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Algorithms for Determining the State-Time Probabilities and the Limit Matrix in Markov Chains

Dmitrii Lozovanu, Stefan Pickl

Abstract. New calculation procedures for finding the probabilities of state transitions of the system in Markov chains based on dynamic programming are developed and polynomial time algorithms for determining the limit state matrix in such processes are proposed. Computational complexity aspects and possible applications of the proposed algorithms for the stochastic optimization problems are characterized.

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1 Introduction and Preliminary Results

In this paper we develop a dynamic programming approach for finite Markov processes and propose polynomial time algorithms for determining the limit state matrix in Markov chains. A characterization of a simple Markov process and the basic definitions related to determining the probabilities of state transitions of the system in such processes can be found in [3–5, 9, 10]. Here, for the finite Markov processes, we consider the problem of determining the probability of system's transition from a starting state to a final one when the final state is reached at the time-moment which belongs to a given interval of time. For such a specific case, we develop dynamic programming algorithms. Furthermore, the asymptotic behavior of the proposed algorithms are analyzed. Such a characterization of the problem allows us to apply a new approach for studying Markov chains and to elaborate polynomial time algorithms for determining the limit state probabilities of the dynamical system in such processes. We show that for non-ergodic Markov chains the limit probability matrix can be found in polynomial time. Therefore, we propose two polynomial time algorithms. The computational complexity of the first algorithm is $O(n^4)$ and of the second one is $O(n^3)$. Note that the well-known algorithm from [9] (see also [4, 5, 10]) in the worst case uses $O(n^4)$ elementary operations. Comparing this algorithm with proposed ones we can conclude that the approach described below allows us to ground new efficient algorithms for determining the limit state matrix in Markov chains. Additionally, we develop dynamic programming procedures for the calculation of the state probability transitions in the non-stationary discrete Markov processes. The proposed calculation procedures and algorithms can

be used for studying and solving the stochastic version of classical discrete optimal control problems [2–9].

In this paper we consider discrete Markov processes with a finite set of states [3, 5, 7]. We denote the set of states of the dynamical system in such processes by $X = \{x_1, x_2, \ldots, x_n\}$. At the moment of time $t = 0$ the state of the system is x_{i_0}. For an arbitrary state $x \in X$ the probabilities $p_{x,y}$ of system's transitions from x to another states $y \in X$ such that $\sum\limits_{y \in X} p_{x,y} = 1$ are given. So, we assume that the Markov process is determined by the stochastic matrix of probabilities $P = (p_{x,y})$ and the starting state x_{i_0} of the dynamical system. The probability $P_{x_{i_0}}(x, t)$ of system's transitions from the state x_{i_0} to an arbitrary state $x \in X$ by using t transitions is defined and calculated on the basis of the following recursive formula [3]

$$P_{x_{i_0}}(x, \tau + 1) = \sum_{y \in X} P_{x_{i_0}}(y, \tau) p_{y,x}, \quad \tau = 0, 1, 2, \ldots, t - 1,$$

where $P_{x_{i_0}}(x_{i_0}, 0) = 1$ and $P_{x_{i_0},0}(x, 0) = 0$ for $x \in X \setminus \{x_{i_0}\}$. This formula can be represented in the matrix form by

$$\pi(\tau + 1) = \pi(\tau) P, \quad \tau = 0, 1, 2, \ldots, t - 1. \tag{1}$$

Here $\pi(\tau) = (\pi_1(\tau), \pi_2(\tau), \ldots, \pi_n(\tau))$ is the vector, where the component i expresses the probability of the system L to reach from $x_{x_{i_0}}$ the state x_i at the moment of time τ, i.e. $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given and its components are defined by $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. If for given starting vector $\pi(0)$ we apply our formula for $t = 0, 1, 2, \ldots, t-1$, then we obtain

$$\pi(t) = \pi(0) P^{(t)}$$

where $P^{(t)} = P \times P \times \cdots \times P$. So, an arbitrary element $p^{(t)}_{x,y}$ of this matrix expresses the probability of system L to reach the state y from x by using t units of times.

Formula (1) can be applied for the calculation of the state probabilities of the system in finite Markov processes. In the case $\tau \to \infty$ this formula leads to the relation $\pi = \pi P$ which together with the condition $\sum\limits_{i=1}^{n} \pi_i = 1$ allows us to determine the limit state probabilities in ergodic Markov chains.

2 The Main Results

To solve our main problem we need to develop special calculation procedures for determining the probability of system's transitions from a starting state to a final one when the final state is reached at the time-moment from given interval of time. We describe such calculation procedures which will allow us to ground polynomial time algorithms for finding the limit state matrix in aperiodic Markov chains.

