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Convex Solids with Hyperplanar Midsurfaces for Restricted Families of Chords

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Abstract. We provide new characteristic properties of convex quadrics in $\mathbb{R}^n$ in terms of hyperplanarity of midsurfaces of convex solids for restricted families of chords. These properties are based on various auxiliary characterizations of convex quadrics that involve hyperplane supports and plane quadric sections.

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1 Introduction

A classical result of convex geometry states that a convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is a solid ellipsoid (solid ellipse if $n = 2$) provided the middle points of every family of parallel chords of K lie in a hyperplane (see Brunn [4, pp. 59–61] for $n = 2$, Blaschke [3, p. 159] for $n = 3$, and Busemann [5, p. 92] for all $n \geq 3$). Gruber [7] refined this result by proving, in particular, that a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid if there is an open nonempty subset T of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ such that for every unit vector $e \in T$, the middle points of all chords of K parallel to e belong to a hyperplane. Another refinement was suggested in 2009 by Erwin Lutwak, who posed the following problem: Is it true that a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid provided there is a point $p \in \operatorname{int} K$ and a scalar $\delta > 0$ such that, for every chord $[u, v]$ of K through p, the middle points of all chords of K which are parallel to $[u, v]$ and lie at a distance δ or less from $[u, v]$ belong to a hyperplane?

In this paper, we establish similar characterizations of convex quadric hypersurfaces (briefly, convex quadrics) among all convex hypersurfaces in $\mathbb{R}^n$. By a *convex solid* in $\mathbb{R}^n$ we mean an n-dimensional closed convex set $K \subset \mathbb{R}^n$ distinct from the whole space. A *convex hypersurface* in $\mathbb{R}^n$ is the boundary of a convex solid. This definition includes a hyperplane and a pair of parallel hyperplanes.

In a standard way, a *quadric* (or a *second degree hypersurface*) in $\mathbb{R}^n$ is the locus of points $x = (\xi_1, \ldots, \xi_n)$ that satisfy a quadratic equation

$$F(x) \equiv \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where at least one a_{ik} is distinct from zero and $a_{ik} = a_{ki}$ for all $i, k = 1, \ldots, n$. We say that a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric $Q \subset \mathbb{R}^n$ and a connected component U of $\mathbb{R}^n \setminus Q$ such that U is a convex set and S is the boundary of U. As proved in [17], a convex hypersurface $S \subset \mathbb{R}^n$ is a convex quadric if and only if there is a Cartesian coordinate system $\xi_1, \dots, \xi_n$ for $\mathbb{R}^n$ such that S can be expressed as the locus of points $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ which satisfy one of the equations

$$\begin{aligned}
&a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, && 1 \le k \le n,\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1,\ \xi_1 \ge 0, && 2 \le k \le n,\\
&a_1\xi_1^2 = 0, &&\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0,\ \xi_1 \ge 0, && 2 \le k \le n,\\
&a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 2 \le k \le n,
\end{aligned}$$

where all scalars a_i involved are positive. In particular, convex quadrics in $\mathbb{R}^n$ that contain no lines are ellipsoids, elliptic paraboloids, sheets of elliptic hyperboloids on two sheets, and sheets of elliptic cones. Various characteristic properties of convex quadrics are given in [13, 15–17]. In particular, the following assertions will be of use below.

(A) ([15]) The boundary of a convex solid $K \subset \mathbb{R}^n$, $n \ge 3$, is a convex quadric if and only if there is a point $p \in \operatorname{int} K$ such that every section of $\operatorname{bd} K$ by a 2-dimensional plane through p is a convex quadric curve.

(B) ([16]) Given a line-free convex solid $K \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, $n \ge 3$, all proper bounded sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses if and only if the set $\operatorname{bd} K \setminus \big((p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)\big)$ lies in a convex quadric, where $\operatorname{rec} K$ denotes the recession cone of K (see definitions below).

2 Main Results

We need some definitions to formulate the main results. A *chord* of the convex solid K is a line segment $[u, v]$, $u \ne v$, such that $[u, v] = K \cap \langle u, v\rangle$, where $\langle u, v\rangle$ denotes the line through u and v. We will say that both $[u, v]$ and $\langle u, v\rangle$ are *parallel* to a unit vector $e \in \mathbb{R}^n$ if $u - v$ is a nonzero multiple of e. A convex solid K has chords if and only if it is distinct from a closed halfspace. By a *plane* of dimension m we mean a translate of an m-dimensional subspace of $\mathbb{R}^n$. A plane L *properly* intersects the solid K if L intersects both the boundary $\operatorname{bd} K$ and the interior $\operatorname{int} K$ of K.

The *recession cone* of a convex solid $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{y \in \mathbb{R}^n : x + \alpha y \in K \text{ whenever } x \in K \text{ and } \alpha \ge 0\}.$$

It is well-known that $\operatorname{rec} K$ is a closed convex cone with apex o, the origin of $\mathbb{R}^n$; furthermore, $\operatorname{rec} K$ is distinct from $\{o\}$ if and only if K is unbounded. The subset $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ consists of *non-recessional* unit vectors for K. Equivalently, a unit vector $e \in \mathbb{R}^n$ is non-recessional for K if and only if the intersection of K with any line parallel to e is either bounded or empty. Obviously, K has non-recessional unit vectors if and only if K is distinct from a closed halfspace of $\mathbb{R}^n$.

For any plane $L \subset \mathbb{R}^n$ which is complementary to the *linearity space* of K, defined by

$$\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K),$$

the convex solid K can be expressed as the direct sum

$$K = \operatorname{lin} K \oplus (K \cap L),$$

and $K \cap L$ is a closed convex set containing no lines (see, e. g., [19] for general references on convex sets).

Theorem 1. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 2$, distinct from a closed halfspace of $\mathbb{R}^n$ and an open nonempty subset T of $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$, the following conditions are equivalent:*

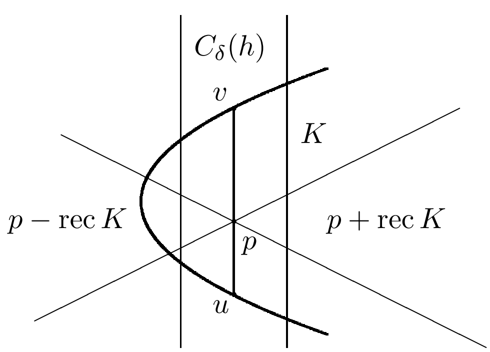
1) *for every unit vector $e \in T$, the middle points of all chords of K which are parallel to e belong to a hyperplane,*

2) $\operatorname{bd} K$ *is a convex quadric.*

Problem 1. Is it true that Theorem 1 still holds if condition 1) is replaced by the following weaker condition:

1′) for every unit vector $e \in T$, there is a scalar $\lambda = \lambda(e) \in (0, 1)$ such that the points dividing in the same ratio λ all chords of K which are parallel to e belong to a hyperplane.

The answer to Problem 1 is affirmative in the following two cases: K is a convex body in $\mathbb{R}^n$ (see [7]), K is a convex solid in $\mathbb{R}^n$ and $T = \mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ (see [15]). The papers [7, 15] also contain results which involve a weaker version of 1′), with $\lambda \in [0, 1]$ instead of $\lambda \in (0, 1)$.



In what follows, we consider double cones $(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)$ with apices $p \in \mathbb{R}^n$, as depicted above.

