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Free topological universal algebras and absolute neighborhood retracts

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Abstract. We prove that for a complete quasivariety $\mathcal{K}$ of topological E-algebras of countable discrete signature E and each submetrizable $\mathsf{ANR}(k_\omega)$-space X its free topological E-algebra $F_{\mathcal{K}}(X)$ in the class $\mathcal{K}$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space.

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1 Introduction

In this paper we study the construction of a free topological universal algebra and show that this construction preserves the class of submetrizable $\mathsf{ANR}(k_\omega)$-spaces.

To give a precise formulation of our main result, we need to recall some definitions related to topological universal algebras. For more detailed information, see [5–7].

Definition 1. Let $(E_n)_{n\in\omega}$ be a sequence of pairwise disjoint topological spaces. The topological sum $E = \bigoplus_{n\in\omega} E_n$ is called *a continuous signature*. The signature is called *discrete* (*countable*) if so is the space E.

A *topological universal algebra of signature* E or briefly, a *topological E-algebra* is a topological space X endowed with a family of continuous maps $e_{n,X} : E_n \times X^n \to X$, $n \in \omega$.

A topological E-algebra $(X, \{e_{n,X}\}_{n\in\omega})$ is called *Tychonoff* if the underlying topological space X is Tychonoff.

Homomorphisms between E-algebras are defined as follows.

Definition 2. A function $h : X \to Y$ between two topological E-algebras $(X, \{e_{n,X}\}_{n\in\omega})$ and $(Y, \{e_{n,Y}\}_{n\in\omega})$ is called an *E-homomorphism* if

$$e_{n,Y}(z, h(x_1), \dots, h(x_n)) = h(e_{n,X}(z, x_1, \dots, x_n))$$

for any $n \in \omega$, $z \in E_n$, and $x_1, \dots, x_n \in X$.

Such a function h is called an *algebraic isomorphism* (*topological isomorphism*) if h is bijective and both functions h and h^{-1} are (continuous) E-homomorphisms of the E-algebras.

Next, we define some operations over E-algebras.

Definition 3. For topological E-algebras X_α, $\alpha \in A$, the Tychonoff product $X = \prod_{\alpha\in A} X_\alpha$ is a topological E-algebra endowed with the structure mappings

$$e_{n,X}(z, x_1, \dots, x_n) = \big(e_{n,X_\alpha}(z, \mathrm{pr}_\alpha(x_1), \dots, \mathrm{pr}_\alpha(x_n))\big)_{\alpha\in A}$$

where $n \in \omega$, $z \in E_n$, $x_1, \dots, x_n \in X$, and $\mathrm{pr}_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is the α-coordinate projection.

Definition 4. A subset $A \subset X$ of a topological E-algebra $(X, \{e_n\}_{n\in\omega})$ is called a *subalgebra* if $e_n(E_n \times A^n) \subset A$ for all $n \in \omega$.

Since for any subalgebras $A_i \subset X$, $i \in \mathcal{I}$, of a topological E-algebra X the intersection $A = \bigcap_{i\in\mathcal{I}} A_i$ is a subalgebra of X, for each subset $Z \subset X$ there is a minimal subalgebra $\langle Z\rangle$ of X that contains Z. This is the subalgebra generated by the set Z. The structure of this subalgebra $\langle Z\rangle$ can be described as follows.

Given a subset $L \subset E$ and a subset Z of a topological E-algebra $(X, \{e_n\}_{n\in\omega})$, let

$$\langle Z\rangle_0^L = Z,$$
$$\langle Z\rangle_{n+1}^L = \langle Z\rangle_n^L \cup \bigcup_{k\in\omega} e_{k,X}\big((E_k \cap L) \times (\langle Z\rangle_n^L)^k\big) \text{ for } n \in \omega, \text{ and}$$
$$\langle Z\rangle_\omega^L = \bigcup_{n\in\omega} \langle Z\rangle_n^L.$$

By induction, one can check that for compact subspaces $L \subset E$ and $Z \subset X$ the subset $\langle Z\rangle_n^L$ of X is compact for every $n \in \omega$. Consequently, $\langle Z\rangle_\omega^L$ is a σ-compact subset of X.