2.1 Calculation of the Probabilities of States Transition of the System with a Given Restriction on the Number of Stages

In this subsection we show how to calculate the probability of system's transitions from the state x_{i_0} to the state x when x is reached at the time moment $T(x)$ such that $T_1 \leq T(x) \leq T_2$ where T_1 and T_2 are given. So, we consider the problem of determining the probability of the system L to reach the state x at least at one of the moments of time $T_1, T_1+1, \ldots, T_2$. We denote this probability by $P_{x_{i_0}}(x, T_1 \leq T(x) \leq T_2)$. Some reflections on this definition allow us to write the following formula

$$P_{x_{i_0}}(x, T_1 \leq T(x) \leq T_2) =$$

$$= P_{x_{i_0}}(x, 0 \leq T(x) \leq T_2) - P_{x_{i_0}}(x, 0 \leq T(x) \leq T_1 - 1).$$

Further we describe some results which allow to calculate the probability $P_x(y, 0 \leq T(y) \leq t)$ for $x, y \in X$ and $t = 1, 2, \ldots$. For this reason we shall give the graphical interpretation of the Markov processes using the graph of state transitions $GR = (X, ER)$ [1, 3, 7, 10]. In this graph each vertex $x \in X$ corresponds to a state of the dynamical system and a possible system passage from one state x to another state y with positive probability $p_{x,y}$ is represented by the directed edge $e = (x, y) \in ER$ from x to y; to directed edges $(x, y) \in ER$ in GR the corresponding probabilities $p_{x,y}$ are associated. It is evident that in the graph GR each vertex x contains at least one leaving edge (x, y) and $\sum\limits_{y \in X} p_{x,y} = 1$. As an example the graph of state transitions $GR = (X, ER)$ for the Markov process with the stochastic matrix of probabilities

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.3 & 0.5 & 0.2 \end{pmatrix}$$

is represented in Fig. 1.

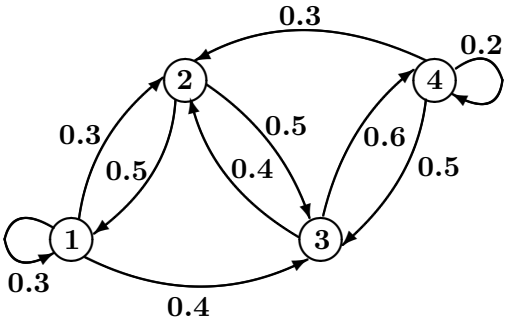

Fig. 1

In general we will consider also the stochastic process which may stop if one of the states from a given subset of states of dynamical system is reached. This means

that the graph of such a process may contain the so-called deadlock vertices. So, we consider the stochastic process for which the graph of transition probabilities may contain the deadlock vertices $y \in X$ and $\sum_{z \in X} p_{x,z} = 1$ for the vertices $x \in X$ which contain at least one leaving directed edge. As an example in Fig. 2 a graph $GR = (X, ER)$ which contains a deadlock vertex is represented.

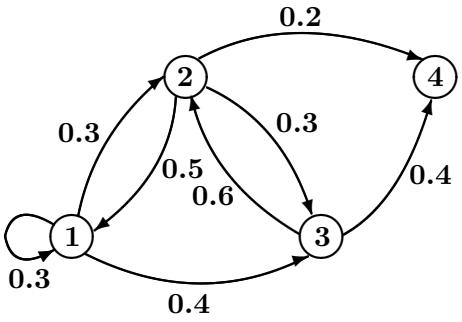

Fig. 2

This graph corresponds to the stochastic process with the following matrix of state transitions

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such graphs do not correspond to a Markov process and the matrix of probability P contains a row with zero components. Nevertheless the probabilities $P_{x_0}(x,t)$ in this case can be calculated on the basis of the recursive formula given above. Note that the matrix P can be easily transformed into a stochastic matrix changing the probabilities $p_{y,y} = 0$ for deadlock states $y \in X$ by the probabilities $p_{y,y} = 1$. This transformation leads to a new graph which corresponds to a Markov process because the obtained graph contains a new directed edge $e = (y, y)$ with $p_e = 1$ for $y \in X$. We call the vertices $y \in X$ in this graph the absorbing vertices and the corresponding states of the dynamical system in Markov process the absorbing states. So, the stochastic process which may stop in a given set of states can be represented either by a graph with deadlock vertices or by a graph with absorbing vertices. In Fig. 3 represents the graph with absorbing vertex $y = 4$ for the Markov process defined by the matrix P given below.

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1.0 \end{pmatrix}.$$

It is easy to see that the stochastic matrix P in this example is obtained from the previous one by changing $p_{4,4} = 0$ with $p_{4,4} = 1$. The corresponding graph with

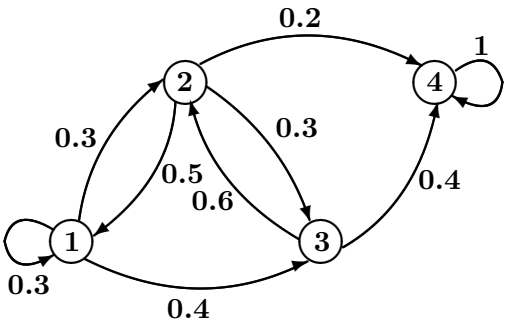

Fig. 3

the absorbing vertex $y = 4$ in this case is obtained from the graph on Fig. 2 by adding the directed edge $e = (4, 4)$ with $p_{4,4} = 1$.

We shall calculate the probabilities $P_x(y, 0 \leq T(y) \leq t)$ by using the graph with absorbing vertices.

Lemma 1. *Let a Markov process be given for which the graph $GR = (X, ER)$ contains an absorbing vertex $y \in X$. Then for an arbitrary state $x \in X$ the following recursive formula holds:*

$$P_x(y, 0 \leq T(y) \leq \tau + 1) = \sum_{z \in X} p_{x,z} P_z(y, 0 \leq T(z) \leq \tau), \quad \tau = 0, 1, 2, \ldots,$$

where $P_x(y, 0 \leq T(y) \leq 0) = 0$ if $x \neq y$ and $P_y(y, 0 \leq T(y) \leq 0) = 1$.

