compactification is a Wallman-Shanin g-compactification. The Wallman compactification $\omega X = \omega_{\mathcal{F}(X)}X$ is a Wallman-Shanin compactification of X. The compactification ωX is a Wallman-Frink compactification if and only if X is a T_1-space. There exists Hausdorff compactifications of discrete spaces which are not Wallman-Shanin compactifications [11, 13].

2 Comparison of the WS-rings

Following [7] and [14] on the family $\mathcal{F}(X)$ of closed subsets of a space X consider the binary relation $\sim$: $A \sim B$ if and only if the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is relatively compact in X, i.e. its closure in X is compact.

For any family $\mathcal{L} \subseteq \mathcal{F}(X)$ we put $m\mathcal{L} = \{F \in \mathcal{F}(X) : F \sim A \text{ for some } A \in \mathcal{L}\}$. A family $\mathcal{L}$ is called maximal if $\mathcal{L} = m\mathcal{L}$.

Lemma 2.1. *If $\mathcal{L}$ is a WS-ring of subsets of a space X, then $m\mathcal{L}$ is a WS-ring too.*

Proof. Follows from the relations $(A \cap B)\triangle(F \cap H) \subseteq (A\triangle F) \cup (B\triangle H)$ and $(A \cup B)\triangle(F \cup H) \subseteq (A\triangle F) \cup (B\triangle H)$.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X. We put $\mathcal{L} \leq \mathcal{M}$ if $\mathcal{L} \subseteq \mathcal{M}$ and for each $\xi \in \omega_{\mathcal{M}}X$ we have $\xi \cap \mathcal{L} \in \omega_{\mathcal{L}}X$.

Lemma 2.2. *Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X and $\mathcal{L} \leq \mathcal{M}$. Then there exists a unique continuous mapping $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ such that $\omega_{\mathcal{L}}(x) = \varphi(\omega_{\mathcal{M}}(x))$ for any $x \in X$, i.e. $\omega_{\mathcal{L}} = \varphi \circ \omega_{\mathcal{M}}$.*

Proof. By definition for any $\xi \in \omega_{\mathcal{M}}X$ we have $\xi \cap \mathcal{L} \in \omega_{\mathcal{L}}X$. We put $\varphi(\xi) = \xi \cap \mathcal{L}$. Thus φ is a mapping of $\omega_{\mathcal{M}}X$ into $\omega_{\mathcal{L}}X$. Obviously, $\varphi(\omega_{\mathcal{M}}X) = \omega_{\mathcal{L}}X$.

If $x \in X$, then $\xi(x, \mathcal{L}) = \xi(x, \mathcal{M}) \cap \mathcal{L}$. Hence $\omega_{\mathcal{L}}(x) = \varphi(\omega_{\mathcal{M}}(x))$. For any $F \in \mathcal{L}$ we have $\varphi^{-1}(\{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}) = \{\eta \in \omega_{\mathcal{L}}X : F \in \eta\}$. Thus the mapping φ is continuous.

If $f : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ is a continuous mapping and $\omega_{\mathcal{L}} = f \circ \omega_{\mathcal{M}}$, then $f^{-1}(\{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}) = \{\eta \in \omega_{\mathcal{L}}X : F \in \eta\}$ for any $F \in \mathcal{L}$. Thus $f = \varphi$. The proof is complete.

Theorem 2.3. *Let $\mathcal{L}$ be a WS-ring and a closed base of a space X and $F \in \mathcal{L}$ for any closed compact subset F of X. Then $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}}) = (\omega_{m\mathcal{L}}X, \omega_{m\mathcal{L}})$. Moreover, $\omega_{\mathcal{L}}X$ is a compactification of the space X.*

Proof. For any $\xi \in \omega_{\mathcal{L}}X$ we put $\varphi(\xi) = \xi \cap \mathcal{L}$.

Claim 1. $\varphi(\xi) \in \omega_{\mathcal{L}}X$.

Let $F \in \mathcal{L}$ and $F \notin \xi$. Then there exists $H \in \xi$ such that $F \cap H = \emptyset$. Since $H \in m\mathcal{L}$, we have $H \sim \Phi$ for some $\Phi \in \mathcal{L}$. Hence, there exists a closed compact subset $\Phi_1 \in \mathcal{L}$ such that $H \triangle \Phi \triangle \Phi_1$.

Case 1. $\Phi_1 \in \xi$.

In this case $\cap\xi \neq \emptyset$ and there exists a point $x \in \Phi_1 \subseteq X$ such that $\xi = \xi(x, m\mathcal{L})$. In this case $\varphi(\xi) = \xi(x, \mathcal{L}) \in \omega_{\mathcal{L}}X$.

Case 2. $\Phi_1 \notin \xi$.