Writing the signature E and the space Z as the unions $E = \bigcup_{n\in\omega} L_n$ and $Z = \bigcup_{n\in\omega} Z_n$ of non-decreasing sequences of subsets, we see that

$$\langle Z\rangle = \bigcup_{n\in\omega} \langle Z_n\rangle_n^{L_n}$$

is the subalgebra of X, generated by Z. If the spaces Z_n and L_n, $n \in \omega$, are compact (finite), then each subset $\langle Z_n\rangle_n^{L_n}$, $n \in \omega$, of X is compact (finite) and hence the algebraic hull $\langle Z\rangle$ of Z in X is σ-compact (at most countable).

Definition 5. A class $\mathcal{K}$ of topological E-algebras is called a *complete quasivariety* if

1) for each topological E-algebra $X \in \mathcal{K}$, each E-subalgebra of X belongs to the class $\mathcal{K}$;

2) for any topological E-algebras $X_\alpha \in \mathcal{K}$, $\alpha \in A$, their Tychonoff product $\prod_{\alpha\in A} X_\alpha$ belongs to the class $\mathcal{K}$;

3) a Tychonoff E-algebra belongs to $\mathcal{K}$ if it is algebraically isomorphic to a topological E-algebra $Y \in \mathcal{K}$.

A complete quasivariety $\mathcal{K}$ is *non-trivial* if it contains a topological E-algebra X that contains more than one point.

Finally, we recall the notion of a free topological E-algebra.

Definition 6. Let $\mathcal{K}$ be a complete quasivariety of topological E-algebras. A *free topological E-algebra in $\mathcal{K}$* over a topological space X is a pair $(F_{\mathcal{K}}(X), \eta)$ consisting of a topological E-algebra $F_{\mathcal{K}}(X) \in \mathcal{K}$ and a continuous map $\eta : X \to F_{\mathcal{K}}(X)$ such that for any continuous map $f : X \to Y$ to a topological E-algebra $Y \in \mathcal{K}$ there is a unique continuous E-homomorphism $h : F_{\mathcal{K}}(X) \to Y$ such that $f = h \circ \eta$.

The construction $F_{\mathcal{K}}(X)$ of a free topological E-algebra has been intensively studied by M. M. Choban [6, 7]. In particular, he proved that for each complete quasivariety $\mathcal{K}$ of topological E-algebras and any topological space X a free topological E-algebra $(F_{\mathcal{K}}(X), \eta)$ exists and is unique up to a topological isomorphism. Also he proved the following important result, see [6, 2.4]:

Theorem 1 (Choban)**.** *If $\mathcal{K}$ is a non-trivial complete quasivariety of topological E-algebras, then for each Tychonoff space X the canonical map $\eta : X \to F_{\mathcal{K}}(X)$ is a topological embedding and $F_{\mathcal{K}}(X)$ coincides with the subalgebra $\langle \eta(X) \rangle$ generated by the image $\eta(X)$ of X in $F(X, \mathcal{K})$.*

Since $\eta : X \to F_{\mathcal{K}}(X)$ is a topological embedding, we can identify a Tychonoff space X with its image $\eta(X)$ in $F_{\mathcal{K}}(X)$ and say that the free E-algebra $F_{\mathcal{K}}(X)$ is algebraically generated by X.

In fact, the construction of a free topological E-algebra $F_{\mathcal{K}}(X)$ determines a functor $F_{\mathcal{K}} : \mathbf{Top} \to \mathcal{K}$ from the category **Top** of topological spaces and their continuous maps to the category whose objects are topological E-algebras from the class $\mathcal{K}$ and morphisms are continuous E-homomorphisms.

In [5–7] a lot of attention was paid to the problem of preservation of various topological properties by the functor $F_{\mathcal{K}}$. In particular, it was shown that the functor $F_{\mathcal{K}}$ preserves (submetrizable) k_ω-spaces provided the signature E is a (submetrizable) k_ω-space, see [7, 4.1.2].