Proof. It is easy to observe that for $\tau = 0$ the theorem holds. Moreover, we can see that here the condition that y is an absorbing state is essential; otherwise for $x = y$ the recursive formula from lemma fails to hold. For $\tau \geq 1$ the correctness of this formula follows from the definition of the probabilities $P_x(y, 0 \leq T(y) \leq \tau + 1), P_z(y, 0 \leq T(z) \leq \tau)$ and from the induction principle on τ. □

The recursive formula from this lemma can be written in matrix form by

$$\pi'(\tau + 1) = P\pi'(\tau), \quad \tau = 0, 1, 2, \ldots.$$

Here P is the stochastic matrix of the Markov process with the absorbing state $y \in X$ and

$$\pi'(\tau) = \begin{pmatrix} \pi'_1(\tau) \\ \pi'_2(\tau) \\ \vdots \\ \pi'_n(\tau) \end{pmatrix}, \quad \tau = 0, 1, 2, \ldots$$

are the column vectors, where an arbitrary component $\pi'_i(\tau)$ expresses the probability of the dynamical system to reach the state y from x_i by using not more than τ unites of times, i.e. $\pi'_i(\tau) = P_{x_i}(y, 0 \leq T(y) \leq \tau)$. At the starting moment of time

$\tau = 0$ the vector $\pi'(0)$ is given: All components are equal to zero except for the component corresponding to the absorbing vertex which is equal to one, i.e.

$$\pi'_i(0) = \begin{cases} 0, & \text{if } x_i \neq y; \\ 1, & \text{if } x_i = y. \end{cases}$$

If we apply this formula for $\tau = 0, 1, 2, \ldots, t-1$, then we obtain

$$\pi'(t) = P^{(t)}\pi'(0), \quad t = 1, 2, \ldots.$$

So, if we denote by j_y the column of the matrix $P^{(t)}$ which corresponds to the absorbing state y then an arbitrary element $p^{(t)}_{i,j_y}$ of this column expresses the probability of the system L to reach the state y from x_i by using not more than t units of time, i.e. $p^{(t)}_{i,j_y} = P_{x_i}(y, 0 \leq T(x) \leq t)$. This allows us to formulate the following lemma:

Lemma 2. *Let a finite Markov process with the absorbing state $y \in X$ be given. Then:*

$$a) \quad P_{x_i}(y, \tau) = P_{x_i}(y, 0 \leq T(y) \leq \tau), \quad for \quad x_i \in X \setminus \{y\}, \quad \tau = 1, 2, \ldots;$$

$$b) \quad P_{x_i}(y, T_1 \leq T(y) \leq T_2) = p^{(T_2)}_{i,j_y} - p^{(T_1-1)}_{i,j_y}.$$

Proof. The condition $a)$ in this lemma holds because

$$P_{x_i}(y, \tau) = p^{(\tau)}_{i,j_y} = P_{x_i}(y, 0 \leq T(y) \leq \tau).$$

The condition $b)$ we obtain from Lemma 1 and the following properties

$$P_{x_i}(y, 0 \leq T(y) \leq T_2) = p^{(T_2)}_{i,j_y}, \quad P_{x_i}(y, 0 \leq T(y) \leq T_1 - 1) = p^{(T_1-1)}_{i,j_y}.$$

□

So, to calculate $P_{x_i}(y, T_1 \leq T(y) \leq T_2)$ it is sufficient to find the matrices $P^{(T_1-1)}$, $P^{(T_2)}$ and then to apply the formula from Lemma 2.

The procedure of the calculation of the probabilities $P_x(y, 0 \leq T(y) \leq t)$ in the case of the Markov process without absorbing states can be easily reduced to the procedure of the calculation of the probabilities in the Markov process with the absorbing state y by using the following transformation of the stochastic matrix P. We put $p_{i_y,j} = 0$ if $j \neq i_y$ and $p_{i_y,i_y} = 1$. It is easy to see that such a transformation of the matrix P does not change the probabilities $P_x(y, 0 \leq T(y) \leq t)$. After such a transformation we obtain a new stochastic matrix for which the recursive formula from the Lemma 21 can be applied. In general for the Markov processes with absorbing state these probabilities can be calculated by using the algorithm which works with the original matrix P without changing its elements. Below such an algorithm is described.

Algorithm 1: Determining the state probabilities of the system with a restriction on number of transitions (stationary case)

Preliminary step (Step 0): Put $P_x(y, 0 \leq T(y) \leq 0) = 0$ for every $x \in X \setminus \{y\}$ and $P_y(y, 0 \leq T(x) \leq 0) = 1$.

General step (Step $\tau + 1, \tau \geq 0$): For every $x \in X \setminus \{y\}$ calculate

$$P_x(y, 0 \leq T(x) \leq \tau + 1) = \sum_{z \in X} p_{x,z} P_z(y, 0 \leq T(y) \leq \tau) \tag{2}$$

and then put

$$P_y(y, 0 \leq T(y) \leq \tau + 1) = 1. \tag{3}$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

Theorem 1. *Algorithm 1 correctly finds the probabilities $P_x(y, 0 \leq T(x) \leq \tau)$ for $x \in X, \quad \tau = 0, 1, 2, \ldots, t - 1$.*

Proof. It is easy to see that the probabilities $P_x(y, 0 \leq T(x) \leq \tau + 1)$ at the general step of the algorithm are calculated on the basis of formula (2) which takes into account condition (3). This calculation procedure is equivalent with the calculation of the probabilities $P_x(y, 0 \leq T(x) \leq \tau + 1)$ with the condition that the state y is an absorbing state. So, the algorithm is correct. □

If in Algorithm 1 we use the notation $\pi'_i(\tau) = P_{x_i}(y, 0 \leq T(y) \leq \tau)$, $\pi_{i_y}(\tau) = P_y(y, 0 \leq T(y) \leq \tau)$ then we obtain the following description of the algorithm in a matrix form:

Algorithm 2: Calculation of the state probabilities of the system in the matrix form (stationary case)

Preliminary step (Step 0): Fix the vector $\pi'(0) = (\pi'_1(0), \pi'_2(0), \ldots, \pi'_n(0))$, where $\pi'_i(0) = 0$ for $i \neq i_y$ and $\pi'_{i_y}(0) = 1$.