In this case there exists $H_1 \in \xi$ such that $H_1 \subseteq H$ and $H_1 \cap \Phi_1 = \emptyset$. Since $\mathcal{L}$ is a base, there exists $H_2 \in \mathcal{L}$ such that $H_1 \subseteq H_2$ and $H_2 \cap \Phi_1 = \emptyset$. Then $H_2 \in \xi$ and $H_2 \cap F = \emptyset$. Thus $H_2 \in \varphi(\xi)$ and $H_2 \cap F = \emptyset$. Hense $\varphi(\xi)$ is a maximal filter in $\mathcal{L}$, i.e. $\varphi(\xi) \in \omega_{\mathcal{L}}X$. Claim 1 is proved.

By virtue of Lemma 2.2, $\varphi : \omega_{m\mathcal{L}}X \to \omega_{\mathcal{L}}X$ is the unique continuous mapping for which $\omega_{\mathcal{L}} = \varphi \circ \omega_{m\mathcal{L}}$.

Claim 2. If $\xi \in \omega_{\mathcal{L}}X$, $F \in \xi$, $H \in \mathcal{L}$, $\cap\xi = \emptyset$ and $F \sim H$, then $H \in \xi$.

There exists a compact subset $\Phi \in \mathcal{L}$ such that $F \triangle H \subseteq \Phi$. Let $H \notin \xi$. Then there exists $L \in \xi$ such that $L \subseteq F, L \cap H = \emptyset$ and $L \cap \Phi = \emptyset$. Then $F \subseteq H \cup \Phi$ and $L \cap (H \cup \Phi) = \emptyset$, a contradiction.

Claim 3. $\varphi : \omega_{m\mathcal{L}}X \to \omega_{\mathcal{L}}X$ is a homeomorphism.

Let $\xi_1, \xi_2 \in \omega_{m\mathcal{L}}X$, $\xi_1 \neq \xi_2$ and $\eta = \varphi(\xi_1) = \varphi(\xi_2)$. In this case $\cap\eta = \emptyset$. Thus there exist $H_1 \in \xi_1 \setminus \xi_2$ and $H_2 \in \xi_2 \setminus \xi_1$ such that $H_1 \cap H_2 = \emptyset$. Since $H_1 \cap H_2 \in m\mathcal{L}$, there exist $F_1, F_2 \in \mathcal{L}$ and a compact subset $\Phi \in \mathcal{L}$ such that $F_1 \sim H_1, F_2 \sim H_2, F_1 \triangle H_1 \subseteq \Phi, F_2 \triangle H_2 \subseteq \Phi$. By virtue of Claim 2, we have $F_1 \in \xi_1$ and $F_2 \in \xi_2$. Then $F_1, F_2 \in \eta$ and $F_1 \cap F_2 \subseteq \Phi$, i.e. $\Phi \in \eta$, a contradiction. Therefore φ is a one-to-one mapping. Let $H \in m\mathcal{L}$ and $H_1 = \{\xi \in \omega_{m\mathcal{L}}X : H \in \xi\}$. Assume that $\eta \in \omega_{m\mathcal{L}}X$ and $H \notin \eta$.

Case 1. $\cap\eta \neq \emptyset$.

In this case $\eta = \xi(x, m\mathcal{L})$ for some $x \in X$ and $x \notin H$. Since $\mathcal{L}$ is a base of X, there exists $F \in \mathcal{L}$ such that $x \notin F$ and $H \subseteq F$. Thus $\varphi(\eta) \notin cl_{\omega_{\mathcal{L}}X}\varphi(H_1)$.

Case 2. $\cap\eta = \emptyset$.

In this case there exists $F \in \mathcal{L}$ such that $H \sim F$. We can assume that $H \subseteq F$. Then $F \notin \eta$ and $\varphi(\eta) \notin cl_{\omega_{\mathcal{L}}X}\varphi(H_1) \subseteq \{\xi \in \omega_{\mathcal{L}}X : F \in \xi\}$. Therefore the set $\varphi(H_1)$ is closed for any $H \in m\mathcal{M}$. Since $\{H_1 : H \in m\mathcal{M}$ is a closed base of $\omega_{\mathcal{L}}X$, the mapping φ is closed. Hence φ is a homeomorphism. The proof is complete.

Let $\mathcal{L}$ and $\mathcal{M}$ be WS-rings of closed subsets of a space X. We put $\mathcal{L} << \mathcal{M}$ if for any two sets $F_1, F_2 \in \mathcal{L}$, with the empty intersection $F_1 \cap F_2 = \emptyset$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of X, $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$. If $\mathcal{L} << \mathcal{M}$ and $\mathcal{M} << \mathcal{L}$, then we put $\mathcal{L} \approx \mathcal{M}$.

