BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 3–27
ISSN 1024–7696

# Serial rings and their generalizations

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**Abstract.** We give a survey of results on the theory of semiprime semidistributive rings, in particular, serial rings. Besides this we prove that a serial ring is Artinian if and only if some power of its Jacobson radical is zero. Also we prove that a serial ring is Noetherian if and only if the intersection of all powers of Jacobson radical is zero. These two theorems hold for semiperfect semidistributive rings.

**Mathematics subject classification:** 16L30.
**Keywords and phrases:** Serial ring, SPSD-ring, quiver of ring.

*Dedicated to Professor Alexei I. Kashu*
*on the occasion of his seventieth birthday*

## 1 Introduction

Artinian uniserial, or primary decomposable serial rings were first introduced and studied by G. Köthe in the paper [9], where he proved that any module over such a ring is a direct sum of cyclic modules (he called such rings "Einreihige Ringen"). This result was generalized for Artinian serial rings by T. Nakayama, who called these rings "generalized uniserial rings" (see [16] and [17]). In these papers T. Nakayama proved that any module over such a ring is a direct sum of uniserial submodules each of which is a homomorphic image of an ideal generated by a primitive idempotent. T. Nakayama also showed that, conversely, these are the only rings whose indecomposable finitely generated modules (both left and right) are homomorphic images of ideals generated by primitive idempotents.

Artinian principal ideal rings were studied in papers of G. Köthe and K. Asano (see [1] and [2]), where it was proved that any Artinian principal right ideal ring is right uniserial. In fact, K. Asano proved that an Artinian ring is uniserial if and only if each ideal is a principal right ideal and a principal left ideal. The classical proof of this theorem is given in the book [7]. For such rings K. Asano also proved an analogue of the Wedderburn-Artin theorem, namely, he proved that any Artinian uniserial ring can be decomposed into a direct sum of full matrix rings of the form $M_n(A)$, where $A$ is a local Artinian ring with a cyclic radical. A one-sided characterization of Artinian principal ideal rings and its connection with primary decomposable serial rings is given in theorem 2.1 of the paper [4]

L. A. Skornyakov studied serial rings, which he called "semi-chain rings", in his paper [18]. There he proved that A is a right and left Artinian serial ring if and only if every left A-module is a direct sum of uniserial modules. His result generalizes a theorem proved by K. R. Fuller (see [5]), to the effect that if each left module over a ring A is a direct sum of uniserial modules, then A is a serial left Artinian ring.

The first serial non-Artinian rings were studied and described by R. B. Warfield and V. V. Kirichenko. In particular, they gave a full description of the structure of serial Noetherian rings. We follow the papers [12] and [10], where the technique of quivers was used systematically.

It is well known that many important classes of rings are naturally characterized by the properties of modules over them. As examples, we mention semisimple Artinian rings, uniserial rings, semiprime hereditary semiperfect rings and semidistributive rings.

There is the following chain of strict inclusions:

semisimple Artinian rings $\subset$ generalized uniserial rings $\subset$ serial rings $\subset$ semidistributive rings.

In this chain the first three classes of rings are semiperfect. The example of the ring of integers $\mathbb{Z}$ shows that a distributive ring is not necessarily semiperfect.

The reduction theorem for SPSD-rings and decomposition theorem for semiprime right Noetherian SPSD-rings were proved in the paper [14].

Quivers and prime quivers of SPSD-rings were studied in [13].

A semilocal ring $A$ is called **semiperfect** if idempotents of the ring $A$ can be lifted modulo $R$.

Semiperfect rings were introduced by H. Bass in 1960.

To understand the definition of a semilocal ring we need some additional definitions and propositions.

A nonzero ring $A$ is called **local** if it has the unique maximal right ideal.

The intersection of all maximal right ideals of a ring $A$ is called the **Jacobson radical** of $A$. The Jacobson radical is denoted $R = rad\ A$.

The following theorem contains the list of properties which are equivalent for any local ring.