General step (Step $\tau + 1, \tau \geq 0$): For given τ calculate

$$\pi'(\tau + 1) = P\pi'(\tau)$$

and then put

$$\pi'_{i_y}(\tau + 1) = 1.$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

Note that the condition $\pi'_{i_y}(\tau + 1) = 1$ in the algorithm allows us to preserve the value $\pi'_{i_y}(t) = 1$ at every moment of time t in the calculation process. This condition reflects the property that the system remains in the state y at every time-step t if the state y is reached. We can modify this algorithm for determining the probability $P_x(y, 0 \leq T(y) \leq 0)$ in a more general case if we assume that the system will remain at every time step t in the state y with the probability $\pi'_{i_y}(t) = q(y)$, where $q(y)$ may differ from 1, i.e, $q(y) \leq 1$. In the following we can see that this modification

is very important for determining the matrix of limit probabilities in finite Markov processes. So, $q(y) \leq 1$, and we can use the following algorithm:

Algorithm 3: Calculation of the state probabilities of the system with a given probability of its remaining in the final state (stationary case)

Preliminary step (Step 0): Fix the vector $\pi'(0) = (\pi'_1(0),\ \pi'_2(0),\ \ldots,\ \pi'_n(0))$, where $\pi'_i(0) = 0$ for $i \neq i_y$ and $\pi'_{i_y}(0) = q(y)$.

Genrral step (Step $\tau + 1, \tau \geq 0$): For given τ calculate

$$\pi'(\tau + 1) = P\pi'(\tau)$$

and then put

$$\pi'_{i_y}(\tau + 1) = q(y).$$

If $\tau < t - 1$ then go to next step; otherwise STOP.

Remark 1. All results and algorithms described above are also valid for the stochastic processes in the case when $\sum\limits_{z \in X} p_{x,z} = r(x) \leq 1$ for $x \in X$.

2.2 Polynomial time algorithms for determining the limit state matrix in Markov chains

Denote by $S = (s_{i,j})$ the limit matrix of probabilities for the Markov chain induced by stochastic matrix $P = (p_{x,y})$. We denote the vector columns of the matrix S by

$$S^j = \begin{pmatrix} s_{1,j} \\ s_{2,j} \\ \vdots \\ s_{n,j} \end{pmatrix}, \quad j = 0, 1, 2, \ldots, n,$$

and the row vectors of the matrix S we denote by $S_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,n}), \quad i = 1, 2, \ldots, n$. To describe the algorithms for finding the limit matrix S for non-ergodic Markov process we need to analyze the structure of the graph of transition probabilities and to study the behavior of the algorithms from the previous subsection in the case $t \to \infty$. First of all we note that for the ergodic Markov chain the graph GR is strongly connected and all vector rows $S_i, \quad i = 1, 2, \ldots, n$, are the same. In this case the limit state probabilities can be found by solving the system of linear equations

$$\pi = \pi P, \quad \sum_{J=1}^{n} \pi_j = 1,$$

i.e. $S_i = \pi, \quad i = 1, 2, \ldots, n$. In general, such an approach can be used for an arbitrary ergodic Markov process if the limit state probabilities exist.

In the multichain Markov processes the graph $GR = (X, ER)$ consists of several strongly connected components $G^1 = (X^1, E^1)$, $G^2 = (X^2, E^2)$, ..., $G^k = (X^k, E^k)$ where $\bigcup\limits_{i=1}^{k} X^i = X$. Additionally, among these components, there are such strongly

connected components $G^{i_p} = (X^{i_p}, E^{i_p}),\quad p = 1, 2, \ldots, q$, which do not contain a leaving directed edge $e = (x, y)$ where $x \in X^{i_p}$ and $y \in X \setminus X^{i_p}$. We call such components G^{i_p} *deadlock* components in GR. A characterization of the ergodic classes (recurrence classes) in the Markov process can be made in terms of a graph of transition probabilities using the deadlock components.

Lemma 3. *If $G^{i_p} = (X^{i_p}, E^{i_p})$ is a strongly connected deadlock component in GR then X^{i_p} is an ergodic class (recurrence chain) of the Markov process; if $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ then x is a transient state of the system in the Markov process.*

Lemma [3] reflects the well known properties of the Markov chains from [3–5, 10] in the terms of graphs of transition probabilities. The proof of the lemma follows from [3–5, 10].

Below we give some auxiliary results which can be obtained from the algorithmic procedure from the previous subsection in the case $t \to \infty$. Let a Markov process with a finite set of states X be given. For an arbitrary state $x_j \in X$ we denote by X_j the subset of states $x_k \in X$ for which in GR there exists at least a directed path from x_k to x_j. Additionally, we denote $N = \{1, 2, \ldots, n\},\quad I(X_j) = \{k | x_k \in X_j\}$.