Proposition 2.4. *Let $\mathcal{L}$, $\mathcal{M}$ be two WS-rings and closed bases of a space X and $F \in \mathcal{L} \cap \mathcal{M}$ for any closed compact subset F of X. The next assertions are equivalent:*

1. *$\mathcal{L} << \mathcal{M}$.*
2. *$m\mathcal{L} << m\mathcal{M}$.*
3. *$m\mathcal{L} << \mathcal{M}$.*

Proof. Let $\mathcal{L} << \mathcal{M}$. Assume that $F_1, F_2 \in m\mathcal{L}$ and $F_1 \cap F_2 = \emptyset$. By virtue of Theorem 2.3, we have $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_1 = \emptyset$. Then there exist two sets $L_1, L_2 \in \mathcal{L}$ such that $L_1 \cap L_2 = \emptyset$, $F_1 \subseteq L_1$ and $F_2 \subseteq L_2$. Since $\mathcal{L} << \mathcal{M}$, there exist two sets $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2$ is a compact subset of X, $F_1 \subseteq L_1 \subseteq H_1$ and $F_2 \subseteq L_2 \subseteq H_2$. Therefore $m\mathcal{L} << \mathcal{M}$ and $m\mathcal{L} << m\mathcal{M}$. The implications $1 \to 3 \to 2 \to 3$ are proved. Theorem 2.3 completes the proof.

Proposition 2.5. *Let $\omega_{\mathcal{L}}X$ and $\omega_{\mathcal{M}}X$ be Hausdorff compactifications of a space X and $F \in \mathcal{M}$ for any compact subset F of X. The next assertions are equivalent:*

1. *There exists a continuous mapping $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ such that $\varphi(x) = x$ for any $x \in X$.*
2. *$\mathcal{L} << \mathcal{M}$.*
3. *$m\mathcal{L} << \mathcal{M}$.*
4. *$m\mathcal{L} << m\mathcal{M}$.*

Proof. Let $\varphi : \omega_{\mathcal{M}}X \to \omega_{\mathcal{L}}X$ be a continuous mapping and $\varphi(x) = x$ for any $x \in X$. Fix $F_1, F_2 \in \mathcal{L}$ such that $F_1 \cap F_2 = \emptyset$. Then $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_2 = \emptyset$. Since φ is a continuous mapping, then $cl_{\omega_{\mathcal{L}}X}F_1 \cap cl_{\omega_{\mathcal{L}}X}F_2 = \emptyset$. The family $\{cl_{\omega_{\mathcal{M}}X}H : H \in \mathcal{M}\}$ is a closed base of a compact space $\omega_{\mathcal{M}}X$. Thus there exists $H_1, H_2 \in \mathcal{M}$ such that $H_1 \cap H_2 = \emptyset, F_1 \subseteq H_1, F_2 \subseteq H_2$. Implication $1 \to 2$ is proved.

Assume that $\mathcal{L} << \mathcal{M}$. There exist two continuous mappings $f : \beta X \to \omega_{\mathcal{L}}X$ and $g : \beta X \to \omega_{\mathcal{M}}X$ of the Stone-*Čech* compactification βX of X such that $f(x) = g(x) = x$ for any $x \in X$. It is sufficient to prove that $\varphi(x) = f(g^{-1}(x))$ is a singleton for any $x \in \omega_{\mathcal{M}}X$. Let $y \in \omega_{\mathcal{L}}X$ and $x_1, x_2 \in \varphi(y)$ be two distinct points of $\omega_{\mathcal{L}}X$. Obviously, $y \in \omega_{\mathcal{M}}X \setminus X$ and $\varphi(y) \subseteq \omega_{\mathcal{L}}X$. There exists $F_1, F_2 \in \mathcal{L}$ such that $x_1 \in cl_{\omega_{\mathcal{L}}}F_1, x_2 \in cl_{\omega_{\mathcal{L}}}F_2$ and $F_1 \cap F_2 = \emptyset$. Let $H_1, H_2 \in \mathcal{M}$, $H = H_1 \cap H_2$ be a compact subset of X, $F_1 \subseteq H_1$, $F_2 \subseteq H_2$. Then $H = cl_{\omega_{\mathcal{M}}X}H_1 \cap cl_{\omega_{\mathcal{M}}X}H_2$. Let $\Phi_1 = f^{-1}(x_1)$ and $\Phi_2 = f^{-1}(x_2)$. Then $y \in g(\Phi_1) \cap g(\Phi_2)$. Since $\Phi_1 \subseteq cl_{\beta X}F_1$ and $g(cl_{\beta X}F_1) = cl_{\omega_{\mathcal{M}}X}F_1$ we have $g(\Phi_1) \subseteq cl_{\omega_{\mathcal{M}}X}F_1 \subseteq cl_{\omega_{\mathcal{M}}X}H_1$ and $g(\Phi_2) \subseteq cl_{\omega_{\mathcal{M}}X}F_2 \subseteq cl_{\omega_{\mathcal{M}}X}H_2$. Hence $Y \in H \subseteq X$, a contradiction. Implication $2 \to 1$ is proved. Proposition 2.4 completes the proof.