A Hausdorff topological space X is called *a k_ω-space* if $X = \varinjlim X_n$ is the *direct limit* of a non-decreasing sequence of compact subsets $(X_n)_{n\in\omega}$ of X in the sense that $X = \bigcup_{n\in\omega} X_n$ and a subset $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n for each $n \in \omega$. Such a sequence $(X_n)_{n\in\omega}$ is called a *k_ω-sequence* for X.

An *s_ω-space* is a direct limit $\varinjlim X_n$ of a k_ω-sequence $(X_n)_{n\in\omega}$ consisting of second countable compact subspaces of X. It is easy to see that a k_ω-space X is an s_ω-space if and only if it is *submetrizable* in the sense that X admits a continuous metric.

Theorem 2 (Choban)**.** *Let $\mathcal{K}$ be a complete quasivariety of topological E-algebras whose signature E is a (submetrizable) k_ω-space. Then for each (submetrizable) k_ω-space X the free topological E-algebra $F_{\mathcal{K}}(X)$ is a (submetrizable) k_ω-space. Moreover, if $E = \varinjlim L_n$ and $X = \varinjlim X_n$ for some k_ω-sequences $(L_n)_{n\in\omega}$ and $(X_n)_{n\in\omega}$, then $(\langle \eta(X_n) \rangle_n^{L_n})_{n\in\omega}$ is a k_ω-sequence for $F_{\mathcal{K}}X$ and thus $F_{\mathcal{K}}X = \varinjlim \langle \eta(X_n) \rangle_n^{L_n}$.*

The principal result of this paper asserts that the functor $F_{\mathcal{K}}$ preserves $\mathsf{ANR}(k_\omega)$-spaces.

Definition 7. A k_ω-space X is called an *absolute neighborhood retract in the class of k_ω-spaces* (briefly, an $\mathsf{ANR}(k_\omega)$) if X is a neighborhood retract in each k_ω-space that contains X as a closed subspace.

In Theorem 10 we shall show that a submetrizable k_ω-space X is an $\mathsf{ANR}(k_\omega)$-space if and only if each map $f : B \to X$ defined on a closed subspace of a (metrizable) compact space A extends to a continuous map $\bar{f} : N(B) \to X$ defined on a neighborhood $N(B)$ of B in A.

A topological space X is called *compactly finite-dimensional* if each compact subset of X is finite-dimensional.

The following theorem is the main result of this paper.

Theorem 3. *If $\mathcal{K}$ is a complete quasivariety of topological E-algebras of countable discrete signature E, then for each submetrizable* (*compactly finite-dimensional*) *$\mathsf{ANR}(k_\omega)$-space X so is its free topological E-algebra $F_{\mathcal{K}}X$ in the quasivariety $\mathcal{K}$.*

2 $\mathsf{ANR}(k_\omega)$-spaces

In this section we collect some information about $\mathsf{ANR}(k_\omega)$-spaces. Such spaces are tightly connected with ANE-spaces.

Following [11] we define a topological space X to be an *absolute neighborhood extensor* for a class $\mathbf{C}$ of topological spaces (briefly, an $\mathsf{ANE}(\mathbf{C})$-*space*) if each map $f : B \to X$ defined on a closed subspace B of a topological space $C \in \mathbf{C}$ has a continuous extension $\bar{f} : N(B) \to X$ defined on some neighborhood $N(B)$ of B in C. If any such f can be extended to the whole space C, then X is called an *absolute extensor* for the class $\mathbf{C}$.

By the Dugundji-Borsuk Theorem [8],[4] each convex subset of a locally convex linear topological space, is an absolute extensor for the class of metrizable spaces. This theorem was generalized by Borges [3] who proved that a convex subset of a locally convex space is an absolute extensor for the class of stratifiable spaces. This class contains all metrizable spaces and all submetrizable k_ω-spaces, and is closed with respect to many countable topological operations, see [3],[10].