**Theorem 1.1.** *The following properties of a ring $A$ with the Jacobson radical $R$ are equivalent:*

1. *$A$ is local;*
2. *$R$ is the unique maximal right ideal in $A$;*
3. *all non-invertible elements of $A$ form a proper ideal;*
4. *$R$ is the set of all non-invertible elements of $A$;*
5. *the quotient ring $A/R$ is a division ring.*

**Proposition 1.2.** *Let $e^2 = e \in A$. Then $rad(eAe) = eRe$, where $R$ is the radical of $A$.*

Recall that a module $M$ is called distributive if for any submodules $K, L, N$

$$K \cap (L + N) = K \cap L + K \cap N.$$

Clearly, a submodule and a quotient module of a distributive module are distributive. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring $A$ is called **right (left) semidistributive** if the right (left) regular module $A_A$ $({}_AA)$ is semidistributive. A right and left semidistributive ring is called **semidistributive**.

Obviously, every uniserial module is a distributive module and every serial module is a semidistributive module.

**Example 1.1.** *Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a finite poset with ordering relation $\leqslant$ and let $D$ be a division ring. Denote by $A(S, D)$ the following subring of $M_n(D)$:*

$$A(S, D) = \left\{ \sum_{\alpha_i \leq \alpha_j} d_{ij} e_{ij} \,|\, d_{ij} \in D \right\}.$$

*It is not difficult to check that $A(S, D)$ is a semidistributive Artinian ring.*

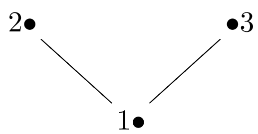
*In particular, the hereditary semidistributive ring*

$$A_3 = \left\{ \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \,|\, d_{ij} \in D \right\}$$

*is of the form:*

$$A_3 = A(P_3, D),$$

*where $P_3$ is the poset with the diagram*




*It is also clear that $A_3$ is a semidistributive ring which is left serial, but not right serial.*

**Proposition 1.3.** *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if all submodules of $M$ with two generators are distributive modules.*

*Proof.* Suppose that all two-generated submodules of $M$ are distributive modules. Let $K, L, N$ be submodules of $M$ and $k = l + n \in K \cap (L + M)$; $l \in L$, $n \in N$. Obviously, $kA \subset lA + nA$ and $KA = kA \cap (lA + nA) = KA \cap lA + kA \cap nA$. Therefore, $k \in K \cap L + K \cap N$, i.e. $K \cap (L + N) \subseteq K \cap L + K \cap N$. The inclusion $K \cap L + K \cap N \subseteq K \cap (L + N)$ is always valid. □

**Lemma 1.4.** *Let $M$ be a distributive module over a ring $A$. Then for any $m, n \in M$ there exist $a, b \in A$ such that $1 = a + b$ and $maA + nbA \subset mA \cap nA$.*

*Proof.* Write $t = m + n$ and $H = mA \cap nA$. Obviously, $tA \subseteq mA + nA$ and $Ta \cap (mA + nA) = tA = (tA + mA) \cap (tA + nA)$. So there exist $b, d \in A$ such that $tb \in mA$, $td \in nA$ and $t = tb+td$. Then $nb = tb-mb \in H$ and $md = td-nd \in H$. Let $a = 1-b$ and $g = 1-b-d$. He have $tg = t-tb-td = 0$ and $ng = tg-mg = -mg \in H$. So $ma = md + mg \in H$ and $maA + nbA \subseteq mA \cap nA$. □

**Lemma 1.5.** *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if for any $m, n \in M$ there exist four elements $a, b, c, d$ of $A$ such that $1 = a + b$ and $ma = nc$, $nb = md$.*

*Proof.* Necessity follows from Lemma 1.4. Conversely, let $k \in K \cap (L + N)$, where $K, L, N$ are submodules of $M$. Then $k = m + n$, where $m \in L$ and $n \in N$. By assumption there exist $a, b \in A$ such that $1 = a+b$ and $ma \in mA \cap nA$, $nb \in mA \cap nA$. Consequently, $ka = ma + na \in kA \cap nA$ and $kb = mb + nb \in kA \cap mA$. Therefore, $k = ka + kb \in (kA \cap nA) + (kA \cap mA) \subset K \cap L + K \cap N$, i.e., $K \cap (L + N) = K \cap L + K \cap N$. □