Definition 1. Let δ be a positive scalar, $K \subset \mathbb{R}^n$ a convex solid, p a point in $\operatorname{int} K$, and $h = [u, v]$ a chord of K through p. Denote by $C_\delta(h)$ the closed circular cylinder of radius δ centered about the line $\langle u, v\rangle$, and by $\mathcal{F}_\delta(h)$ the family of all chords of K which are parallel to h and lie in $C_\delta(h)$. Furthermore, let

$$\Omega_\delta(p) = \cup\,(C_\delta(h) \cap \operatorname{bd} K),$$

where the union is taken over all chords h of K that contain p.

Clearly, $\Omega_\delta(p)$ is a closed neighborhood of $\operatorname{bd} K \setminus \big((p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)\big)$ in $\operatorname{bd} K$.

Theorem 2. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 2$, distinct from a closed halfspace of $\mathbb{R}^n$, a point $p \in \operatorname{int} K$, and a scalar $\delta > 0$, the following conditions are equivalent:*

1) *for every chord h of K that contains p, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane,*

2) *the set $\Omega_\delta(p)$ lies in a convex quadric.*

If K is a convex body in $\mathbb{R}^n$, then $\operatorname{rec} K = \{o\}$, implying the equality $\Omega_\delta(p) = \operatorname{bd} K$ for any given point $p \in \operatorname{int} K$. Therefore Theorem 2 implies the following corollary, which gives an affirmative solution to Lutwak's problem.

Corollary 1. *A convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is a solid ellipsoid if and only if there is a point $p \in \operatorname{int} K$ and a scalar $\delta > 0$ such that for every chord h of K which contains p, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.*

Remark 1. We observe that the scalar δ in Theorem 2 and Corollary 1 cannot be chosen as a function of h. Indeed, if K is a 3-dimensional octahedron, given by

$$K = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| + |\xi_2| + |\xi_3| \leq 1\},$$

then for any chord h of K that contains the origin o, there is a scalar $\delta = \delta(h) > 0$ such that the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a plane through o.

Problem 2. Is it true that Theorem 2 still holds if condition 1) is replaced by the following weaker condition:

1′′) for any chord h of K that contains p, there is a scalar $\lambda = \lambda(e) \in (0, 1)$ such that the points dividing in the same ratio λ all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.

The proofs of Theorem 1 and 2 are based on some auxiliary statements. The first one complements Theorem 1 from [17] by giving new characteristic properties of quadrics $Q \subset \mathbb{R}^n$ with at least one convex connected component of $\mathbb{R}^n \setminus Q$ in terms of local convexity and local supports. In what follows, a quadric $Q \subset \mathbb{R}^n$ is called *proper* provided its complement $\mathbb{R}^n \setminus Q$ has two or more connected components, which happens when Q, given by (1), is a hyperplane or both sets

$$\{x \in \mathbb{R}^n : F(x) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^n : F(x) < 0\}$$

are nonempty.

We will say that a proper quadric $Q \subset \mathbb{R}^n$ is *locally convex* at a point $u \in Q$ if there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ with center u and radius $\rho > 0$ such that $Q \cap U_\rho(u)$ is a piece of a convex hypersurface. Similarly, a proper quadric $Q \subset \mathbb{R}^n$ is *locally supported* at $u \in Q$ if there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$ through u such that $Q \cap U_\rho(u)$ lies in a closed halfspace of $\mathbb{R}^n$ bounded by H.

Theorem 3. *For a proper quadric $Q \subset \mathbb{R}^n$, $n \geq 2$, the following conditions are equivalent:*

1) *Q is locally convex at a certain point $u \in Q$,*

2) *Q is locally supported at a certain point $u \in Q$,*

3) *at least one of the connected components of $\mathbb{R}^n \setminus Q$ is a convex set,*

4) *Q is the union of at most four convex quadrics,*

5) *there is a Cartesian coordinate system $\xi_1, \dots, \xi_n$ for $\mathbb{R}^n$ such that Q can be expressed as the locus of points $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ which satisfy one of the equations*

$$F_1(x) \equiv a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, \qquad 1 \leq k \leq n, \tag{2}$$

$$F_2(x) \equiv a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1, \qquad 2 \leq k \leq n, \tag{3}$$

$$F_3(x) \equiv a_1\xi_1^2 = 0, \tag{4}$$

$$F_4(x) \equiv a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0, \qquad 2 \leq k \leq n, \tag{5}$$

$$F_5(x) \equiv a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 2 \leq k \leq n, \tag{6}$$

where all scalars a_i involved are positive.

There is a certain analogy between Theorem 3 and respective properties of convex hypersurfaces. Indeed, if S is the boundary of an open nonempty connected set $X \subset \mathbb{R}^n$, then S is a convex hypersurface provided X is locally supported at every point $u \in S$ (see [6]). Similarly, S is a convex hypersurface if X is locally convex at every point $u \in S$ (see [10, 18]). On the other hand, Theorem 3 deals with local convexity and local support of Q at a *single* point.

The next two results characterize convex quadrics in terms of their 2-dimensional planar sections.

Theorem 4. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a convex solid, p a point in* $\operatorname{int} K$, *and T an open nonempty subset of $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$. The following conditions are equivalent:*

1) $\operatorname{bd} K$ *is a convex quadric,*

2) *for every* 2*-dimensional plane* L *through* p *which properly intersects* K *such that the subspace* $L - p$ *meets* T*, the section* $L \cap \operatorname{bd} K$ *is a convex quadric curve.*

Remark 2. Theorem 4 refines, with essential modifications of proofs, the respective statements from [15], given there for the case $T = \mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$. It is unknown whether Theorem 4 remains true for any choice of the point p in $\mathbb{R}^n$ (compare with Problem 1 from [17]).

Obvious changes in the proof of Theorem 4 allow us to generalize the following assertion of Petty [12]: the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid provided there is a line $l \subset \mathbb{R}^n$ such that all proper sections of $\operatorname{bd} K$ by 2-dimensional planes parallel to l are ellipses. Given a line $l \subset \mathbb{R}^n$ and a scalar $\delta > 0$, denote by $\mathcal{P}_\delta(l)$ the family of all 2-dimensional planes in $\mathbb{R}^n$ which are parallel to l and whose distance from l is less than δ.

Theorem 5. *Let* $K \subset \mathbb{R}^n$*,* $n \geq 3$*, be a convex solid,* l *a line that meets* $\operatorname{int} K$ *and is parallel to a unit vector from* $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$*, and* δ *a positive scalar. The following conditions are equivalent:*

1) $\operatorname{bd} K$ *is a convex quadric,*

2) *for any* 2*-dimensional plane* $L \in \mathcal{P}_\delta(l)$ *properly intersecting* K*, the section* $L \cap \operatorname{bd} K$ *is a convex quadric curve.* $\square$

Remark 3. The condition that l is parallel to a unit vector from $\mathbb{S}^{n-1} \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ is essential in Theorem 5. Indeed, let C be the unit cube in the coordinate hyperplane $\xi_1 = 0$ of $\mathbb{R}^n$ and l be the ξ_1-axis of $\mathbb{R}^n$. Denote by K the Cartesian product of C and l. Clearly, K is a convex solid with $\operatorname{rec} K = l$ and any proper section of $\operatorname{bd} K$ by a 2-dimensional plane parallel to l is a pair of parallel lines, which is a degenerate convex quadric curve.