An important example of an $\mathsf{ANR}(k_\omega)$-space is the space

$$Q^\infty = \{(x_i)_{i\in\omega} \in \mathbb{R}^\infty : \sup_{i\in\omega} |x_i| < \infty\}$$

of bounded sequences, endowed with the direct limit topology $\varinjlim [-n,n]^\omega$ generated by the k_ω-sequence $([-n,n]^\omega)_{n\in\mathbb{N}}$ consisting of the Hilbert cubes. Being a locally convex linear topological space, Q^∞ is an absolute extensor for the class of stratifiable spaces.

A topological space X is called a Q^∞*-manifold* if X is Lindelöf and each point $x \in X$ has a neighborhood homeomorphic to an open subset of Q^∞. The theory of Q^∞-manifolds was developed by K.Sakai [12],[13] who established the following fundamental results:

Theorem 4 (Characterization). *A topological space X is homeomorphic to (a manifold modeled on) the space Q^∞ if and only if X is a submetrizable k_ω-space such that each embedding $f : B \to X$ of a closed subset B of a compact metrizable space A can be extended to a topological embedding of (an open neighborhood of B in) the space A into X.*

Theorem 5 (Open Embedding). *Each Q^∞-manifold is homeomorphic to an open subset of Q^∞.*

Theorem 6 (Closed Embedding). *Each submetrizable k_ω-space is homeomorphic to a closed subspace of Q^∞.*

Theorem 7 (Classification). *Two Q^∞-manifolds are homeomorphic if and only if they are homotopically equivalent.*

Theorem 8 (Triangulation). *Each Q^∞-manifold X is homeomorphic to $K \times Q^\infty$ for some countable locally finite simplicial complex K.*

Theorem 9 (ANR-Theorem). *For each submetrizable $\mathsf{ANR}(k_\omega)$-space X the product $X \times Q^\infty$ is a Q^∞-manifold.*

We shall use these theorems in the proof of the following (probably known as a folklore) characterization of submetrizable $\mathsf{ANR}(k_\omega)$-spaces.

Theorem 10. *For a submetrizable k_ω-space X the following conditions are equivalent:*

1) *X is an $\mathsf{ANR}(k_\omega)$-space;*
2) *X is an ANE for the class of k_ω-spaces;*
3) *X is an ANE for the class of compact metrizable spaces;*
4) *X is an ANE for the class of stratifiable spaces;*
5) *X is a retract of a Q^∞-manifold.*

The equivalent conditions (1)–(5) hold if $X = \varinjlim X_n$ is the direct limit of a k_ω-sequence consisting of compact ANR's.

Proof. (1) ⇒ (5) Assume that X is an $\mathsf{ANR}(k_\omega)$-space. By the Closed Embedding Theorem 6, we can identify the submetrizable k_ω-space X with a closed subspace of Q^∞. Being an $\mathsf{ANR}(k_\omega)$, X is a retract of an open neighborhood $N(X) \subset Q^\infty$. Since $N(X)$ is a Q^∞-manifold, X is a retract of a Q^∞-manifold.

(5) ⇒ (4) Assume that X is a retract of a Q^∞-manifold M. By the Open Embedding Theorem 5, M can be identified with an open subspace of Q^∞. By the

Borges' Theorem [3], the locally convex space Q^∞ is an absolute extensor for the class of stratifiable spaces. Then the open subspace M of Q^∞ is an ANE for this class and so is its retract X.

The implication (4) $\Rightarrow$ (3) is trivial since each metrizable space is stratifiable.