Corollary 2.6. *Let $\omega_{\mathcal{L}}X$ and $\omega_{\mathcal{M}}X$ be Hausdorff compactifications of a space X. Then $\omega_{\mathcal{L}}X = \omega_{\mathcal{M}}X$ if and only if $\mathcal{L} \approx \mathcal{M}$.*

3 On compressed compactification

Teorem 3.1. *If (Y, f) is a compressed g-compactification of a space X, then (Y, f) is a Wallman-Shanin g-compactification of the space X.*

Proof. Let τ be a cardinal number for which:

– $f(X)$ is dense in $P_\tau Y$;

– the closed G_τ-subsets of Y form a closed base of the space Y.

We put $Z = f(X)$. Denote by $\mathcal{F}_\tau(Y)$ the family of all closed G_τ-subsets of Y. By construction, $\mathcal{L} = \{f^{-1}(H) : H \in \mathcal{F}_\tau(Y)\}$ is a WS-ring of closed subsets of the space X.

Claim 1. If $H \in \mathcal{F}_\tau(Y)$, then $H = cl_Y(H \cap Z)$.

Obviously, $cl_Y(H \cap Z) \subseteq H$. Let $y \in H$, U be an open subset of Y and $y \in U$. Then $Y = U \cap H$ is a G_τ-subset of Y. Since Z is G_τ-dense in Y, we have $V \cap Z \neq \emptyset$. Hence $U \cap (H \cap Z) \supseteq V \cap Z \neq \emptyset$ and $y \in cl_Y(H \cap Z)$. Claim is proved.

Claim 2. $(Y, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$.

Let $\xi \in \omega_{\mathcal{L}}X$.

Case 1. $\cap\xi \neq \emptyset$.

There exists $x \in X$ such that $\xi = \xi(x, \mathcal{L})$. In this case we put $\varphi(\xi) = f(x)$

Case 2. $\cap\xi = \emptyset$.

In this case ξ is an $\mathcal{L}$-ultrafilter. Let $\xi = \{L_\alpha : \alpha \in A\}$. For each $\alpha \in A$ there exists a unique $H_\alpha \in \mathcal{F}_\tau(Y)$ such that $L_\alpha = f^{-1}(H_\alpha)$. By construction, $\eta = \{H_\alpha : \alpha \in A\}$ is an $\mathcal{F}_\tau(Y)$-ultrafilter and $\cap\eta = \emptyset$. There exists a unique point $y \in Y \setminus Z$ such that $y \in \cap\eta$. We put $\varphi(\xi) = y$.

The mapping $\varphi : \omega_{\mathcal{L}}X \to Y$ of $\omega_{\mathcal{L}}X$ onto Y is constructed. Obviously, the mapping φ is one-to-one. By construction, $\varphi(\omega_{\mathcal{L}}(x)) = f(x)$ for any $x \in X$ and $\varphi(\{\xi \in \omega_{\mathcal{L}}X : f^{-1}(H) \in \xi\} = H$ for each $H \in \mathcal{F}_\tau(Y)$. Hence the mapping φ is a homeomorphism. The proof is complete.

Corollary 3.2. *Let Y be a Hausdorff compactification of a space X and the space X is G_τ-dense in Y. Then Y is a Wallman-Frink compactification of the space X.*

Corollary 3.3 (R. A. Alo, H. L. Shapiro [1], E. Wajch [14]). *Let X be a pseudo-compact space. Then any Hausdorff compactification Y of X is a Wallman-Frink compactification.*

Corollary 3.4. *Let (Y, f) be a Hausdorff g-compactification of a feebly compact space X. Then (Y, f) is a Wallman-Frink g-compactification of the space X.*

For any discrete uncountable space the family of compressed Hausdorff compactifications is large. Moreover, this fact is valid for Hausdorff paracompact locally compact non-Lindelöf spaces.