Alonso and Martín [1] proved that if $L_1, L_2, L_3 \subset \mathbb{R}^n$, $n \geq 3$, are three pairwise distinct $(n-1)$-dimensional subspaces and $K \subset \mathbb{R}^n$ a centrally symmetric convex body such that every proper section of $\operatorname{bd} K$ by a hyperplane parallel to one of these subspaces is an $(n-1)$-dimensional ellipsoid, then $\operatorname{bd} K$ is an ellipsoid itself. They also observed that the assumption on central symmetry of K here cannot be omitted. Indeed, if $K_\alpha \subset \mathbb{R}^3$, $0 < |\alpha| \leq 2$, is a convex body, given by

$$K_\alpha = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + \alpha xyz \leq 1, \ \max\{|x|, |y|, |z|\} \leq 1\},$$

then any proper section of $\operatorname{bd} K_\alpha$ by a plane parallel to one of the coordinate subspaces $x = 0$, $y = 0$, and $z = 0$ is an ellipse (see [1] for other examples). Our next theorem extends the result of Alonso and Martín to the case of any convex body in $\mathbb{R}^n$.

Theorem 6. *If $L_1, L_2, L_3, L_4 \subset \mathbb{R}^n$, $n \geq 3$, are four pairwise distinct $(n-1)$-dimensional subspaces and $K \subset \mathbb{R}^n$ a convex body such that every proper section of* $\operatorname{bd} K$ *by a hyperplane parallel to one of these subspaces is an $(n-1)$-dimensional ellipsoid, then* $\operatorname{bd} K$ *is an ellipsoid itself.*

It would be interesting to generalize Theorem 6 to the case of convex quadrics. In what follows, $\operatorname{rbd} M$ and $\operatorname{rint} M$ denote, respectively, the relative boundary and the relative interior of a closed convex set $M \subset \mathbb{R}^n$.

3 Auxiliary Lemmas

If a proper quadric $Q \subset \mathbb{R}^n$ is given by (1), then a point $u \in Q$ is called *regular* provided the gradient vector

$$\nabla F(u) = \left(\tfrac{\partial F(u)}{\partial \xi_1}, \dots, \tfrac{\partial F(u)}{\partial \xi_n}\right),$$

the *normal* to Q at u, is distinct from the zero vector o; otherwise u is *singular*. The standard classification of quadrics in $\mathbb{R}^n$ (see, e. g., [2]) immediately implies that a description of a *proper* quadric $Q \subset \mathbb{R}^n$, given by (1), can be reduced to one of the canonical equations

$$\begin{aligned}
&a_1\xi_1^2 + \dots + a_k\xi_k^2 = 1, && 1 \leq k \leq n, && (7)\\
&a_1\xi_1^2 + \dots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \dots - a_k\xi_k^2 = 1, && 1 \leq r < k \leq n, && (8)\\
&a_1\xi_1^2 = 0, && && (9)\\
&a_1\xi_1^2 + \dots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \dots - a_k\xi_k^2 = 0, && 1 \leq r < k \leq n, && (10)\\
&a_1\xi_1^2 + \dots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 1 < k \leq n, && (11)\\
&a_1\xi_1^2 + \dots + a_r\xi_r^2 - a_{r+1}\xi_{r+1}^2 - \dots - a_{k-1}\xi_{k-1}^2 = \xi_k, && 1 \leq r < k-1 < n, && (12)
\end{aligned}$$

where all scalars a_i involved are positive. The following lemma routinely follows from (7)–(12).

Lemma 1. *A proper quadric $Q \subset \mathbb{R}^n$ has singular points if and only if its canonical equation is expressed by* (9) *or* (10)*. The set of singular points of Q is given by $\xi_1 = 0$ if Q is described by* (9) *and by $\xi_1 = \dots = \xi_k = 0$ if Q is described by* (10)*.* □

If $u = (\mu_1, \dots, \mu_n)$ is a regular point of a proper quadric $Q \subset \mathbb{R}^n$, then the linear equation in $x = (\xi_1, \dots, \xi_n)$,

$$\nabla F(u)\cdot(x - u) \equiv \tfrac{\partial F(u)}{\partial \xi_1}(\xi_1 - \mu_1) + \dots + \tfrac{\partial F(u)}{\partial \xi_n}(\xi_n - \mu_n) = 0, \tag{13}$$

defines the hyperplane through u which is orthogonal to $\nabla F(u)$; it is called *tangent* to Q at u. Since a proper quadric is differentiable at any regular point, we immediately obtain the following lemma.

Lemma 2. *If a proper quadric $Q \subset \mathbb{R}^n$ is locally supported by a hyperplane G at a regular point $u \in Q$, then G is tangent to Q at u.* $\square$

Lemma 3. *The middle points of all chords of a quadric $Q \subset \mathbb{R}^n$ which are parallel to a given chord $[a, c]$ of Q belong to a hyperplane.*

Proof. Assume that Q is given by (1). The line $l = \langle a, c\rangle$ can be expressed as

$$l = \{z + tv \in \mathbb{R}^n : t \in \mathbb{R}\}, \quad v \neq o,$$

where z is the middle point of $[a, c]$ and $v = a - c$. Equivalently, $x = (\xi_1, \dots, \xi_n) \in l$ if and only if

$$\xi_i = \phi_i + t\nu_i, \quad t \in \mathbb{R}, \quad i = 1, \dots, n, \tag{14}$$

where $z = (\phi_1, \dots, \phi_n)$ and $v = (\nu_1, \dots, \nu_n)$. To determine the values of t for which $x \in l \cap Q$, we substitute $\xi_1, \dots, \xi_n$ from (14) into (1) and arrange the powers of t. The result is a quadratic equation in t,

$$A(v)\, t^2 + 2B(v, z)\, t + C(z) = 0, \tag{15}$$

where

$$A(v) = \sum_{i,k=1}^{n} a_{ik}\nu_i\nu_k, \quad B(v, z) = \tfrac{1}{2}\sum_{i=1}^{n} \tfrac{\partial F(z)}{\partial \xi_i}\, \nu_i, \quad C(z) = F(z). \tag{16}$$

Then a and c correspond to opposite non-zero solutions t_0 and $-t_0$ of (15), which is possible if and only if $A(v)\, C(z) < 0$ and $B(v, z) = 0$. Hence

$$\sum_{i=1}^{n} \Big(\sum_{k=1}^{n} a_{ik}\phi_k + b_i\Big)\nu_i = \tfrac{1}{2}\sum_{i=1}^{n} \tfrac{\partial F(z)}{\partial \xi_i}\nu_i = B(v, z) = 0.$$

Equivalently,

$$\sum_{k=1}^{n} \Big(\sum_{i=1}^{n} a_{ik}\nu_i\Big)\phi_k + \sum_{i=1}^{n} b_i\nu_i = 0. \tag{17}$$

Interpreted as an equation in $\phi_1, \dots, \phi_n$, (17) describes a hyperplane, H, because at least one of the scalars

$$c_k = \sum_{i=1}^{n} a_{ik}\nu_i, \quad k = 1, \dots, n,$$

is distinct from zero. Indeed, assuming $c_1 = \cdots = c_n = 0$, we would obtain

$$A(v) = c_1\nu_1 + \cdots + c_n\nu_n = 0,$$

which is impossible because of $A(v)\, C(z) < 0$. If $[a', c']$ is a chord of Q that is parallel to $[a, c]$, then v is a nonzero multiple of $a' - c'$, which implies that

$$\langle a', c'\rangle = \{z' + tv \in \mathbb{R}^n : t \in \mathbb{R}\},$$

where $z' = (\phi'_1, \dots, \phi'_n)$ is the middle point of $[a', c']$. Repeating the argument above, we obtain that $\phi'_1, \dots, \phi'_n$ satisfy (17), which gives $z' \in H$. $\square$

4 Proof of Theorem 3

1) $\Rightarrow$ 2) Assume that Q is locally convex at a point $u \in Q$; that is, $Q \cap U_\rho(u)$ is a piece of a convex hypesurface $S \subset \mathbb{R}^n$ for a suitable scalar $\rho > 0$. By a convexity argument, there is a hyperplane H supporting S at u. Therefore, $Q \cap U_\rho(u)$ lies in a closed halfspace of $\mathbb{R}^n$ bounded by H, which implies that Q is locally supported at u.