(3) $\Rightarrow$ (2) Assume that X is an ANE for the class of compact metrizable spaces. First we prove that X is an ANE for the class of compact Hausdorff spaces. Let $f : B \to X$ be a continuous map defined on a closed subspace B of a compact Hausdorff space A. Embed the compact space A into a Tychonoff cube I^κ. The image $f(B)$, being a compact subspace of the submetrizable space X, is metrizable. By [9, 2.7.12], the function f depends on countably many coordinates, which means that there is a countable subset $C \subset \kappa$ such that $f = f_C \circ \mathrm{pr}_C$ where $\mathrm{pr}_C : I^\kappa \to I^C$ is the projection onto the face I^C of the cube I^κ and $f_C : \mathrm{pr}_C(B) \to f(B) \subset X$ is a suitable continuous map. Since X is an ANE for compact metrizable spaces, the map f_C has a continuous extension $\tilde{f}_C : U \to X$ defined on an open neighborhood U of $\mathrm{pr}_C(B)$ in the cube I^C. It follows that $V = \mathrm{pr}_C^{-1}(U) \cap A$ is an open neighborhood of B in A and $\tilde{f} = \tilde{f}_C \circ \mathrm{pr}_C|V : V \to X$ is a continuous extension of the map f, witnessing that X is an ANE for the class of compact Hausdorff spaces.

Next, we show that X is an ANE for the class of k_ω-spaces. Let $f : B \to X$ be a continuous map defined on a closed subset B of a k_ω-space A. Then $A = \varinjlim A_n$ for some k_ω-sequence $(A_n)_{n\in\omega}$ of compact subsets of A. Let $A_{-1} = \emptyset$. By induction, for each $n \in \omega$ we can construct a continuous map $f_n : N_n(A_n \cap B) \to X$ defined on a closed neighborhood $N(B \cap A_n)$ of $B \cap A_n$ in A_n and such that

- $N_n(B \cap A_n) \supset N_{n-1}(B \cap A_{n-1})$,
- $f_n|B \cap A_n = f|B \cap A_n$ and
- $f_n|N_{n-1}(B \cap A_n) = f_{n-1}$.

The inductive step can be done because X is an ANE for the class of compact Hausdorff spaces. After completing the inductive construction, consider the set $N(B) = \bigcup_{n\in\omega} N_n(B \cap A_n)$ and the map $\tilde{f} = \bigcup_{n\in\omega} f_n : N(B) \to X$, which is a desired continuous extension of f onto the open neighborhood $N(B)$ of B in A.

The implication (2) $\Rightarrow$ (1) trivially follows from the definitions of $\mathsf{ANR}(k_\omega)$ and $\mathsf{ANE}(k_\omega)$-spaces.

Now assume that $X = \varinjlim X_n$ is the direct limit of a k_ω-sequence $(X_n)_{n\in\omega}$ consisting of compact ANR's. We claim that X is an ANE for the class of compact metrizable spaces. Let $f : B \to X$ be a continuous map defined on a closed subspace B of a compact metrizable space A. Since X carries the direct limit topology $\varinjlim X_n$, the compact subset $f(B)$ lies in some set X_n, $n \in \omega$. Since X_n is an ANR, the map $f : B \to X_n$ has a continuous extension $\tilde{f} : N(B) \to X_n \subset X$ defined on a neighborhood $N(B)$ of B in A. □

3 Some subfunctors of the functor $F_{\mathcal{K}}$

In the proof of Theorem 3 we shall apply a deep Basmanov's result on the preservation of compact ANR's by monomorphic functors of finite degree in the category **Comp** of compact Hausdorff spaces and their continuous maps. Let **C** be a full subcategory of the category **Top**, containing all finite discrete spaces.

We say that a functor $F : \mathbf{C} \to \mathbf{Top}$

- is *monomorphic* if F preserves monomorphisms (which coincide with injective continuous maps in the category **Top** and its full subcategory **C**);

- has *finite supports* (*degree* $\deg F \le n$) if for each object X of the category **C** and each element $a \in FX$ there is a map $f : A \to X$ of a finite discrete space A (of cardinality $|A| \le n$) such that $a \in Ff(FA)$;

The smallest number $n \in \omega$ such that $\deg F \le n$ is called the *degree* of F and is denoted by $\deg F$. If no such number $n \in \omega$ exists, then we put $\deg F = \infty$.

The following improvement of the classical Basmanov's theorem [2] was recently proved in [1].