Lemma 4. *Let a Markov process with a finite set of states X be given and assume that x_j is an absorbing state. Let π^j be a solution of the following system of linear equations*

$$\pi^j = P\pi^j; \quad \pi_{j,j} = 1; \quad \pi_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j), \tag{4}$$

where

$$\pi^j = \begin{pmatrix} \pi_{1,j} \\ \pi_{2,j} \\ \vdots \\ \pi_{n,j} \end{pmatrix}.$$

Then $\pi^j = S^j$, i.e. $\pi_{i,j} = s_{i,j},\quad i = 1, 2, \ldots, n$. If x_j is a unique absorbing state of the Markov process and if x_j in GR is attainable from every $x_i \in X$ (i.e. $I(X_j) = N$) then $\pi_{i,j} = s_{i,j} = 1,\quad i = 1, 2, \ldots, n$.

Proof. We apply Algorithm 2 with respect to a given absorbing state x_j ($y_j = x_j$) when $t \to \infty$. Then $\pi(t)' \to \pi^j$ and therefore we obtain $\pi^j = P\pi^j$ where $\pi_{j,j} = 1$ and $\pi_{i,j} = 0$ for $i \in N \setminus I(Y_j^+)$. The correctness of the second part of the lemma corresponds to the case when $I(X_j) = N$ and therefore we obtain that the vector π^j with the components $\pi_{i,j} = 1,\quad i = 1, 2, \ldots, n$ is the solution of the system $\pi^j = P\pi^j, \pi_{j,j} = 1$. So, Lemma 4 holds. □

Remark 2. If x_j is not an absorbing state then Lemma 4 may fail to hold.

Remark 3. Lemma 4 can be extended for the case when $\sum_{y \in X} p_{x_i,y} = r(x_i) \leq 1$ for some states $x_i \in X$. The solution of the system (4) in this case also coincides with the vector of limit probabilities S^j if such a vector of limit probabilities exists.

However, if $N = I(X_j)$ then some components $\pi_{i,j}$ of the solution π_{ij} may be less than 1.

Let us show that the result formulated above allows us to find the vector of limit probabilities S^j of the matrix S if the diagonal element $s_{j,j}$ of S is known. We consider the subset of the states $Y^+ = \{x_j | s_{j,j} \geq 0\}$. It is easy to observe that $Y^+ = \bigcup_{p=1}^{q} X^{i_p}$; we denote the corresponding set of indexes of this set by $I(Y^+)$. For each $j \in I(Y^+)$ we define the set X_j in the same way as we introduced it above.

Lemma 5. *If a non-zero diagonal element $s_{j,j}$ of the limit matrix S in the non-ergodic Markov process is known, i.e. $s_{j,j} = q(x_j)$, then the corresponding vector S^j of the matrix S can be found by solving the following systems of linear equations:*

$$S^j = PS^j; \quad s_{j,j} = q(x_j); \quad s_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j)$$

Proof. We apply Algorithm 3 with respect to the fixed final state $y_j = x_j \in X$ with $q(y_j) = s_{j,j}$ when $t \to \infty$. Then for a given $y_j = x$ we have $\pi(t)' \to S^j$ and therefore we obtain $S^j = PS^j$ where $q(y_j) = s_{j,j}$ and $s_{i,j} = 0$ for $i \in N \setminus I(X_j)$. So, Lemma 5 holds. □

Basing on this lemma and Algorithm 3 we can prove the following result.

Theorem 2. *The limit state matrix S for aperiodic Markov chains can be found by using the following algorithm:*

1) For each ergodic class X^{i_p} solve the system of linear equations

$$\pi^{i_p} = \pi^{i_p} P^{i_p}, \qquad \sum_{j \in I(X^{i_p})} \pi_j^{i_p} = 1,$$

where π^{i_p} is the row vector with components $\pi_j^{i_p}$ for $j \in I(X^{i_p})$ and P^{i_p} is the submatrix of P induced by the class X^{i_p}. Then for every $j \in I(X^{i_p})$ put $s_{j,j} = \pi_j^{i_p}$; for each $j \in I(X \setminus \bigcup_{p=1}^{q} X^{i_p})$ set $s_{j,j} = 0$;

2) For every $j \in I(Y^+)$, $\quad Y^+ = \bigcup_{p=1}^{q} X^{i_p}$ solve the system of linear equations

$$S^j = PS^j; \quad s_{j,j} = \pi_{j,j}; \quad s_{i,j} = 0 \quad for \quad i \in N \setminus I(X_j)$$

and determine the vector S^j. For every $j \in I(X \setminus Y^+)$ set $S^j = \mathbf{0}$, where $\mathbf{0}$ is the vector row with zero components.

The algorithm finds the matrix S using $O(n^4)$ elementary operations.