2) $\Rightarrow$ 3) Choosing a suitable orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, we may suppose that Q is described by one of the equations (7)–(12). Put $u = (\mu_1, \ldots, \mu_n)$ and denote by H a hyperplane that supports $Q \cap U_\rho(u)$ for a suitable choice of $\rho > 0$.

(a) If Q is expressed by (7), then Q itself is a convex quadric and the connected component $\{x \in \mathbb{R}^n : F_1(x) < 1\}$ of $\mathbb{R}^n \setminus Q$ is an open convex set.

(b) Suppose that Q is given by (8). From Lemma 1 it follows that u is a regular point of Q. Choosing suitable orthogonal bases $e'_1, \ldots, e'_r$ and $e'_{r+1}, \ldots, e'_n$ for the subspaces span $(e_1, \ldots, e_r)$ and span $(e_{r+1}, \ldots, e_n)$, respectively, we may assume that Q is still expressed by (8) and

$$u = (\mu_1, 0, \ldots, 0, \mu_{r+1}, 0, \ldots, 0), \quad \mu_1 > 0, \ \mu_{r+1} \ge 0,$$

with $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 1$. The section of Q by the 2-dimensional subspace $L_1 =$ span (e_1, e_{r+1}) is a hyperbola, whose arm E_1 containing u is given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 = 1, \ \xi_1 > 0, \ \ \xi_2 = \cdots = \xi_r = \xi_{r+2} = \cdots = \xi_n = 0.$$

By Lemma 2, H is tangent to Q at u. Due to (13), H is expressed as

$$a_1\mu_1(\xi_1 - \mu_1) - a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) = 0.$$

Equivalently,

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} = 1.$$

We are going to show that $r = 1$. Indeed, assume for a moment that $r \ge 2$. Then the section of Q by the r-dimensional plane

$$L_2 = \{(\xi_1, \ldots, \xi_n) : \xi_{r+1} = \mu_{r+1}, \xi_{r+2} = \cdots = \xi_n = 0\}$$

is the r-dimensional ellipsoid, E_2, described by

$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 = 1 + a_{r+1}\mu_{r+1}^2, \quad \xi_{r+1} = \mu_{r+1}, \ \xi_{r+2} = \cdots = \xi_n = 0.$$

From $a_1\xi_1^2 + \cdots + a_r\xi_r^2 = a_1\mu_1^2$ it follows that $|\xi_1| \le \mu_1$.

We state that E_1 and E_2 lie in the opposite closed halfspaces of $\mathbb{R}^n$ determined by H. Indeed, since the set $B_1 \subset L_1$ given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 \ge 1, \ \xi_1 > 0, \ \ \xi_2 = \cdots = \xi_r = \xi_{r+2} = \cdots = \xi_n = 0,$$

is strictly convex, the point

$$(\tfrac{\xi_1 + \mu_1}{2}, 0, \ldots, 0, \tfrac{\xi_{r+1} + \mu_{r+1}}{2}, 0, \ldots, 0)$$

belongs to $\operatorname{rint} B_1$ provided the point $x = (\xi_1, 0, \dots, 0, \xi_{r+1}, 0, \dots, 0) \in E_1$ is distinct from u. Hence

$$a_1\big(\tfrac{\xi_1+\mu_1}{2}\big)^2 - a_{r+1}\big(\tfrac{\xi_{r+1}+\mu_{r+1}}{2}\big)^2 \geq 1,$$

which results in

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} \geq 1,$$

with equality if and only if $\xi_1 = \mu_1$ and $\xi_{r+1} = \mu_{r+1}$.

If $x \in E_2$, then from $|\xi_1| \leq \mu_1$ and $\xi_{r+1} = \mu_{r+1}$ we obtain

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} \leq a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 1,$$

with equality if and only if $\xi_1 = \mu_1$. Summing up, $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in the opposite closed halfspaces of $\mathbb{R}^n$ bounded by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence $r = 1$, and, by proved in [17], the connected component $\{x \in \mathbb{R}^n : F_2(x) > 1\}$ of $\mathbb{R}^n \setminus Q$ is an open convex set.

(c) If Q is given by (9), then Q is the hyperplane described by $\xi_1 = 0$ and both open halfspaces $\xi_1 > 0$ and $\xi_1 < 0$ are the connected components of $\mathbb{R}^n \setminus Q$.

(d) Assume that Q is expressed by (10). Since any point

$$x = (0, \dots, 0, \xi_{k+1}, \dots, \xi_n) \in Q$$

is the apex of a "double cone" $Q \cap \operatorname{span}(e_{k+1}, \dots, e_n)$, which cannot be locally supported at x, at least one of the coordinates $\mu_1, \dots, \mu_k$ of u must be distinct from 0. From Lemma 1 it follows that u is a regular point of Q. By Lemma 2, H is tangent to Q at u. Choosing suitable orthogonal bases $e_1', \dots, e_r'$ and $e_{r+1}', \dots, e_n'$ for the subspaces $\operatorname{span}(e_1, \dots, e_r)$ and $\operatorname{span}(e_{r+1}, \dots, e_n)$, respectively, we may assume that Q is still expressed by (10) and

$$u = (\mu_1, 0, \dots, 0, \mu_{r+1}, 0, \dots, 0), \quad \mu_1 > 0,\ \mu_{r+1} > 0,$$

with $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = 0$. Clearly, the section of Q by the 2-dimensional subspace $L_1 = \operatorname{span}(e_1, e_{r+1})$ is a double cone. Denote by E_1 the arm of this cone given by

$$a_1\xi_1^2 - a_{r+1}\xi_{r+1}^2 = 0,\ \xi_1 > 0,\ \xi_{r+1} > 0,\quad \xi_2 = \cdots = \xi_r = \xi_{r+2} = \cdots = \xi_n = 0.$$

Then $u \in E_1$. Hence H is given by

$$a_1\mu_1(\xi_1 - \mu_1) - a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) = 0,$$

or

$$a_1\mu_1\xi_1 - a_{r+1}\mu_{r+1}\xi_{r+1} = 0.$$

We are going to show that $r = 1$. Indeed, assume for a moment that $r \geq 2$. Then the section of Q by the r-dimensional plane

$$L_2 = \{(\xi_1, \dots, \xi_n) : \xi_{r+1} = \mu_{r+1}, \xi_{r+2} = \cdots = \xi_n = 0\}$$

is the r-dimensional ellipsoid, E_2, described by

$$a_1\xi_1^2 + \cdots + a_r\xi_r^2 = a_{r+1}\mu_{r+1}^2, \quad \xi_{r+1} = \mu_{r+1},\ \xi_{r+2} = \cdots = \xi_n = 0.$$

Similarly to case (b) above, one can show that $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in distinct closed halfspaces of $\mathbb{R}^n$ determined by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence $r = 1$. As shown in [17], the connected component $\{x \in \mathbb{R}^n : F_4(x) > 0\}$ of $\mathbb{R}^n \setminus Q$ is a convex set.