Let $M$ be an $A$-module. Given two elements $m, n \in M$ we set

$$(m : n) = \{a \in A | na \in mA\}.$$

**Theorem 1.6** (**W. Stephenson**). *A module $M$ is distributive if and only if*

$$(m : n) + (n : m) = A$$

*for all $m, n \in M$.*

*Proof.* This immediately follows from Lemma 1.5. □

A module $M$ has the **square-free socle** if its socle contains at most one copy of each simple module.

**Theorem 1.7** (**V. Camillo**). *Let $M$ be an $A$-module. Then $M$ is a distributive module if and only if $M/N$ has the square-free socle for every submodule $N$.*

*Proof.* Necessity. Every quotient and submodule of a distributive module are distributive, so that if $M/N$ contains a submodule of the form $U \oplus U$, then $M$ is not a distributive module. Simply because $U \oplus U$ is not a distributive module. Indeed, for the diagonal $D(U \oplus U) = \{(u, u) | n \in U\}$ of $U \oplus U$ we have $D(U) \cap (U \oplus U) = D(U)$ and $D(U) \cap (U \oplus 0) = 0$ and $D(U) \cap (0 \oplus U) = 0$.

Conversely. Let $m, n \in M$. We show that $(m : n) + (n : m) = A$. Let $K$ be a maximal right ideal of $A$ and $U = A/K$. Consider the quotient module $mA + nA/mK + nK$. The socle of $mA + nA/mK + nK$ doesn't contain $U \oplus U$ if one of the following conditions holds:

(1) $m \in nA + mK + nK = nA + mK$;

(2) $m \in nA + mK + nK = nA + mK$;

In the case (1) we have $m = na+nK$ or $m(1 \oplus k) = na$. So $(1 \oplus k) \in (n : m)$. Since $(1 \oplus k) \notin K$, we have $(n : m) \not\subseteq K$. In the case (2) analogously $(m : n) \not\subseteq K$. □

**Theorem 1.8.** *A semiprimary right semidistributive ring $A$ is right Artinian.*

*Proof.* It is sufficient to show that each indecomposable projective $A$-module $P = eA$ is Artinian ($e$ is a nonzero idempotent of $A$). Let $m$ be the minimal natural number with $PR^m = 0$. Since the module $P$ is distributive, by Theorem 1.7, the quotient module $PR^i/PR^{i+1}$ decomposes into a finite direct sum of simple modules ($i = 1, \dots, m-1$). Thus, the module $P$ possesses a composition series and the module $P$ is Artinian. □

We write **SPSDR-ring** (**SPSDL-ring**) for a semiperfect right (left) semidistributive ring and **SPSD-ring** for a semiperfect semidistributive ring.

**Theorem 1.9 (A. Tuganbaev).** *A semiperfect ring $A$ is right (left) semidistributive if and only if for any local idempotents $e$ and $f$ of the ring $A$ the set $eAf$ is a uniserial right $fAf$-module (uniserial left $eAe$-module) ([6], Theorem 14.2.1).*

## 2 Q-lemma and Annihilation lemma

Recall the definition of the Gabriel quiver for a finite dimensional algebra $A$ over a field $k$. We can restrict ourselves to **basic** split algebras. (An algebra A is called basic if $A/R$ is isomorphic to a product of division algebras, where R is the Jacobson radical of A. An algebra A over a field $k$ is called **split** if $A/R \simeq M_{n_1}(k) \times M_{n_2}(k) \times \ldots \times M_{n_s}(k)$.) All algebras over algebraically closed fields are split.