Theorem 3.5. *Let X be a Hausdorff locally compact space which contains an uncountable discrete family of open non-empty subsets. Assume that $dimX = 0$. Then the family $\mathcal{B}$ of all compressed Hausdorff compactifications of X is uncountable and $\beta X = sup\mathcal{B}$.*

Proof. Fix $n \geq 2$. There exists a family $\{X_\mu : \mu \in M\}$ of open-and-closed subsets of X such that for any $\mu \in M$ the set X_μ is compact and there exist n distinct points $b_{1\mu}, b_{2\mu}, ..., b_{n\mu} \in X_\mu$. The sets $\{B_i = \{b_{i\mu} : \mu \in M\} : i \leq n\}$ are closed and disjoint. Fix n distinct points $b_1, b_2, ..., b_n \in \beta X \setminus X$. Since the sets $\{cl_{\beta X} B_i : i \leq n\}$ are disjoint we can assume that $b_{im} \notin \cup\{cl_{\beta X} B_j : j \leq n, j \neq i\}$ for any $i \leq n$. Fix n open-and-closed subsets $\{H_i : i \leq n\}$ of $\beta X \setminus X$ such that $b_i \in H_i, H_i \cap H_j = \emptyset$, $H_i \cap cl_{\beta X} B_j = \emptyset$ for any $i \neq j$ and $\beta X \setminus X = \cup\{H_i : i \leq n\}$. Then there exists a compactification Y of X and a continuous mapping $f : \beta X \to Y$ such that $f(x) = x$ for any $x \in X$ and $f^{-1}(f(b_i)) = H_i$ for any $i \leq n$. The compactification Y is compressed. By construction, the compressed compactifications $\mathcal{B}$ of X separate the points of βX. Thus $\beta X = sup\mathcal{B}$. The proof is complete.

4 Cartesian products of compactifications

Let A be a non-empty set, $\{X_\alpha : \alpha \in A\}$ be a family of non-empty spaces, $X = \Pi\{X_\alpha : \alpha \in A\}$, $(b_\alpha X_\alpha, \varphi_\alpha)$ be a family of g-compactifications of given spaces X_α. Then $bX = \Pi\{b_\alpha X_\alpha : \alpha \in A\}$ and the mapping $\varphi : X \to bX$, where $\varphi((x_\alpha : \alpha \in A)) = (\varphi_\alpha(x_\alpha) : \alpha \in A)$ for any $(x_\alpha : \alpha \in A) \in X$, is a g-compactification of the space X. If each $b_\alpha X_\alpha$ is a compactification of the space X_α, then bX is a compactification of the space X. Let $\mathcal{L}_\alpha$ be a WS-ring of closed sets of the space X_α. We put $\mathcal{L}' = \{\Pi\{H_\alpha : \alpha \in A\} : H_\alpha \in \mathcal{L}_\alpha, \alpha \in A\}$, $\mathcal{L} = \{H_1 \cup H_2 \cup ... \cup H_n : H_1, H_2, ..., H_n \in \mathcal{L}', n \in \mathbb{N}\}$.

Now we put $\mathcal{L} = \otimes\{\mathcal{L}_\alpha : \alpha \in A\}$.

Theorem 4.1. *The family $\mathcal{L}$ of closed subsets of the space X is a WS-ring and $\omega_{\mathcal{L}} X = \Pi\{\omega_{\mathcal{L}_\alpha} X_\alpha : \alpha \in A\}$.*

Proof. Obviously, $\mathcal{L}$ is a WS-ring of closed subsets of the space X.

Let ξ be an $\mathcal{L}'$-filter. Obviously $e(\xi) = \{H \cup F : H \in \xi, F \in \mathcal{L}\}$ is a $\mathcal{L}$-filter. Moreover, ξ is an $\mathcal{L}'$-ultrafilter if and only if $e(\xi)$ is an $\mathcal{L}$-ultrafilter.

Let $x = (x_\alpha : \alpha \in A\}$, $\xi_\alpha \subseteq \mathcal{L}_\alpha$ and $\xi = \{\Pi\{H_\alpha : \alpha A\} : H_\alpha \in \xi_\alpha, \alpha \in A\}$. Then:
– if any ξ_α is an $\mathcal{L}_\alpha$-filter, then ξ is an $\mathcal{L}'$-filter;
– ξ_α is an $\mathcal{L}_\alpha$-ultrafilter if and only if ξ is an $\mathcal{L}'$-ultrafilter;
– $\xi_\alpha = \xi(x_\alpha, \mathcal{L}_\alpha)$ for any $\alpha \in A$ if and only if $\xi = \xi(x, \mathcal{L}')$ and $e(\xi) = \xi(x, \mathcal{L})$.
These facts complete the proof.

Theorem 4.2. *If $|A| \geq 2$, $\omega X = \Pi\{\omega X_\alpha : \alpha \in A\}$ and any X_α is an infinite T_1-space, then any space X_α is countably compact.*

Proof. Fix $\beta \in A$. Assume that the space X_β is not countably compact. Then X_β contains an infinite, discrete and closed subset $F = \{b_n : n \in \mathbb{N}\}$.

Since $\omega Z_1 = cl_{\omega Z} Z_1$ for any closed subspace Z_1 of a T_0-space Z, we can assume that $X_\beta F$.