(e) If Q is expressed by (11), then Q itself is a convex quadric and the connected component $\{x \in \mathbb{R}^n : F_5(x) < \xi_k\}$ of $\mathbb{R}^n \setminus Q$ is a convex set.

(f) Finally, assume that Q is expressed by (12). From Lemma 1 it follows that u is a regular point of Q. So, H is tangent to Q at u. Choosing suitable orthogonal bases $e_1', \dots, e_r'$ and $e_{r+1}', \dots, e_n'$ for the subspaces span$\,(e_1, \dots, e_r)$ and span$\,(e_{r+1}, \dots, e_n)$, respectively, we may assume that Q is still expressed by (12) and

$$u = (\mu_1, 0, \dots, 0, \mu_{r+1}, 0, \dots, 0, \mu_k, 0, \dots, 0),$$

where $a_1\mu_1^2 - a_{r+1}\mu_{r+1}^2 = \mu_k$. Due to (13), H is expressed as

$$2a_1\mu_1(\xi_1 - \mu_1) - 2a_{r+1}\mu_{r+1}(\xi_{r+1} - \mu_{r+1}) - (\xi_k - \mu_k) = 0.$$

Equivalently,

$$\xi_k = 2a_1\mu_1\xi_1 - 2a_{r+1}\mu_{r+1}\xi_{r+1}.$$

The section of Q by the 2-dimensional plane $L_1 = u + \mathrm{span}\,(e_1, e_k)$ is a parabola, E_1, given by

$$\xi_k = a_1\xi_1^2 - a_{r+1}\mu_{r+1}^2,\ \xi_{r+1} = \mu_{r+1},\ \xi_i = 0 \text{ for all } i \in \{1, \dots, n\} \setminus \{1, r+1, k\}.$$

Similarly, the section of Q by another 2-dimensional plane, $L_2 = u + \mathrm{span}\,(e_{r+1}, e_k)$ also is a parabola, E_2, given by

$$\xi_k = a_1\mu_1^2 - a_{r+1}\xi_{r+1}^2,\ \xi_1 = \mu_1,\ \xi_i = 0 \text{ for all } i \in \{1, \dots, n\} \setminus \{1, r+1, k\}.$$

Clearly, $E_1 \cap U_\rho(u)$ and $E_2 \cap U_\rho(u)$ lie in distinct closed halfspaces of $\mathbb{R}^n$ determined by H such that $E_1 \cap H = E_2 \cap H = \{u\}$, in contradiction with the choice of $U_\rho(u)$. Hence Q cannot be given by (12).

Equivalence of conditions 1), 3)–5) follows from the proof of Theorem 1 from [17].

5 Proof of Theorem 4

The statement 1) $\Rightarrow$ 2) immediately follows from the fact that a proper section of a convex quadric by a 2-dimensional plane is a convex quadric curve. Conversely, assume that condition 2) of the theorem holds. Translating K on the vector $-p$, we may suppose that $o = p \in \mathrm{int}\, K$. We observe that K is distinct from a halfspace, since otherwise $\mathrm{rec}\, K \cup -\mathrm{rec}\, K = \mathbb{R}^n$ in contradiction with the choice of T. Also, we eliminate the trivial case when K is a slab between two parallel hyperplanes (implying that $\mathrm{bd}\, K$ is a degenerate convex quadric). Therefore we may assume that $\dim(\mathrm{lin}\, K) \le n-2$.

We observe that the proof of 2) $\Rightarrow$ 1) can be reduced to the case $\dim(\mathrm{lin}\, K) = 0$; that is, to the case when K contains no lines. Indeed, assuming the inequality $\dim(\mathrm{lin}\, K) \ge 1$, choose a subspace $M \subset \mathbb{R}^n$ complementary to $\mathrm{lin}\, K$ and intersecting T. Put $K' = M \cap K$ and $T' = T \cap M$. Clearly, $\mathrm{lin}\, K' = M \cap \mathrm{lin}\, K = \{o\}$ and T' is an open nonempty subset of $(M \cap \mathbb{S}^{n-1}) \setminus (\mathrm{rec}\, K' \cup -\mathrm{rec}\, K')$. Choose a 2-dimensional subspace $L \subset M$ that meets T' and properly intersects K'. From the equality $K = \mathrm{lin}\, K \oplus K'$, we obtain $L \cap \mathrm{rbd}\, K' = L \cap \mathrm{bd}\, K$. Hence condition 2) implies that $L \cap \mathrm{rbd}\, K'$ is a convex quadric curve. Therefore, K' satisfies condition 2) of the theorem (with M and T' instead of $\mathbb{R}^n$ and T, respectively). Finally, the equality $\mathrm{bd}\, K = \mathrm{lin}\, K \oplus \mathrm{rbd}\, K'$ shows that $\mathrm{bd}\, K$ is a degenerate convex quadric provided $\mathrm{rbd}\, K'$ is a convex quadric.

Our further consideration of the case $\dim(\mathrm{lin}\, K) = 0$ is organized by induction on $n\,(\ge 3)$. Let $n = 3$. Since K is line-free, there is a 2-dimensional subspace L' through l properly intersecting K such that $L' \cap K$ is bounded. Choose a pair of distinct planes L_1 and L_2 both containing l and placed so close to L' that the sets $L_1 \cap K$ and $L_2 \cap K$ are bounded. By condition 2), both sections $E_1 = L_1 \cap \mathrm{bd}\, K$ and $E_2 = L_2 \cap \mathrm{bd}\, K$ are convex quadric curves, whence they are ellipses. Let c be the midpoint of the line segment $l \cap K$ and c_1 and c_2 the centers of E_1 and E_2, respectively. Applying a suitable affine transformation, we may assume that both E_1 and E_2 are circles and the planes L_1 and L_2 are orthogonal. Clearly, the image of K under this transformation, also denoted by K, satisfies condition 2) of the theorem. Let 2δ be the length of $l \cap K$.

Choose a coordinate system (ξ_1, ξ_2, ξ_3) such that l is the ξ_3-axis, the points c, c_1, c_2 lie in the coordinate plane $\xi_3 = \sigma_3$, where σ_3 is a suitable scalar, and

$$c_1 = (\sigma_1, 0, \sigma_3), \quad c_2 = (0, \sigma_2, \sigma_3), \quad c = (0, 0, \sigma_3), \quad \sigma_1, \sigma_2, \sigma_3 \ge 0.$$

Then E_1 and E_2 are described as

$$\begin{aligned} E_1 &= \{(\xi_1, 0, \xi_3) : (\xi_1 - \sigma_1)^2 + (\xi_3 - \sigma_3)^2 = \sigma_1^2 + \delta^2\}, \\ E_2 &= \{(0, \xi_2, \xi_3) : (\xi_2 - \sigma_2)^2 + (\xi_3 - \sigma_3)^2 = \sigma_2^2 + \delta^2\}. \end{aligned}$$

Clearly, L_1 and L_2 are given by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively.

Choose a point $v \in \mathrm{bd}\, K \setminus (L_1 \cup L_2)$ so close to l that $v/\|v\| \in T$ and a certain 2-dimensional plane L through $\langle o, v\rangle$ meets K along a bounded set and intersects each of the ellipses E_1, E_2 at precisely two points. As above, $L \cap \mathrm{bd}\, K$ is an ellipse.