Let $P_1, \dots, P_s$ be all pairwise nonisomorphic principal right A-modules. Write $R_i = P_iR$ ($i = 1, \dots, s$) and $W_i = R_i/R_iR$. Since $W_i$ is a semisimple module, $W_i = \bigoplus\limits_{j=1}^{s} U_j^{t_{ij}}$, where $U_j = P_j/R_j$ are simple modules. It is equivalent to the isomorphism $P(R_i) \simeq \bigoplus\limits_{j=1}^{s} P_j^{t_{ij}}$. To each module $P_i$ assign a point $i$ in the plane and join the point $i$ with the point $j$ by $t_{ij}$ arrows. The so constructed graph is called the quiver of $A$ in the sense of P.Gabriel and denoted by $Q(A)$.

A semiperfect ring $A$ is called **reduced** if its quotient ring by the Jacobson radical $R$ is a direct sum of division rings.

Let $A$ be a semiperfect ring such that $A/R^2$ is a right Artinian ring. The quiver of the ring $A/R^2$ is called the **quiver** of the ring $A$ and is denoted by $Q(A)$.

**Theorem 2.1.** *Let $A$ be an arbitrary ring with an idempotent $e^2 = e \in A$. Set $f = 1 - e$, $eAf = X$, $fAe = Y$, and let*

$$A = \begin{pmatrix} eAe & X \\ Y & fAf \end{pmatrix}$$

*be the corresponding two-sided Peirce decomposition of the ring $A$. Then the ring $A$ is right Noetherian (Artinian) if and only if the rings $eAe$ and $fAf$ are right Noetherian (Artinian), $X$ is a finitely generated $fAf$-module and $Y$ is a finitely generated $eAe$-module.*

For further reasonings we will need the following proposition.

**Proposition 2.2.** *Let $A$ be a ring. For an $A$-module $P$ the following statements are equivalent:*

*1) $P$ is projective;*

*2) every short exact sequence $0 \to N \to M \to P \to 0$ splits;*

*3) $P$ is a direct summand of a free $A$-module $F$.*

Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring $A$ into a direct sum of principal right $A$-modules and let $1 = f_1 + \ldots + f_s$ be the corresponding decomposition of the identity of $A$ into a sum of pairwise orthogonal idempotents, i.e., $f_i A = P_i^{n_i}$ . Then ${}_A A = Af_1 \oplus \ldots \oplus Af_s = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$ is the decomposition of the semiperfect ring $A$ into a direct sum of principal left $A$-modules, i.e. $Af_i = Q_i^{n_i}$ , where $Q_i$ is an indecomposable projective left $A$-module ($i = 1, \ldots, s$). Now consider the two-sided Peirce decomposition of the ring $A$

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

Consider also the two-sided Peirce decomposition of the Jacobson radical $R$ of $A$ : $R = \bigoplus_{i,j} f_i R f_j$. Since $R$ is a two-sided ideal, $f_i R f_j \subset R$ for all $i, j$. By Proposition 1.2 we have $R_{ii} = f_i R f_i = rad(f_i A f_i)$ for $i = 1, \ldots, n$. We shall show that $f_i R f_j = f_i A f_j$ for $i \neq j$. Indeed, multiplying on the left elements from $f_j A$ by an element $f_i a f_j$ we obtain a homomorphism $\varphi_{ji}$ of the module $f_j A$ to $f_i A$. If $Im(\varphi_{ji}) = f_i A$, then $\varphi_{ji}$ is an epimorphism. Since $f_i A = P_i^{n_i}$, $f_j A = P_j^{n_j}$ are projective modules, by Proposition 2.2, and $P_i^{n_i}$ is isomorphic to a direct summand of the module $P_j^{n_j}$. But this is impossible, since the indecomposable modules $P_i$ and $P_j$ are non-isomorphic. Therefore $Im(\varphi_{ji}) \subset f_i A$. We can write the homomorphism $\varphi_{ji}$ in the form of a matrix $\varphi_{ji} = (\varphi_{ji}^{rs})$, where $\varphi_{ji}^{rs} : P_j \to P_i$ are homomorphisms of indecomposable non-isomorphic projective modules $P_j$ and $P_i$ for $r = 1, \ldots, n_i$, $s = 1, \ldots, n_j$. Since $Im(\varphi_{ji}^{rs}) \neq P_i$, we have $Im(\varphi_{ji}^{rs}) \subset P_i R$. Therefore $Im(\varphi_{ji}^{rs}) \subseteq f_i AR = f_i R$, i.e., $f_i A f_j \subseteq f_i R$. Hence $A_{ij} = f_i A f_j = f_i R f_j$ for $i \neq j$. Thus, we obtain the following result.