Proof. Let us show that the algorithm finds correctly the limit matrix S. Item 1) of the algorithm finds the limit probabilities $s_{j,j}$. This item is based on Lemma 3 and on the conditions which each ergodic class X^{i_p} and each transient state $x_{\in} X \setminus Y^+$

should satisfy. So, item 1) correctly finds the limit probabilities $s_{i,j}$ for $j \in N$. Item 2) of the algorithm is based on Lemma 5 and therefore determines correctly the vectors S^j of the matrix S when the diagonal elements $s_{j,j}$ are known. So, the algorithm finds correctly the limit matrix S of the non-ergodic Markov processes if such a limit matrix exists. The computational complexity of the algorithm is determined by the computational complexity of solving $q \leq n$ equations for each ergodic class X^{i_p}(item 1) and the computational complexity of solving not more than n systems of linear equations for determining the vectors S^j (item 2). So, the running time of the algorithm is $O(n^4)$. □

Basing on this theorem we can find the limit matrix S using algorithm from Theorem 2. In the worst case the running time of the algorithm is $O(n^4)$ however intuitively it is clear that the upper bound of this estimation couldn't be reached. Practically this algorithm efficiently finds the limit matrix S. In the following we show that for determining the limit matrix in aperiodoc Markov chains there exists an algorithm with computational complexity $O(n^3)$.

2.3 An algorithm for the calculation of the limit matrix in aperiodic Markov chains with running time $O(n^3)$

We describe another algorithm for finding the limit matrix for aperiodic Marcov chains which in the most part takes into account the structure properties of the random graph of the Markov process. We can see that such an approach allows us to ground an algorithm with computational complexity $O(n^3)$.

Algorithm 4: Determining the limit state matrix for non-ergodic Markov processes

The algorithm consists of two parts: The first part determines the limit probabilities $s_{x,y}$ for $x \in \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X$. The second procedure calculates the limit probabilities $s_{x,y}$ for $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X$.

Procedure 1:

1. For each ergodic class X^{i_p} we solve the system of linear equations:

$$\pi^{i_p} = \pi^{i_p} P^{i_p}, \quad \sum_{y \in X^{i_p}} \pi_y^{i_p} = 1,$$

where P^{i_p} is the matrix of probability transitions corresponding to the ergodic class X^{i_p}, i.e. P^{i_p} is a submatrix of P, and π^{i_p} is a row vector with the components $\pi_y^{i_p}$ for $y \in X^{i_p}$. If $\pi_y^{i_p}$ are known then $s_{x,y}$ for $x \in X^{i_p}$ and $y \in X$ can be calculated as follows:

Set $s_{x,y} = \pi_y^{i_p}$ if $x, y \in X^{i_p}$ and $s_{x,y} = 0$ if $x \in X^{i_p}$, $y \in X \setminus X^{i_p}$.

Procedure 2:

1. We construct an auxiliary acyclic directed graph $GA = (XA, EA)$ which is obtained from the graph $GR = (X, ER)$ by using the following transformations:

   We contract each set of vertices X^{i_p} into one vertex z^{i_p} where X^{i_p} is a set of vertices of a strongly connected deadlock component $G^{i_p} = (X^{i_p}, E^{i_p})$ in GR. If the obtained graph contains parallel directed edges $e^1 = (x, z)$, $e^2 = (x, z)$, …, $e^r = (x, z)$ with the corresponding probabilities $p^1_{x,z}$, $p^2_{x,z}$, …, $p^r_{x,z}$ then we change them by one directed edge $e = (x, z)$ with the probability $p_{x,z} = \sum_{i=1}^{r} p^i_{x,z}$; after this transformation of each vertex z^i_p we put equivalently a directed edge of the form $e = (z^p, z^p)$ with the probability $p'_{z^p,z^p} = 1$.

2. We fix the directed graph $GA = (XA, EA)$ obtained by the construction principle from step 1 where $XA = \Big(X \setminus (\bigcup_{p=1}^{q} X^{i_p})\Big) \cup Z^p$, $\quad Z^p = (z^1, z^2, \dots, z^q)$. Additionally, we fix the new probability matrix $P' = (p'_{x,y})$ which corresponds to this random graph GA.

3. For each $x \in XA$ and every $z^i \in Z^p$ we find the probability $\pi'_x(z^i)$ of the system transaction from the state x to the state z^p. The probabilities $\pi'_x(z^i)$ can be found by solving the following p systems of linear equations:

$$P'\pi'(z^1) = \pi'(z^1), \quad \pi'_{z^1}(z^1) = 1, \quad \pi'_{z^2}(z^1) = 0, \quad \dots, \quad \pi'_{z^q}(z^1) = 0;$$

$$P'\pi'(z^2) = \pi'(z^2), \quad \pi'_{z^1}(z^2) = 0, \quad \pi'_{z^2}(z^2) = 1, \quad \dots, \quad \pi'_{z^q}(z^2) = 0;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$P'\pi'(z^q) = \pi'(z^q), \quad \pi'_{z^1}(z^q) = 0, \quad \pi'_{z^2}(z^q) = 0, \quad \dots, \quad \pi'_{z^q}(z^q) = 1,$$

   where $\pi'(z^i)$, $\quad i = 1, 2 \dots, p$ are the column vectors with components $\pi'_x(z^i)$ for $x \in XA$. So, each vector $\pi'_x(z^i)$ defines probabilities of system transitions from states $x \in XA$ to the ergodic class X^i.

4. We put $s_{x,y} = 0$ for every $x, y \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $s_{x,y} = \pi'_x(z^p)\pi^{i_p}_y$ for every $x \in X \setminus \bigcup_{p=1}^{q} X^{i_p}$ and $y \in X^{i_p}$, $X^{i_p} \subset X$. If $x \in X^{i_p}$ and $y \in X \setminus X^{i_p}$ then we fix $s_{x,y} = 0$.