We state the existence of a quadric surface $Q \subset \mathbb{R}^3$ that contains $\{v\} \cup E_1 \cup E_2$. For this, consider the family of quadrics $Q(\mu) \subset \mathbb{R}^3$, given by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \mu\xi_1\xi_2 - 2\sigma_1\xi_1 - 2\sigma_2\xi_2 - 2\sigma_3\xi_3 + \sigma_3^2 - \delta^2 = 0,$$

where μ is a scalar parameter. Obviously, $E_i = L_i \cap Q(\mu)$, $i = 1, 2$, for all $\mu \in \mathbb{R}$. If $v = (v_1, v_2, v_3)$, then $v \notin L_1 \cup L_2$ if and only if $v_1 v_2 \ne 0$. Hence $v \in Q = Q(\mu_0)$,

where
$$\mu_0 = \frac{\delta^2 - \sigma_3^2 + 2\sigma_1 v_1 + 2\sigma_2 v_2 + 2\sigma_3 v_3 - v_1^2 - v_2^2 - v_3^2}{v_1 v_2}.$$

Next, we observe that $L \cap \mathrm{bd}\, K \subset Q$. Indeed, a planar quadric curve is uniquely determined by any five points which do not belong to a line (see, e. g., [11, pp. 395–397]). Hence the ellipse $L \cap \mathrm{bd}\, K$ is uniquely determined by the five-point set $\{v\} \cup (E_1 \cap L) \cup (E_2 \cap L)$. Since $L \cap Q$ is a quadric curve containing $\{v\} \cup (E_1 \cap L) \cup (E_2 \cap L)$, one has $L \cap \mathrm{bd}\, K = L \cap Q \subset Q$.

Slightly rotating L about the line $\langle o, v\rangle$, we obtain a family of ellipses $L \cap \mathrm{bd}\, K$ that cover an open subset V of $\mathrm{bd}\, K$, which consists of two open "lenses" with a common endpoint v. As above, $V \subset Q$. Repeating this argument for the points $w \in V \cap Q$ with $\langle o, w\rangle$ sufficiently close to l, we obtain that both endpoints q_1 and q_2 of the line segment $K \cap l$ are interior to an open set $W \subset \mathrm{bd}\, K$ such that $W \cap Q = W$.

Finally, to show the inclusion $\mathrm{bd}\, K \subset Q$, choose any point $x \in \mathrm{bd}\, K$ and denote by N the 2-dimensional subspace through $\{x\} \cup l$. Since the quadric curve $N \cap Q$ and the convex quadric curve $N \cap \mathrm{bd}\, K$ coincide along the non-collinear set $N \cap W$ and are uniquely determined by this set, one has $N \cap \mathrm{bd}\, K \subset N \cap Q$. Varying N about l, we conclude that $\mathrm{bd}\, K \subset Q$. Since Q is locally convex at any point $x \in \mathrm{bd}\, K$, Theorem 3 implies that $\mathrm{bd}\, K$ is a convex quadric.

Let $n \geq 4$. As above, we assume that $o \in \mathrm{int}\, K$. To prove that $\mathrm{bd}\, K$ is a convex quadric in $\mathbb{R}^n$, it suffices to show that the intersection of $\mathrm{bd}\, K$ with any 2-dimensional subspace $L \subset \mathbb{R}^n$ is a convex quadric curve (see statement (A) from the introduction). Choose a vector $e \in T \setminus L$ and put $M = \mathrm{span}\,(e \cup L)$. Then M is a 3-dimensional subspace of $\mathbb{R}^n$. Since the set $T \cap M$ is open in $\mathbb{S}^{n-1} \cap M$, there is a scalar $\varepsilon > 0$ such that any 2-dimensional subspace N of M that forms with $\langle o, e\rangle$ an angle of size less than ε intersects $T \cap M$. By condition 2), $N \cap \mathrm{bd}\, K$ is a convex quadric curve. From the case $n = 3$ it follows that $M \cap \mathrm{bd}\, K$ is a 3-dimensional convex quadric. Hence $L \cap \mathrm{bd}\, K$ $(= L \cap M \cap \mathrm{bd}\, K)$ is a convex quadric curve. Therefore $\mathrm{bd}\, K$ is a convex quadric.

6 Proof of Theorem 1

Since Lemma 3 shows that 2) $\Rightarrow$ 1), it remains to prove the converse assertion. In what follows, given a vector $e \in T$, denote by $H(e)$ a hyperplane that contains the middle points of all chords of K which are parallel to e.

First, we consider the case $n = 2$. Choose a vector $e_0 \in T$ and a chord $[p_0, q_0]$ of K in direction e_0. Then $[p_0, q_0]$ cuts K into two planar convex solids, say K_0 and K_0'. If both K_0 and K_0' are unbounded, then, as easily seen, K is a closed slab between a pair of parallel lines, which implies that $\mathrm{bd}\, K$ is a degenerate convex quadric. Assume that at least one of the sets K_0 and K_0' is bounded. Denote by P a closed halfplane of $\mathbb{R}^2$ determined by the line $\langle p_0, q_0\rangle$ for which $K \cap P$ is bounded. Let e_m, $m \geq 1$, be the unit vector forming with e_0 an angle of size π/m such that the chord $[p_0, q_1(m)]$ of K in direction e_m lies in P. Clearly, there is a positive integer m_0 with the property that $e_m \in T$ for all $m \geq m_0$. Denote by $p_1(m)$ and $q_2(m)$ the points in $P \cap \mathrm{bd}\, K$ so that $[p_1(m), q_1(m)]$ and $[p_1(m), q_2(m)]$ have directions e_0 and e_m, respectively. By the assumption, $H(e_0)$ contains the middle points of the chords $[p_0, q_0]$ and $[p_1(m), q_1(m)]$, while $H(e_m)$ contains the middle points of the chords $[p_0, q_1(m)]$ and $[p_1(m), q_2(m)]$.

Since the set

$$X_5(m) = \{p_0, q_0, p_1(m), q_1(m), q_2(m)\}$$

does not belong to a line, there is a unique quadric curve $Q(m)$ containing $X_5(m)$ (see, e. g., [11, pp. 395–397]). If a point $q_k(m), k \geq 2$, is chosen and the line through $q_k(m)$ in direction e_0 intersects $H(e_0) \cap K$, then denote by $p_k(m)$ the point in $\mathrm{bd}\, K$ for which the line segment $[p_k(m), q_k(m)]$ has direction e_0. Similarly, if a point $p_k(m), k \geq 2$, is chosen and the line through $p_k(m)$ in direction e_m intersects $H(e_m) \cap K$, then denote by $q_{k+1}(m)$ the point in $\mathrm{bd}\, K$ for which $[p_k(m), q_{k+1}(m)]$ has direction e_m. By Lemma 3 and condition 1) of the theorem, the set

$$X_{2k+1}(m) = \{p_0, q_0, p_1(m), q_1(m), \ldots, p_k(m), q_k(m), q_{k+1}(m)\}$$

belongs to $Q(m) \cap \mathrm{bd}\, K$. Clearly, there is an increasing sequence of positive integers $k(m)$, $m \geq m_0$, such that $X_{2k(m)+1}(m)$ exists and the sequence of sets

$$X_{2k(m_0)+1}(m_0), X_{2k(m_0+1)+1}(m_0+1), \ldots,$$

tends to a dense subset of $P \cap \mathrm{bd}\, K$. Hence the arcs $P \cap Q(m_0), P \cap Q(m_0+1), \ldots$ converge to $P \cap \mathrm{bd}\, K$, which shows that the arc $P \cap \mathrm{bd}\, K$ is a piece of a quadric curve. Continuously translating $[p_0, q_0]$ away from P, we express $\mathrm{bd}\, K$ as the union of an increasing sequence of convex quadrics, implying that $\mathrm{bd}\, K$ is itself a convex quadric.