**Proposition 2.3.** *Let $A = P^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring $A$ into a direct sum of principal right $A$-modules and let $1 = f_1 + \ldots + f_s$ be a corresponding decomposition of the identity of $A$ into a sum of pairwise orthogonal idempotents, i.e., $f_i A = P_i^{n_i}$. Then the Jacobson radical of the ring $A$ has a two-sided Peirce decomposition of the following form:*

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{pmatrix}$$

*where* $R_{ii} = rad(f_iAf_i)$, $A_{ij} = f_iAf_j$ *for* $i, j = 1, \ldots, n$.

**Lemma 2.4. (Annihilation lemma)** *Let* $1 = f_1 + \ldots + f_s$ *be a canonical decomposition of* $1 \in A$. *For every simple right* $A$*-module* $U_i$ *and for each* $f_j$ *we have* $U_if_j = \delta_{ij}U_i$, $i, j = 1, \ldots, s$. *Similarly, for every simple left* $A$*-module* $V_i$ *and for each* $f_j$, $f_jV_i = \delta_{ij}V_i$, $i, j = 1, \ldots, s$.

*Proof.* We shall give the proof for the case of right modules. From the previous proposition we obtain that $f_iRf_j = f_iAf_j$ for $i \neq j$. Hence $P_i^{n_i}f_j \subset f_iR$. But $f_iA/f_iR \simeq U_i^{n_i}$. Therefore $U_i^{n_i}f_j = 0$ and so $U_if_j = 0$ for $i \neq j$.

We are going to show that $U_if_i = U_i$. Let $u \in U_i$. Then $u \cdot 1 = u(f_1 + \ldots + f_s) = uf_i$ since $uf_j = 0$ for $i \neq j$. The lemma is proved. □

Let A be a reduced semiperfect ring, and let $1 = e_1 + \ldots + e_s$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents.

Set $U_i = e_iA/e_iR$ and $V_i = Ae_i/Re_i$.

**Lemma 2.5. (Q-Lemma)** *The simple module* $U_k$ *(resp.* $V_k$*) appears in the direct sum decomposition of the module* $e_iR/e_iR^2$ *(resp.* $Re_i/R^2e_i$*) if and only if* $e_iR^2e_k$ *(resp.* $e_kR^2e_i$*) is strictly contained in* $e_iRe_k$ *(resp.* $e_kRe_i$*).*

*Proof.* If $U_k$ is a direct summand of the module $W_i = e_iR/e_iR^2$, then by Lemma 2.4, $Wie_k \neq 0$. Therefore $e_iRe_k$ does not equal $e_iR^2e_k$ and the inclusion $e_iRe_k \supset e_iR^2e_k$ is strict.

Conversely, suppose that $e_iR^2e_k$ is strictly contained in $e_iRe_k$. Consider a submodule $X_k$ contained in $e_iR$,

$$X_k = e_iRe_i \oplus \ldots \oplus e_iRe_{k-1} \oplus e_iR^2e_k \oplus e_iRe_{k+1} \oplus \ldots \oplus e_iRe_s$$

(here the direct sum sign denotes a direct sum of Abelian groups).

From the inclusions $e_iR \supset X_k \supset e_iR^2$ it follows that $e_iR/X_k$ is a semisimple module. We have the equalities $e_iR/X_k = e_iRe_k/e_iR^2e_k = (e_iR/X_k)e_k$. By Lemma 2.4 the module $e_iR/X_k$ decomposes into a direct sum of some copies of the module $U_k$. Since $e_iR/X_k$ is isomorphic to a direct summand $W_i$, the module $U_k$ is contained in $W_i$ as a direct summand.