Theorem 3. *The algorithm correctly finds the limit state matrix S and the running time of the algorithm is $O(|X|^3)$.*

Proof. The correctness of Procedure 1 of the algorithm follows from the definition of the ergodic Markov class (recurrence chain). So, Procedure 1 finds the probabilities $s_{x,y}$ for $x \in \bigcup\limits_{p=1}^{q} X^{i_p}$ and $y \in X$. Let us show that Procedure 2 correctly finds the rest elements $s_{x,y}$ of the matrix S. Indeed, each vertex $x \in X \setminus \bigcup\limits_{p=1}^{q} X^{i_p}$ in GA corresponds to a transient state of the Markov chain and therefore we have $s_{x,y} = 0$ for every $x, y \in X \setminus \bigcup\limits_{p=1}^{q} X^{i_p}$. If $x \in X^{i_p}$ then the system couldn't reach a state $y \in X \setminus X^{i_p}$ and therefore for arbitrary two states x, y we have $s_{x,y} = 0$. Finally, we show that the algorithm correctly determines the limit probability $s_{x,y}$ if $x \in X \setminus \bigcup\limits_{p=1}^{q} X^{i_p}$ and $y \in X^{i_p}$. In this case the limit probability $s_{x,y}$ is equal to the limit probability of the system to reach the ergodic class X^{i_p} multiplied by the limit probability of the system to remain in the state $y \in X^{i_p}$, i.e. $s_{x,y} = \pi'_x(z^p)\pi_y^{i_p}$. Here $\pi_y^{i_p}$ is the probability of the system to remain in the state $y \in X^{i_p}$ and $\pi_x(z^{i_p})$ is the limit probability of the system to reach the absorbing state z_{i_p} in GA. The value $\pi_x(z^{i_p})$ according to the construction of auxiliary graph GA coincides with the limit probability of the system to reach the ergodic class X^{i_p}. The correctness of this fact can easily be obtained from Lemma 3 and Theorem 2. According to Lemma 3 the probabilities $\pi_x(z^p)$ for $x \in X \setminus \bigcup\limits_{p=1}^{q} X^{i_p}$ can be found by solving the following system of linear equations

$$P'\pi'(z^p) = \pi'(z^p), \quad \pi'_{z^1}(z^p) = 0, \quad \pi'_{z^2}(z^p) = 0, \quad \ldots, \quad \pi'_{z^p}(z^p) = 1,$$

which determined them correctly. So, the algorithm correctly finds the limit state matrix S.

Now let us show that the running time of the algorithm is $O(n^3)$. We obtain this estimation in the item 4 solving $q \leq n$ systems of linear equations. Each of these systems contains no more than n variables. All these systems have the same left part and therefore they can be solved simultaneously applying Gaussian method. The simultaneous solution of these q systems with the same left part by using Gaussian method uses $O(n^3)$ elementary operations. □

3 Determining the State Probabilities of the Dynamical System in Non-Stationary Markov Processes

In the case when the probabilities of system's transitions from one state to another depend on time we have a non-stationary process defined by a dynamic matrix $P(t) = (p_{x,y}(t))$ which describes this process. If this matrix is stochastic for every moment of time $t = 1, 2, \ldots$, then the state probabilities $P_{x_{i_0}}(x, t)$ can be defined and calculated by using a similar formula obtained from Section 1 changing $p_{x,y}$ by

$p_{x,y}(\tau)$, i.e.

$$P_{x_{i_0}}(x, \tau+1) = \sum_{y \in X} P_{x_{i_0}}(y, \tau) p_{y,x}(\tau), \quad \tau = 0, 1, 2, \ldots, t-1$$

where $P_{x_{i_0}}(x_{i_0}, 0) = 1$ and $P_{x_{i_0}}(x, 0) = 0$ for $x \in X \setminus \{x_{i_0}\}$. In the matrix form this formula can be represented as follows

$$\pi(\tau+1) = \pi(\tau)P, \quad \tau = 0, 1, 2, \ldots, t-1$$

where $\pi(\tau) = (\pi_1(\tau), \pi(\tau), \ldots, \pi_n(\tau))$ is the vector with the components $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given in the same way as for the stationary process, i.e. $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. If for a given starting vector $\pi(0)$ and $\tau = 0, 1, 2, \ldots, t-1$ we apply this formula then we obtain

$$\pi(t) = \pi(0)P(0)P(1)P(2)\ldots P(t-1).$$

So, an arbitrary element $q_{x_i,x_j}(t)$ of the matrix

$$Q(t) = P(0)P(1)P(2)\ldots P(t-1)$$

expresses the probability of system L to reach the state x_j from x_i by using t units of times.

Now let us show how to calculate the probability $P_{x_{i_0}}(y, T_1 \leq T(y) \leq T_2)$ in the case of non-stationary Markov processes. In the same way as for the stationary case we consider the non-stationary Markov process with given absorbing state $y \in X$. So, we assume that the dynamic matrix $P(t)$ is given which is stochastic for every $t = 0, 1, 2, \ldots$ and $p_{y,y}(t) = 1$ for arbitrary t is given. Then the probabilities $P_x(y, 0 \leq T(y) \leq t)$ for $x \in X$ can be determined if we tabulate the values $P_x(y,\ t - \tau \leq T(y) \leq t), \quad \tau = 0, 1, 2 \ldots, t$, using the following recursive formula:

$$P_x(y,\ t-\tau-1 \leq T(y) \leq t) = p_{x,z}(t-\tau-1)P_z(y,\ t-\tau \leq T(y) \leq t)$$

where for $\tau = 0$ we fix

$P_x(y,\ t \leq T(y) \leq t) = 0$ if $x \neq y$ and $P_y(y,\ t \leq T(y) \leq t) = 1$.