Let $n \geq 3$. Choose a point $p \in \mathrm{int}\, K$, and let L be a 2-dimensional plane through p which properly intersects K such that the subspace $L - p$ meets T. Then $L \cap T$ is an open subset of $L \cap (\mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K))$. If $e \in L \cap (\mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K))$, then, by condition 2) of the theorem, the middle points of all chords of K in direction e belong to a hyperplane $H(e)$. Clearly, $L \cap H(e)$ is a line in L such that the middle points of all chords of $K \cap L$ in direction e belong to $L \cap H(e)$. By the proved above, $L \cap \mathrm{bd}\, K$ is a convex quadric curve. Theorem 4 implies that $\mathrm{bd}\, K$ is a convex quadric.

7 Proof of Theorem 2

2) $\Rightarrow$ 1) Translating K on $-p$, we may assume that $p = o$. Denote by h is a chord of K which contains o. Then h is parallel to a unit vector $e \in \mathbb{S}^{n-1} \setminus (\mathrm{rec}\, K \cup -\mathrm{rec}\, K)$. If $\Omega_\delta(p)$ is the neighborhood of $\mathrm{bd}\, K \setminus \big((p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)\big)$ in $\mathrm{bd}\, K$ that lies in a convex quadric, Q, then the cylinder $C_\delta(h)$ of radius δ centered about the line $\langle o, e\rangle$ intersects $\mathrm{bd}\, K$ within Q. By Lemma 3, the middle points of all chords from $\mathcal{F}_\delta(h)$ belong to a hyperplane.

1) $\Rightarrow$ 2) As above, we can reduce our consideration to the case when $p = o$. Furthermore, we may suppose that K is a line-free. Indeed, assume that $\dim(\mathrm{lin}\, K) \geq 1$.

Choose a chord h of K that contains o. Let $M \subset \mathbb{R}^n$ be a subspace which is complementary to $\operatorname{lin} K$ and contains h. Put $K' = M \cap K$. Clearly, $\operatorname{lin} K' = M \cap \operatorname{lin} K = \{o\}$. If H is a hyperplane that contains the middle points of all chords from $\mathcal{F}_\delta(h)$, then $H \cap M$ contains the middle points of these chords that lie in M. So, if we prove the existence of the neighborhood $\Omega'_\delta(o)$ of the set $\operatorname{rbd} K' \setminus (\operatorname{rec} K' \cup - \operatorname{rec} K')$ in $\operatorname{rbd} K'$ which lies in a convex quadric $Q' \subset M$, then from the equality $\operatorname{bd} K = \operatorname{lin} K \oplus \operatorname{rbd} K'$ we will conclude that the neighborhood $\Omega_\delta(o)$ of $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ in $\operatorname{bd} K$ lies in the convex quadric $\operatorname{lin} K \oplus Q'$.

First, we consider the case $n = 2$. Choose a chord $h = [p_0, q_0]$ of K that contains o and denote by e_0 the unit vector which is a positive scalar of $q_0 - p_0$. As above, $C_\delta(h)$ stands for the closed slab of $\mathbb{R}^2$ of width 2δ centered about the line $\langle p_0, q_0 \rangle$. Denote by e_m, $m \geq 1$, the unit vector forming with e_0 an angle of size π/m. Clearly, there is a positive integer m_0 with the property that both chords $[p_0, q_1(m)]$ and $[p_{-1}(m), q_0]$ of K in direction e_m lie within $C_\delta(h)$ for all $m \geq m_0$.

Denote by $p_1(m)$, $m \geq m_0$, the point in $C_\delta(h) \cap \operatorname{bd} K$ so that $[p_1(m), q_1(m)]$ has direction e_0. By condition 1), there is a line $H(e_0)$ which contains the middle points of the chords $[p_0, q_0]$ and $[p_1(m), q_1(m)]$. Similarly, there is a line $H(e_m)$ containing the middle points of the chords $[p_{-1}, q_0(m)]$ and $[p_0, q_1(m)]$.

Since the set

$$Y_5(m) = \{p_0, q_0, p_{-1}(m), p_1(m), q_1(m)\}$$

does not belong to a line, there is a unique quadric curve $Q(m)$ containing $Y_5(m)$ (see, e. g., [11, pp. 395–397]). If a point $q_k(m), k \geq 2$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $q_k(m)$ in direction e_0 intersects $H(e_0) \cap K$, then let $p_k(m)$ be the point in $C_\delta(h) \cap \operatorname{bd} K$ for which the line segment $[p_k(m), q_k(m)]$ has direction e_0. If a point $p_k(m), k \geq 2$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $p_k(m)$ in direction e_m intersects both $H(e_m) \cap K$ and $C_\delta(h) \cap \operatorname{bd} K$, then denote by $q_{k+1}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which $[p_k(m), q_{k+1}(m)]$ has direction e_m.

Similarly, if a point $p_{-k}(m), k \geq 1$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $p_{-k}(m)$ in direction e_0 intersects $H(e_0) \cap K$, then denote by $q_{-k}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which the line segment $[p_{-k}(m), q_{-k}(m)]$ has direction e_0. If a point $q_{-k}(m), k \geq 1$, is chosen in $C_\delta(h) \cap \operatorname{bd} K$ and the line through $q_{-k}(m)$ in direction e_m intersects both $H(e_m) \cap K$ and $C_\delta(h) \cap \operatorname{bd} K$, then denote by $p_{-k-1}(m)$ the point in $C_\delta(h) \cap \operatorname{bd} K$ for which $[p_{-k-1}(m), q_{-k}(m)]$ has direction e_m.

By Lemma 3 and condition 1) of the theorem, the set

$$\begin{aligned} Y_{2k+2}(m) = \{&p_0, q_0, p_1(m), q_1(m), \dots, p_k(m), q_k(m), \\ &p_{-1}(m), q_{-1}(m), \dots, p_{-k}(m), q_{-k}(m)\} \end{aligned}$$

belongs to $Q(m) \cap C_\delta(h) \cap \operatorname{bd} K$. Clearly, there is an increasing sequence of positive integers $k(m)$, $m \geq m_0$, such that $Y_{2k(m)+2}(m)$ exists for all $m \geq m_0$, and the sets

$$Y_{2k(m_0)+2}(m_0), Y_{2k(m_0+1)+2}(m_0+1), \dots,$$

tend to a dense subset of $C_\delta(h) \cap \operatorname{bd} K$. Hence the curves

$$C_\delta(h) \cap Q(m_0),\ C_\delta(h) \cap Q(m_0+1), \dots$$

converge to $C_\delta(h) \cap \operatorname{bd} K$, which shows that $C_\delta(h) \cap \operatorname{bd} K$ is a piece of a quadric curve (consisting of one or two arcs). Continuously rotating h about o, we cover $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ by the family of overlapping pieces $C_\delta(h) \cap \operatorname{bd} K$ of the same quadric curve. Hence the neighborhood $\Omega_\delta(o)$ of $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ in $\operatorname{bd} K$ lies in a convex quadric.