We put $Y_\beta = \Pi\{\omega X_\alpha : \alpha \in A \setminus \{\beta\}\}$. Obviously, $X = X_\beta \times Y_\beta$ and $\omega X = \omega X_\beta \times \omega Y_\beta$.

If the space Y_β is not countably compact, then Y_β contains a discrete infinite space Z and $\omega(X_\beta \times Z) = cl_{\omega X}(X_\beta \times Z) = (\omega X_\beta \times \omega Z)$, a contradiction with the Glicksberg's theorem ([6], Problem 3.12.20(d)). Thus we can assume that the space Y_β is countably compact.

In the space Y_β fix a set $L = \{c_n : n \in \mathbb{N}\}$, where $c_n \neq c_m$ for distinct $n, m \in \mathbb{N}$. The set $\Phi = \{(b_n, c_n) : n \in \mathbb{N}\}$ is closed and discrete in X. Projection $p : X_\beta \times Y_\beta \to X_\beta$ is a continuous closed mapping. Fix an ultrafilter ξ of closed subsets of the space X for which $\Phi \in \xi$ and $\cap\xi = \emptyset$. Then $p(\xi) = \{p(H) : H \in \xi\}$ is an ultrafilter of closed subsets of the space X_β. If $\cap p(\xi) \neq \emptyset$, then there exists a unique point $b \in X_\beta$ for which $\{b\} = \cap p(\xi)$. In this case $\{b\} \times Y_\beta \in \xi$ and $\cap\xi = \emptyset$. Since $\Phi \in \xi$, there exists a unique $n \in \mathbb{N}$ such that $b = b_n$ and $(b_n, c_n) \in H \cap (\{b\} \times Y_\beta)$ for each $H \in \xi$, a contradiction with $\cap\xi = \emptyset$. Thus $\cap p(\xi) = \emptyset$. Hence there exists a unique $b \in \omega X_\beta \setminus X_\beta$ for which $\{b\} = \cap\{cl_{X_\beta} H : H \in p(\xi)\}$.

Since $\omega X = \omega X_\beta \times \omega Y_\beta$, there exists a unique $c \in \omega Y_\beta \setminus Y_\beta$ such that $(b, c) \in \cap\{cl_{\omega X} H : H \in \xi\}$. In this case $X_\beta \times \{c\} \in \xi$. There exists a unque $n \in \mathbb{N}$ and some $H \in \xi$ such that $(b_n, c_n) \in \Phi \cap (X_\beta \times \{c\})$ and $H \cap (X_\beta \times \{c\}) = \emptyset$. Then $(b, c) \in cl_{\omega X} H \cap cl_{\omega X}(X_\beta \times \{c\})$ and $cl_{\omega X} H \cap cl_{\omega X}(X_\beta \times \{c\})$, a contradiction. The proof is complete.

Theorem 4.3. *Let $f : X \to Y$ be a continuous closed mapping of a space X onto a space Y. Then there exists a unique continuous mapping $\omega f : \omega X \to \omega Y$ such that $f = \omega f | X$. Moreover, the mapping ωf is closed.*

Proof. If ξ is an ultrafilter of closed subsets of X, then $\omega f(\xi) = \{f(H) : H \in \xi\}$ is an ultrafilter of closed subsets of Y. The mapping ωf is constructed.

Let τ be an infinite cardinal number. A space X is called initial τ-compact if any open cover γ of X of the cardinality $\leq \tau$ contains a finite subcover.

We say that the sequential character $s\chi(X) < \tau$ if for any non-closed subset H of X there exist a subset $Y \subseteq X$ and a point $x \in X \setminus H$ such that $x \in Y$, $x \in cl_X(H \cap Y)$ and $\chi(Y, x) < \tau$. A space X is sequential if and only if $s\chi(X) \leq \aleph_0$.

Theorem 4.4. *Let τ be an infinite cardinal number, X be an initial τ-compact space, Y be a compact space and $s\chi(Y) \leq \tau$. Then:*

1. The projection $p : X \times Y \to Y$, where $p(x, y) = y$ for each $(x, y) \in X \times Y$, is a continuous closed-and-open mapping.

2. There exists a continuous bijection $\varphi : \omega(X \times Y) \to \omega X \times Y$ such that $\varphi(x, y) = (x, y)$ for all $(x, y) \in X \times Y$.

Proof. It is well known that the projection p is continuous and open.