For left modules $V_k$ the statement is proved analogously. The lemma is proved. □

**Lemma 2.6.** *Let* $A$ *be a semiperfect ring, and* $e, f$ *be nonzero idempotents of the ring* $A$ *such that* $\overline{e} = \overline{f} \in \overline{A}$. *Then there exists an invertible element* $a \in A$ *such that* $f = aea^{-1}$.

The quiver $Q(A)$ of a ring $A$ is called connected if it cannot be represented in the form of a union of two nonempty disjoint subsets $Q_1$ and $Q_2$ which are not connected by any arrows.

**Theorem 2.7.** *The following conditions are equivalent for a semiperfect Noetherian ring $A$:*

*(a) $A$ is an indecomposable ring;*

*(b) $A/R^2$ is an indecomposable ring;*

*(c) the quiver of $A$ is connected.*

*Proof.* Obviously, the conditions of the theorem are preserved by passing to the Morita equivalent rings. Therefore we can assume that the ring $A$ is reduced.

(a) $\Rightarrow$ (b). Let $\overline{A} = A/R^2 \simeq \overline{A}_1 \times \overline{A}_2$ and let $\overline{1} = \overline{P}+1+\overline{P}_2$ be the corresponding decomposition of the identity of the ring $A/R^2$ into a sum of orthogonal idempotents. Let $g_1, g_2 \in A$ be elements such that $g_1 + R^2 = \overline{f}_1$ and $g_2 + R^2 = \overline{f}_2$. There are idempotents $f_1, f_2 \in A$ such that $f_1 = g_1 + r_1$ and $f_2 = g_2 + r_2$, where $r_1, r_2 \in R^2$. Since $\overline{f}_1\overline{A}\overline{f}_2 = 0$ and $\overline{f}_2\overline{A}\overline{f}_1 = 0$, we have $g_1ag_2 \in R^2$ and $g_2ag_1 \in R^2$ for any $a \in A$. Clearly, $f_i = f_ig_if_i + f_ir_if_i$ $(i = 1, 2)$. Then the element $f_1af_2 = f_1g_1f_1af_2g_2f_2 + f_1g_1f_1af_2r_2f_2 + f_1r_1f_1af_2g_2f_2 + f_1r_1f_1af_2r_2f_2$ belongs to $R^2$ for any $a \in A$. This is immediate from Proposition 2.3. Exactly in the same way $f_2Af_1 \in R^2$. Therefore $f_2Af_1 = f_2R_2f_1$ and $f_1Af_2 = f_1R^2f_2$. By Proposition 2.3, the two-sided Peirce decomposition of $R$ has the form: $R = \begin{pmatrix} R_1 & A_{12} \\ A_{21} & R_2 \end{pmatrix}$, where $R_i = Rad(f_iAf_i)$ $(i = 1, 2)$ and $A_{ij} = f_iAf_j$ for $i \neq j$. Calculating $R^2$ we obtain

$$R^2 = \begin{pmatrix} R_1^2 + A_{12}A_{21} & R_1A_{12} + A_{12}R_2 \\ A_{21}R_1 + R_2A_{21} & A_{21}A_{12} + R_2^2 \end{pmatrix}.$$

From the above we have: $A_{12} = R_1A_{12} + A_{12}R_2$ and $A_{21} = R_2A_{21} + A_{21}R_1$. By Theorem 2.1, taking into account Nakayama's lemma, we obtain that $A_{12} = 0$ and $A_{21} = 0$ and therefore $A = A_{11} \times A_{22}$, where $A_{ii} = f_iAf_i$ $(i = 1, 2)$.