Theorem 11. *Let $F : \mathbf{Comp} \to \mathbf{Comp}$ be a monomorphic functor of finite degree $n = \deg F$ such that the space Fn is finite. Then the functor F preserves the class of compact finite-dimensional ANR-spaces.*

We shall apply this theorem to the subfunctors $\langle\cdot\rangle_n^L$ of the functor $F_{\mathcal{K}}$. We recall that $\mathcal{K}$ is a non-trivial complete quasivariety of topological E-algebras of countable discrete signature E. By Theorem 2, $F_{\mathcal{K}}$ can be thought as a functor $F_{\mathcal{K}} : \mathbf{K}_\omega \to \mathbf{K}_\omega$ in the category $\mathbf{K}_\omega$ of k_ω-spaces and their continuous maps. By Theorem 2.4 of [6], for each Tychonoff space X the free topological E-algebra $F_{\mathcal{K}}(X)$ is algebraically free in the sense that any bijective map $i : X_d \to X$ from a discrete topological space X_d induces an algebraic isomorphism $F_{\mathcal{K}} i : F_{\mathcal{K}} X_d \to F_{\mathcal{K}} X$. This fact implies:

Lemma 1. *The functor $F_{\mathcal{K}} : \mathbf{Tych} \to \mathbf{Top}$ is monomorphic.*

Proof. Let $f : X \to Y$ be an injective continuous map between Tychonoff spaces and $f_d : X_d \to Y_d$ be the same map between these spaces endowed with the discrete topologies. Let $i_X : X_d \to X$ and $i_Y : Y_d \to Y$ be the identity maps. Let $r : Y_d \to X_d$ be any (automatically continuous) map such that $r \circ f_d = \mathrm{id}_{X_d}$. Thus we obtain the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X_d & \underset{r}{\overset{f_d}{\rightleftarrows}} & Y \end{array}$$

Applying the functor $F_{\mathcal{K}}$ to this diagram we get the diagram

$$\begin{array}{ccc} F_{\mathcal{K}}X & \xrightarrow{F_{\mathcal{K}}f} & F_{\mathcal{K}}Y \\ {\scriptstyle F_{\mathcal{K}}i_X}\uparrow & & \uparrow{\scriptstyle F_{\mathcal{K}}i_Y} \\ F_{\mathcal{K}}X_d & \underset{F_{\mathcal{K}}r}{\overset{F_{\mathcal{K}}f_d}{\rightleftarrows}} & F_{\mathcal{K}}Y \end{array}$$

The "vertical" maps $F_{\mathcal{K}}i_X : F_{\mathcal{K}}X_d \to F_{\mathcal{K}}X$ and $F_{\mathcal{K}}i_Y : F_{\mathcal{K}}Y_d \to F_{\mathcal{K}}Y$ in this diagram are bijective because the algebras $F_{\mathcal{K}}X$ and $F_{\mathcal{K}}Y$ are algebraically free. Taking into account that $F_{\mathcal{K}}r \circ F_{\mathcal{K}}f_d = F_{\mathcal{K}}(r \circ f_d) = F_{\mathcal{K}}\mathrm{id}_{X_d} = \mathrm{id}_{F_{\mathcal{K}}X_d}$, we conclude that the map $F_{\mathcal{K}}f_d$ is injective and so is the map $F_{\mathcal{K}}f : F_{\mathcal{K}}X \to F_{\mathcal{K}}Y$ because of the bijectivity of the maps $F_{\mathcal{K}}i_X$ and $F_{\mathcal{K}}i_Y$. □

Now for every compact subset $L \subset E$ and every $n \in \omega$ consider the functor $\langle\cdot\rangle_n^L$: **Comp** → **Comp** which assigns to each compact Hausdorff space X the subspace $\langle X\rangle_n^L$ of $F_{\mathcal{K}}X$. The functor $\langle\cdot\rangle_n^L$ assigns to each continuous map $f : X \to Y$ between compact Hausdorff spaces the restriction $\langle f\rangle_n^L = F_{\mathcal{K}}f|\langle X\rangle_n^L$ of the homomorphism $F_{\mathcal{K}}f : F_{\mathcal{K}}X \to F_{\mathcal{K}}Y$.