This recursive formula can be represented in the following matrix form

$$\pi''(t-\tau-1) = P(t-\tau-1)\pi''(\tau), \quad t = 0, 1, 2, \ldots t-1.$$

At the starting moment of time $t = 0$ the vector $\pi''(0)$ is given: All components are equal to zero except the component corresponding to the absorbing vertex which is equal to one, i.e.

$$\pi_i''(0) = \begin{cases} 0, & \text{if } x_i \neq y; \\ 1, & \text{if } x_i = y. \end{cases}$$

If we apply this formula for $\tau = 0, 1, 2, \ldots, t-1$ then we obtain

$$\pi''(t) = P(0)P(1)P(2)\cdots P(t-1)\pi''(0), \quad t = 1, 2, \ldots.$$

So, if we consider the matrix $Q = P(0)P(1)P(2)\ldots P(t-1)$ then an arbitrary element q_{i,j_y} of the column j_y in the matrix Q expresses the probability of the system L to reach the state y from x_i by using not more than t units of time, i.e. $q_{i,j_y} = P_{x_i}(y,\, 0 \leq T(x) \leq t)$.

Here the matrix $P(t)$ is stochastic matrix for $t = 0, 1, 2, \ldots$ where $p_{y,y}(t) = 1$ for every t and

$$\pi''(\tau) = \begin{pmatrix} \pi''_1(\tau) \\ \pi''_2(\tau) \\ \vdots \\ \pi''_n(\tau) \end{pmatrix}, \quad \tau = 0, 1, 2, \ldots$$

is the column vector, where an arbitrary component $\pi''_i(\tau)$ expresses the probability of the dynamical system to reach the state y from x_i by using not more than τ unites of times when the system start transitions in the sate x at the moment of time $t-\tau$, i.e. $\pi''_i(\tau) = P_{x_i}(y,\, t-\tau \leq T(y) \leq t)$. This means that in the case when y is an absorbing state the probability $P_x(y,\, T_1 \leq T(y) \leq T_2)$ can be found in the following way:

a) find the matrices

$$Q^1 = P(0)P(1)P(2)\cdots P(T_1-1) \text{ and } Q^2 = P(0)P(1)P(2)\cdots P(T_2-1);$$

b) calculate

$$P_x(y,\, T_1 \leq T(y) \leq T_2) =$$

$$= P_x(y,\, 0 \leq T(y) \leq T_2) - P_x(y,\, 0 \leq T(y) \leq T_1 - 1) = q^2_{i_x j_y} - q^2_{i_x j_y}.$$

The results described above allows to develop algorithms for calculation the probabilities $P_x(y,\, 0 \leq T(y) \leq t)$ for an arbitrary non-stationary Markov process. Such algorithms can be obtained if in the general steps of the algorithms we change the matrix P by the matrix $P(t-\tau-1)$ and $\pi'(\tau)$ by $\pi''(\tau)$.

Below we describe these algorithms which can be grounded in an analogues way as the algorithms in Section 2.

Calculation of the state probabilities of the system in the matrix form (non-stationary case)

Preliminary step (Step 0): Fix the vector $\pi''(0) = (\pi''_1(0),\ \pi''_2(0),\ \ldots,\ \pi''_n(0))$, where $\pi''_i(0) = 0$ for $i \neq i_y$ and $\pi''_{i_y}(0) = 1$.

General step (Step $\tau+1$, $\tau \geq 0$): For given τ calculate

$$\pi''(\tau+1) = P(t-\tau-1)\pi''(\tau)$$

and then put

$$\pi''_{i_y}(\tau+1)=1.$$

If $\tau < t-1$ then go to next step; otherwise STOP.

Calculation of the state probabilities of the system with given probability of its remaining in the final state (non-stationary case)

Preliminary step (Step 0): Fix the vector $\pi''(0)=(\pi''_1(0),\ \pi''_2(0),\ \ldots,\ \pi''_n(0))$, where $\pi''_i(0)=0$ for $i\neq i_y$ and $\pi''_{i_y}(0)=1$.

General step (Step $\tau+1$, $\tau\geq 0$): For given τ calculate

$$\pi''(\tau+1)=P(t-\tau-1)\pi''(\tau)$$

and then put

$$\pi''_{i_y}(\tau+1)=q(y).$$

If $\tau < t-1$ then go to next step; otherwise STOP.

Note that the algorithm finds the probabilities $P_x(y,\ 0\leq T(y)\leq t)$ when the value $q(y)$ is given. We treat this value as the probability of the system to remain in the state y; for the case $q(y)=1$ this algorithm coincides with previous one.

4 Conclusion

A new approach for studying finite Marcov processes and determining the limit matrix of probability transitions in Markov chains is proposed. The proposed approach allows us to develop new algorithms for determining the states probability in the considered Markov processes. Polynomial time algorithms for finding the limit matrix of probability transitions of the system in Markov chains are elaborated. These algorithms can be used for determining the average cost per transaction of dynamical system in decision Markov processes (Markov processes with rewards) [3, 5, 9] and and stochastic discrete optimal control problems [2, 6–8].

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Dmitrii Lozovanu
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chişinău, MD−2028
Moldova
E-mail: *lozovanu@math.md*

Stefan Pickl
Institut fur Angewandte Systemwissenschaften
und Wirtschaftsinformatik
Fakultät fur Informatik, Universität der Bundeswehr
München
E-mail: *stefan.pickl@unibw.de*