(a) $\Rightarrow$ (c). Let the quiver of the ring $A$ be disconnected. Then $Q(A) = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = \varnothing$, and the points of the sets $Q_1$ and $Q_2$ are not connected by any arrows. Renumbering, if necessary, the principal right $A$-modules $P_1, \ldots, P_s$ one may assume that $Q_1 = \{1, \ldots, k\}$ and $Q_2 = \{k+1, \ldots, s\}$. Let $A = P_1 \oplus \ldots \oplus P_s$ be a decomposition of the ring $A$ into a direct sum of principal right $A$-modules (where $P_i = e_iA$, $e_i^2 = e_i \in A$, $1 = e_1 + \ldots + e_s$) and $1 = f_1 + f_2$, where $f_1A = P_1 \oplus \ldots \oplus P_k$ and $f_2A = P_{k+1} \oplus \ldots \oplus P_s$. We set $A_{ij} = f_iAf_j$, $R_i = radA_{ii}$ $(i = 1,\ 2)$. If $A_{12} \neq 0$, then by Theorem 2.1, taking into account Nakayama's lemma, we obtain that the inclusion $A_{12} \supset R_1A_{12} + A_{12}R_2$ is strict. But $R_1A_{12} + A_{12}R_2 = f_1R^2f_2$. Therefore there are local idempotents $e_i$ and $e_j$ such that $e_i$ is a summand of $f_1$ and $e_j$ is a summand of $f_2$ and $e_iR^2e_j$ is strictly contained in $e_iRe_j$. By Lemma 2.5 we obtain that there is an arrow which connects the point $i$ with the point $j$. A contradiction. Analogously it can be proved that $A_{21} = 0$.

(c) $\Rightarrow$ (a). If the ring $A$ is decomposable then $A/R^2$ is also decomposable. Clearly, in this case $Q(A)$ is disconnected.

(b) $\Rightarrow$ (a) is trivial.

The theorem is proved. □

**Remark**. Theorem 2.7 is not true for semiperfect one-sided Noetherian rings. As an example one can consider the ring $A = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. The quiver of this ring consists of two points and one loop near one of them.

As $R^2 = \begin{pmatrix} p^2\mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ then the ring $A/R^2$ decomposes into a direct product of rings:

$$A/R^2 \simeq \mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)} \times \mathbb{Q}.$$

However, the ring A itself is indecomposable into a direct product of rings.

**Theorem 2.8.** *Let the ring $A$ be a serial ring such that the intersection of all powers of its radical $\cap_{n=1}^{\infty} R^n = 0$ is equal to zero. Then $A$ is right and left Noetherian ring.*

*Proof.* Let $M \in P$ and $\cap_{n=1}^{\infty} R^n = 0$.

Then the inclusion $MR \subset M$ is strong. If $M = MR$ then $M \subset R$ and the equality $M = MR^n$ gives that $M \subset R$ for all $n$ i.e. $M = 0$.

Let $e$ be an arbitrary idempotent of the ring $A$. Then $eRe = Rad\ eAe$ and $eAe \subset R$, $(eRe)^n \subset R^n$ and that is why $\cap(eRe)^n = 0$.

So for any local idempotent $e$ the ring $eAe$ is uniserial and the intersection of natural powers of the radical $R$ is equal to 0. That is why the ring $eAe$ is discrete valuated as it is Artinian. Assume that all rings $eAe$ are Artinian. Then $A$ is also Artinian. Let at least one ring of the form $e_iAe_i$ be discrete valuated. Then there exists a local idempotent $e_j$ such that the ring $(e_j + e_i)A(e_j + a_i)$ is of the form $\begin{pmatrix} A_j & X \\ Y & \mathcal{O}_i \end{pmatrix}$, where $X$ is an infinitely generated right $\mathcal{O}_i$-module. According to Lemma 3.28 $\begin{pmatrix} R_1X & X \\ Y & R_2 \end{pmatrix} = \begin{pmatrix} R_1^2 + XY & R_1X + XR_2 \\ YR_1 + R_2Y & YX + R_2 \end{pmatrix}$ and $XR_2 = X$. Consider the following module $M = (XY, X)$, which belongs to $(A_1X)$. It is obviousl that $(XY, X)\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix} = (XY, X)$. This contradicts to the strong inclusion $MR \subset M$, whence $X$ is a finitely generated right $\mathcal{O}_2$-module, and in the same way $Y$ is a finitely generated left $\mathcal{O}_1$-module.

So according to Theorem 2.1 the ring $(e_i + e_j)A(e_i + e_j)$ is right Noetherian.

□

## 3 Semiperfect semidistributive rings

Theorem 1.9 has the following corollary.