Lemma 2. *For every $n \in \mathbb{N}$, $\langle\cdot\rangle_n^L$:* **Comp** → **Comp** *is a well-defined monomorphic functor of finite degree in the category* **Comp**.

Proof. First we check that for each continuous map $f : X \to Y$ between compact Hausdorff spaces, the morphism $\langle f\rangle_n^L = F_{\mathcal{K}}f|\langle X\rangle_n^L$ is well-defined, which means that $F_{\mathcal{K}}f(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L$. This will be done by induction on $n \in \omega$.

For $n = 0$ the inclusion $F_{\mathcal{K}}(\langle X\rangle_0^L) = F_{\mathcal{K}}(X) = f(X) \subset Y = \langle Y\rangle_0^L$ follows from the fact that the homomorphism $F_{\mathcal{K}}$ extends the map f (here we identify X and Y with the subspaces $\eta(X)$ and $\eta(Y)$ in $F_{\mathcal{K}}(X)$ and $F_{\mathcal{K}}(Y)$, respectively).

Assume that the inclusion $F_{\mathcal{K}}f(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L$ has been proved for some $n \in \omega$. By definition,

$$\langle X\rangle_{n+1}^L = \langle X\rangle_n^L \cup \bigcup_{k\in\omega} e_{k,X}((E_k \cap L) \times (\langle X\rangle_n^L)^k).$$

Fix any element $x \in \langle X\rangle_{n+1}^L$. If $x \in \langle X\rangle_n^L$, then

$$F_{\mathcal{K}}(x) \in F_{\mathcal{K}}(\langle X\rangle_n^L) \subset \langle Y\rangle_n^L \subset \langle Y\rangle_{n+1}^L$$

by the inductive assumption.

If $x \in \langle X\rangle_{n+1}^L \setminus \langle X\rangle_n^L$, then $x = e_{k,X}(z, x_1, \dots, x_k)$ for some $k \in \omega$, $z \in E_k \cap L$, and points $x_1, \dots, x_k \in \langle X\rangle_n^L$. Since $F_{\mathcal{K}}f$ is an E-homomorphism, we get

$$\begin{aligned} F_{\mathcal{K}}f(x) = F_{\mathcal{K}}f(e_{k,X}(z, x_1, \dots, x_k)) &= e_{k,Y}(z, F_{\mathcal{K}}f(x_1), \dots, F_{\mathcal{K}}f(x_k)) \in \\ &\in e_{k,Y}((E_k \cap L) \times (\langle Y\rangle_n^L)^k) \subset \langle Y\rangle_{n+1}^L. \end{aligned}$$

Thus for every $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ is well-defined. It is monomorphic as a subfunctor of the monomorphic functor $F_{\mathcal{K}}$.

Next, we show that the functor $\langle\cdot\rangle$ has finite degree. This will be done by induction on $n \in \omega$. Since $\langle X\rangle_0^L = X$, $\deg\langle\cdot\rangle_0^L = 1$.

Assume that for some $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ has finite degree d. Since L is a compact subset of E, there is $m \in \omega$ such that $L \cap E_k = \emptyset$ for all $k \geq m$. We claim that $\deg\langle\cdot\rangle_{n+1}^L \leq m \cdot d$. Take any element $x \in \langle X\rangle_{n+1}^L$. If $x \in \langle X\rangle_n^L$, then by the inductive assumption there is a subset $A \subset X$ of cardinality $|A| \leq d$ such that $x \in \langle A\rangle_n^L$ and we are done. If $x \in \langle X\rangle_{n+1}^L \setminus \langle X\rangle_n^L$, then $x = e_{k,X}(z, x_1, \dots, x_k)$ for some $k \in \omega$, $z \in E_k \cap L$, and points $x_1, \dots, x_k \in \langle X\rangle_n^L$. Since $L \cap E_k \ni z$ is not empty, $k \leq m$. By the inductive assumption, for every $i \leq k$ there is a finite subset $A_i \subset X$ of cardinality $|A_i| \leq d$ such that $x_i \in \langle A_i\rangle_n^L$. Then the union $A = \bigcup_{i=1}^k A_i$ has cardinality $|A| \leq k \cdot d \leq m \cdot d$ and

$$x = e_{k,X}(z, x_1, \dots, x_k) \in e_{k,X}((L \cap E_k) \times (\langle A\rangle_n^L)^k) \subset \langle A\rangle_{n+1}^L$$

witnessing that the functor $\langle\cdot\rangle_{n+1}^L$ has finite degree $\deg\langle\cdot\rangle_{n+1}^L \leq m \cdot d$. □