Let $y_0 \in Y$, W be an open subset of $X \times Y$ and $p^{-1}(y_0) = X \times \{y_0\} \subseteq W$. We put $V = \{y \in Y : p^{-1}(y) \subseteq W\}$. Obviously, $y_0 \in V$. We affirm that the set V is open in Y. Suppose that the set V is not open in Y. Then the set $Y \setminus V$ is not closed in Y. Thus there exist a point $z \in V$ and a subspace $Z \subseteq Y$ such that $z \in Z$, $z \in cl_Z(Z \cap (Y \setminus V))$ and $\chi(Z, z) \leq \tau$. We fix an open base $\{V_\alpha : \alpha \in A\}$ of the space Z at the point z such that $|A| \leq \tau$. For any $\alpha \in A$ consider the set $U_\alpha = \cup\{U : U \textit{ is open } X,\ U \times V_\alpha \subseteq W\}$. Obviously $X = \cup\{U_\alpha : \alpha \in A\}$. Since X is τ-compact and $|A| \leq \tau$, there exists a finite set $B \subseteq A$ such that $X = \cup\{U_\alpha : \alpha \in B\}$. There exists an element $\beta \in A$ for which $V_\beta \subseteq \cap\{V_\alpha : \alpha \in B\}$. Then $U_\beta \supseteq \cup\{U_\alpha : \alpha \in B\} = X$. Hence $U_\beta = X$ and $X \times V_\beta = U_\beta \times V_\beta \subseteq W$. Therefore $z \in V_\beta \subseteq V$ and $z \notin cl_Y(Y \setminus V)$, a contradiction. Assertion 1 is proved.

Consider the projection $f : X \times Y \to X$. The mappings f and p are continuous open-and-closed. Then there exist two continuous closed mappings $\omega f : \omega(X \times Y) \to \omega X$ and $\omega p : \omega(X \times Y) \to \omega Y$ such that$f = \omega f | X \times Y$ and $p = \omega p | X \times Y$. Consider the continuous mapping $\varphi : \omega(X \times Y) \to \omega X \times Y$ for which $\varphi(z) = (\omega f(z), \omega p(z))$ for each $z \in \omega(X \times Y)$. By construction, we have $\varphi(z) = (f(x, y), p(x, y)) = (x, y) = z$ for each $z = (x, y) \in X \times Y \subseteq \omega(X \times Y)$. Fix $z \in \omega(X \times Y) \setminus (X \times Y)$. Then there exists a unique ultrafilter ξ of closed subsets of $X \times Y$ for which $\{z\} = \cap\{cl_{\omega(X \times Y)} H : H \in \xi\}$. The family $p(\xi) = \{g(H) : H \in \xi\}$ is an ultrafilter of closed subsets of the space Y. There exists a unique point $y(z) = \omega g(z) \in \cap\{cl_Y g(H) : H \in \xi\}$. In this case $X(\xi) = X \times \{y(z)\} \in \xi$. Thus $\overline{\xi} = \{H \cap X(\xi) : H \in \xi\} \subseteq \xi$ is an ultrafilter of closed subsets of the subspace $X(\xi)$ of $X \times Y$.

Let ξ, η be two ultrafilters of closed subsets of the space $X \times Y$, $z \in \cap\{cl_{\omega(X \times Y)} H : H \in \xi\}$ and $z' \in \cap\{cl_{\omega(X \times Y)} H : H \in \eta\}$. Assume that $y(z) = y(z')$. Then $X(\xi) = X(\eta)$ and there there exist $H \in \overline{\xi}$ and $L \in \overline{\eta}$ such that $H \cap L = \emptyset$. Since $f | X(\xi) : X(\xi) \to X$ is a homeomorphism, $f(\xi) = f(\overline{\xi})$, $f(\eta) = f(\overline{\eta})$ and $f(H) \cap f(L) = \emptyset$. Thus $f(\xi) \neq f(\eta)$ and $\omega f(z) = \cap\{cl_{\omega X} f(M) : M \in \xi\} \neq \cap\{cl_{\omega X} f(P) : P \in \xi\eta\} = \omega f(z')$. Therefore φ is a bijection. The proof is complete.

Corollary 4.6. *Let τ be an infinite cardinal number, X be an initial τ-compact normal space, Y be a compact Hausdorff space and $s\chi(Y) \leq \tau$. Then:*

1. $\omega(X \times Y) = \omega X \times Y$.

2. $X \times Y$ is an initial τ-compact normal space.

Remark 4.7. Let X be a first countable normal countably compact not paracompact space and $Y = \beta X$. By virtue of Tamano's Theorem (see [6], Theorem 5.1.38), the space $X \times Y$ is not normal. Then $\omega X = \beta X$ and $\omega(X \times Y) \neq \beta(X \times Y) = \omega X \times Y$. Thus the restriction $s\chi(Y) \leq \tau$ in the above assertions is essential.

5 Remainders of compactifications

The main result of the section is the following theorem.