Let $n \geq 3$. Choose any 2-dimensional subspace L such that $L \cap K$ is bounded (this is possible since K is assumed to be line-free). Then $\operatorname{rec}(L \cap K) = \{o\}$. If h is a chord of $L \cap K$ and $H \subset \mathbb{R}^n$ is a hyperplane containing the middle points of all chords of K which are parallel to h and lie within the cylinder $C_\delta(h)$, then $C_\delta(h) \cap L$ is a slab of width 2δ centered about the line containing h and $L \cap H$ is a line that contains the middle points of all chords of $L \cap K$ that belong to $\mathcal{F}_\delta(h)$. Hence $L \cap K$ satisfies condition 1) of the theorem (with L instead of $\mathbb{R}^n$) By the proved above (see the case $n = 2$), $\operatorname{rbd}(L \cap K)$ is a convex quadric; so, it is an ellipse.

Because the argument above holds for any choice of a 2-dimensional subspace L, the set $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ lies in a convex quadric Q (see assertion (B) from the introduction). If K is bounded, then $\operatorname{rec} K = \{o\}$ and the whole hypersurface $\operatorname{bd} K$ is a convex quadric. Assume that K is unbounded and choose a halfline t with endpoint o that lies in $\operatorname{int} K$. Then (see the case $n = 2$) for any 2-dimensional subspace L that contains t, the neighborhood $\Omega_\delta(o)$ of $(L \cap \operatorname{bd} K) \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ in $\operatorname{rbd}(L \cap K)$ lies in $L \cap Q$. Therefore the neighborhood $\Omega_\delta(o)$ of $\operatorname{bd} K \setminus (\operatorname{rec} K \cup -\operatorname{rec} K)$ in $\operatorname{bd} K$ lies in Q.

8 Proof of Theorem 6

The proof is organized by induction on $n\,(\geq 3)$. Let $n = 3$. Consider the 1-dimensional subspace $l = L_1 \cap L_2$ and choose a longest chord $[x, z]$ of K in direction l. Translating K on a suitable vector, we may suppose that the origin o of $\mathbb{R}^3$ is the middle point of $[x, z]$. By the assumption, both sections $E_1 = L_1 \cap \operatorname{bd} K$ and $E_2 = L_2 \cap \operatorname{bd} K$ are ellipses. Due to the choice of $[x, z]$, there are parallel planes M_x and M_z both supporting K such that $K \cap M_x = \{x\}$ and $K \cap M_z = \{z\}$ (see, e. g., [14]). Applying a suitable linear transformation, we may suppose that (i) L_1 and L_2 are orthogonal to each other, (ii) both ellipses E_1 and E_2 are circumferences with diameter $[x, z]$, (iii) both planes M_x and M_z are orthogonal to $[x, z]$. Clearly, the image of K under this transformation still satisfies the hypothesis of the theorem.

Choosing suitable Cartesian coordinates ξ_1, ξ_2, ξ_3 for $\mathbb{R}^3$, we may consider that x shows a positive direction of the ξ_3-axis and

$$E_1 = \{(\xi_1, 0, \xi_3) : \xi_1^2 + \xi_3^2 = \rho^2\}, \quad E_2 = \{(0, \xi_2, \xi_3) : \xi_2^2 + \xi_3^2 = \rho^2\},$$

where $\rho = \|x\|$. Clearly, L_1 and L_2 are given by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively. Furthermore, M_x and M_z are described by $\xi_3 = \rho$ and $\xi_3 = -\rho$.

Choose a point $v \in (L_3 \cap \operatorname{bd} K) \setminus (L_1 \cup L_2)$ and consider the family of quadrics $Q(\mu)$ defined by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \mu\xi_1\xi_2 - \rho^2 = 0,$$

where μ is a scalar parameter. Clearly, $E_i = L_i \cap Q(\mu)$, $i = 1, 2$, for all $\mu \in \mathbb{R}$. If $v = (v_1, v_2, v_3)$, then $v \notin L_1 \cup L_2$ if and only if $v_1 v_2 \neq 0$. Hence $v \in Q = Q(\mu_0)$, where

$$\mu_0 = \frac{\rho^2 - v_1^2 - v_2^2 - v_3^2}{v_1 v_2}.$$

Since v lies within the slab $-\rho \leq \xi_3 \leq \rho$ and does not belong to the interior of $\operatorname{conv}(E_1 \cup E_2)$, the quadric Q is either a cylinder or an ellipsoid.

We state that the ellipse $E_3 = L_3 \cap \operatorname{bd} K$ is symmetric about o and lies in Q. Indeed, if L_3 contains $[x, z]$, then $[x, z]$ is the longest diameter of E_3, which shows that E_3 is uniquely determined by $[x, z]$ and v. In particular, E_3 is symmetric about o. Since $L_3 \cap Q$ is an ellipse containing $\{v, x, z\}$ and supported by both planes M_x and M_z, we conclude that $E_3 = L_3 \cap Q$. If L_3 does not contain $[x, z]$, then L_3 meets each of E_1, E_2 at a pair of points symmetric about o. Because E_3 is uniquely determined by v and the four points of intersection with $E_1 \cup E_2$, the ellipse E_3 is symmetric about o and lies in Q.

Considering separately the cases $l \subset L_3$ and $l \not\subset L_3$, we observe that a certain plane $u_0 + L_4$, $u_0 \in \operatorname{bd} K$, intersects the union $E_1 \cup E_2 \cup E_3$ at precisely six points, which do not belong to a line. Since the ellipse $E_4(u_0) = (u_0 + L_4) \cap \operatorname{bd} K$ is uniquely determined by these six points and since $(u_0 + L_4) \cap Q$ is also an ellipse determined by these points, one has $E_4(u_0) \subset Q$. By continuity, there is a small neighborhood U of u_0 such that the argument above holds for all $u \in U$. Clearly, the ellipses $E_4(u)$, $u \in U$, cover an open "belt" Ω of $\operatorname{bd} K$ which lies in Q. Repeating this consideration for the subspace L_1 and all points $u \in \Omega$, we obtain a wider "belt" of $\operatorname{bd} K$ which also lies in Q. Since the whole $\operatorname{bd} K$ can be expressed as the union of an increasing sequence of such "belts" obtained by the alternating consideration of translates of L_1 and L_2, we conclude that $\operatorname{bd} K \subset Q$. Therefor Q is a bounded convex quadric; that is, $\operatorname{bd} K = Q$ is an ellipsoid.

Let $n \geq 4$. Assume that the theorem is true for all $\mathbb{R}^m$, $m \leq n-1$, and let $K \subset \mathbb{R}^n$ be a convex body which satisfies its hypothesis. Translating K on a suitable vector, we may suppose that $o \in \operatorname{int} K$. Choose an $(n-1)$-dimensional subspace $P \subset \mathbb{R}^n$ such that the $(n-2)$-dimensional subspaces $P \cap L_i$, $i = 1, 2, 3, 4$, are pairwise distinct. From the hypothesis it follows that all proper sections of $P \cap \operatorname{bd} K$ by translates of the subspaces $P \cap L_i$, $i = 1, 2, 3, 4$, within P are $(n-2)$-dimensional ellipsoids. The inductive assumption gives that $P \cap \operatorname{bd} K$ is an $(n-1)$-dimensional ellipsoid. Because the family of $(n-1)$-dimensional subspaces $P \subset \mathbb{R}^n$ with the property

$$P \cap L_i \neq P \cap L_j, \quad i \neq j, \quad i, j \in \{1, 2, 3, 4\},$$

is dense in the family of all $(n-1)$-dimensional subspaces of $\mathbb{R}^n$, we obtain, by continuity, that every section of $\operatorname{bd} K$ by an $(n-1)$-dimensional subspace is an $(n-1)$-dimensional ellipsoid. This implies that $\operatorname{bd} K$ is an ellipsoid itself (see [5]).

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