Lemma 3. *If $L \subset E$ is finite, then for each $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ preserves finite spaces.*

Proof. Let X be a finite space. By induction on $n \in \omega$ we shall show that the space $\langle X\rangle_n^L$ is finite. This is clear for $n = 0$. Assume that for some $n \in \omega$ the space $\langle X\rangle_n^L$ is finite. Since $L \subset E$ is finite there is $m \in \omega$ such that $L \cap E_n = \emptyset$ for all $k > m$. Then

$$\langle X\rangle_{n+1}^L = \langle X\rangle_n^L \cup \bigcup_{k \leq m} e_{k,X}((E_k \cap L) \times (\langle X\rangle_n^L)^k)$$

is finite as the finite union of finite sets. □

Combining Lemmas 2, 3 with Theorem 11, we get

Corollary 1. *For any finite subset $L \subset E$ and every $n \in \omega$ the functor $\langle\cdot\rangle_n^L$ preserves (finite-dimensional) compact ANR's.*

4 Proof of Theorem 3

Without loss of generality, the quasivariety $\mathcal{K}$ is non-trivial (otherwise, $F_{\mathcal{K}}(X)$ is a singleton and hence is an $\mathsf{ANR}(k_\omega)$-space for each non-empty space X).

Let X be a submetrizable $\mathsf{ANR}(k_\omega)$-space. By the ANR-Theorem 9, the product $X \times Q^\infty$ is a Q^∞-manifold. By the Triangulation Theorem 8, $X \times Q^\infty$ is homeomorphic to $T \times Q^\infty$ for a countable locally finite simplicial complex T. This implies that $X \times Q^\infty$ can be written as the direct limit $X \times Q^\infty = \varinjlim X_n$ of a k_ω-sequence $(X_n)_{n\in\omega}$ of compact ANR's.

Write the countable discrete space E as the direct limit $E = \varinjlim L_n$ of a k_ω-sequence $(L_n)_{n\in\omega}$ of finite subsets of E. By Choban's Theorem 2, the space $F_{\mathcal{K}}(X \times Q^\infty)$ is the direct limit $\varinjlim \langle X_n\rangle_n^{L_n}$ of the k_ω-sequence $\langle X_n\rangle_n^{L_n}$. By Corollary 1, each space $\langle X_n\rangle_n^{L_n}$, $n \in \omega$, is a compact metrizable ANR. Consequently, $F_{\mathcal{K}}(X \times Q^\infty) =$

$\varinjlim \langle X_n \rangle_n^{L_n}$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space by Theorem 10. Since X is a retract of $X \times Q^\infty$, the space $F_\mathcal{K}X$ is a retract of $F_\mathcal{K}(X \times Q^\infty)$ and hence $F_\mathcal{K}X$ is a submetrizable $\mathsf{ANR}(k_\omega)$-space.

Now assume that X is a compactly finite-dimensional s_ω-space. Then $X = \varinjlim X_n$ is the direct limit of finite-dimensional compact metrizable spaces. By the Choban's Theorem 2, the space $F_\mathcal{K}(X \times Q^\infty)$ is the direct limit $\varinjlim \langle X_n \rangle_n^{L_n}$ of the k_ω-sequence $\langle X_n \rangle_n^{L_n}$. Corollary 1 implies that each compact space $\langle X_n \rangle_n^{L_n}$ is metrizable and finite-dimensional. Then the space $F_\mathcal{K}X = \varinjlim \langle X_n \rangle_n^{L_n}$ is compactly finite-dimensional, being the direct limit of finite-dimensional compact spaces.

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