Theorem 5.1 *For any space Y the following assertions are equivalent:*

1. Y is a T_1-space.

2. There exists a T_0-space X such that the spaces Y and $\omega X \setminus X$ are homeomorphic.

3. There exists a T_1-space X such that the spaces Y and $\omega X \setminus X$ are homeomorphic.

Proof. Let X be a T_0- space and $Y = \omega X \setminus X$. Any ultrafilter of closed sets ξ represents a point $\xi \in \omega X$ for which the set $\{\xi\}$ is closed in ωX. Thus Y is a T_1-space. Implication $2 \to 1$ is proved. Implication $3 \to 2$ is obvious.

Let Y be a non-empty T_1-space. If Y is compact, then we put $Z = Y$. Let Y be a non-compact space. Consider a point $b \notin Y$. In this case Y is an open subspace of the space $Z = Y \cup \{b\}$, where the base of the space Z at the point b is the family $\{Z \setminus \Phi : \Phi$ *is a closed compact subset of* $Y\}$. By construction Z is a compact T_1-space. Fix an infinite cardinal number $\tau \geq w(Z)$. Denote by $W(\tau^+)$ the space of all ordinal numbers of the cardinality $\leq \tau$ in the topology generated by the linear order. Then $W(\tau^+)$ is a normal initial τ-compact space and $\omega W(\tau^+) \setminus (\tau^+) = \{c\}$ is a singleton.

If the space Y is compact, we consider the space $X = W(\tau^+) \times Y$ as a subspace of the compact space $\omega W(\tau^+) \times Z$. Further, if the space Y is not compact, then we consider the space $X = (W(\tau^+) \times Y) \cup \{(c, b)\}$ as a subspace of the compact space $\omega W(\tau^+) \times Z$.

Since the space X is initial τ-compact and $s\chi(Z) \leq \tau$, the mapping $g : X \longrightarrow Z$, where $g(z, y) = y$ for any $(z, y) \in X$, is continuous and open-and-closed. Hence $\omega X = \omega W(\tau^+) \times Z$. By construction, the spaces $\omega X \setminus X = \{c\} \times Y$ and Y are homeomorphic. The proof is complete.

Any Hausdorff locally compact space is a Wallman remainder of some normal space.

Question 1. *Under which conditions a completely regular space is a Wallman remainder of some normal space?*

Question 2. *Under which conditions a T_1-space is a Wallman remainder of some completely regular (regular, Hausdorff) space?*

Other problems about remainders of spaces have been examined recently in [2–4].

References

[1] Alo R. A., Shapiro H. L. *Wallman compact and realcompact spaces*. Contribs. Extens. Theory Topol. Struct., Proceed. Sympos. Berlin 1967, Berlin, 1969, 9–14.

[2] Arhangel'skii A. V. *A study of remainders of topological groups*, Fund. Math. 2009, **203**, 165–178.

[3] Arhangel'skii A. V., Choban M. M. *Remainders of rectifiable spaces*, Topology and Appl. 2010, **157**, 789–799.

[4] Arhangel'skii A. V., Choban M. M. *Some addition theorems for rectifiable spaces*, Buletinul Acad. de Ştiinţe a Rep. Moldova, Matematica 2011, No. 2(66).

[5] Choban M. M., Calmuţchi L. I. *Some problems of the theory of compactifications of topological spaces*. ROMAI Journal, 2006, **2:1**, 25–51.

[6] Engelking E. *General Topology*. PWN, Warszava, 1977.

[7] Faulkner G. D., Vipera M. C. *Remainders of compactifications and their relation to a quotient lattice of the topology*. Proc. Amer. Math. Soc. 1994, **122**, 931–942.

[8] Frink O. *Compactifications and semi-normal spaces*. Amer. J. Math. 1964, **86**, 602–607.

[9] Gillman L., Jerison M. *Rings of continuous functions*. Springer-Verlag, New York, 1976.

[10] Shanin N. A. *On special extension of topological spaces*. Dokl. AN SSSR 1943, **38**, 7–11 (in Russian).

[11] Shapiro L. B. *Reduction of the basic problem on bicompact extension of the Wallman type*. Soviet Math. Dokl. 1974, **15**, 1020–1023 (Russian original: Dokl. AN SSSR 1974, **217**, 38–41).

[12] Steiner E,Ḟ. *Wallman spaces and compactifications*. Fund. Math. 1968, **61**, 295–304.

[13] Uljanov V. M. *Solution of a basic problem on compactification of Wallman type*. Soviet Math. Dokl. 1977, **18**, 567–571 (Russian original: Dokl. AN SSSR 1977, **233**, 1056–1059).

[14] Wajch E. *Complete rings of functions and Wallman-Frink compactifications*. Colloq. Math. 1988, **56:2**, 281–290.

L. I. Calmuţchi, M. M. Choban
Department of Mathematics
Tiraspol State University
5 Gh. Iablocichin str.
Chişinău, MD−2069, Moldova.
E-mail: *lcalmutchi@gmail.com, mmchoban@gmail.com*

Received July 27, 2011