**Corollary 3.1.** *Let $A$ be a semiperfect ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents. The ring $A$ is right (left) semidistributive if and only if for any idempotents $e_i$ and $e_j$ of the above decomposition, the set $e_iAe_j$ is a uniserial right $e_jAe_j$-module (left $e_iAe_i$-module).*

**Corollary 3.2. (Reduction Theorem for SPSD-rings)** *Let $A$ be a semiperfect ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ in a sum of mutually orthogonal local idempotents. The ring $A$ is right (left) semidistributive if and only if for any idempotents $e_i$ and $e_j$ $(i \neq j)$ of the above decomposition the ring $(e_i + e_j)A(e_i + e_j)$ is right (left) semidistributive.*

*Proof.* It is sufficient to prove the corollary for a reduced ring $A$. If $A$ is a right semidistributive, then $e_iAe_j$ is right uniserial $e_jAe_j$-module $(i \neq j)$ and the ring $e_iAe_i$ is right uniserial for $i = 1, \ldots, n$. By Corollary 3.1, the ring $(e_i + e_j)A(e_i + e_j)$ is right semidistributive. Conversely, if the ring $(e_i + e_j)A(e_i + e_j)$ is right semidistributive, then, by Theorem 1.9, the set $e_iAe_j$ is a uniserial right $A_{jj}$-module and, by Corollary 3.1, the ring $A$ is right semidistributive. □

**Corollary 3.3.** *Let $A$ be a Noetherian SPSD-ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity $1 \in A$ into a sum of mutually orthogonal local idempotents, let $A_{ij} = e_iAe_j$ and let $R_i$ be the Jacobson radical of the ring $A_{ii}$. Then $R_iA_{ij} = A_{ij}R_{jj}$ for $i, j = 1, \ldots, n$.*

**Example 3.1.** *Consider*

$$A = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$$

*as an $\mathbb{R}$-algebra ($\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers). The Peirce decomposition of the Jacobson radical $R = R(A)$ has the form*

$$R = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}$$

*and the $\mathbb{R}$-algebra $A$ is right serial, i.e., right semidistributive.*

*The left indecomposable projective $Q_2 = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ has the socle $\begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}$, which is a direct sum of two copies of the left simple module $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$. Consequently, by Proposition 1.3, the $\mathbb{R}$-algebra $A$ is an SPSDR-ring but it is not an SPSDL-ring.*

## 3.1 Quivers of SPSD-rings

Recall that a quiver without multiple arrows and multiple loops is called a simply laced quiver. Let $A$ be an SPSD-ring. By Theorem 1.8, the quotient ring $A/R^2$ is right Artinian and its quiver $Q(A)$ is defined by $Q(A) = Q(A/R^2)$.

**Theorem 3.4.** *The quiver $Q(A)$ of an SPSD-ring $A$ is simply laced. Conversely, for any simply laced quiver $Q$ there exists an SPSD-ring $A$ such that $Q(A) = Q$.*

*Proof.* We may assume that $A$ is reduced and $R^2 = 0$. Let $A_A = P_1 \oplus \ldots \oplus P_s$, where $P_1, \ldots, P_s$ are indecomposable. Then $P_iR$ is a semisimple A-module:

$$P_iR = \bigoplus_{j=1}^{s} U_j^{t_{ij}}$$

where $U_j = P_j/P_jR$ are simple. The $A$-module $P_iR$ is a submodule of a distributive A-module and, therefore, $P_iR$ is distributive. By the definition of $Q(A)$ we have $[Q(A)] = (t_{ij})$ and, by Theorem 1.7, $0 \le t_{ij} \le 1$. So $Q(A)$ is a simply laced quiver.

Conversely, let $kQ$ be the path $k$-algebra of a simply laced quiver $Q$ and $J$ be its fundamental ideal, i.e., the ideal generated by all arrows of $Q$. Write $B = kQ/J^2$ and $\pi : kQ \to B$ for the natural epimorphism. Let $\pi(\varepsilon_i) = e_i$, where $\varepsilon_1, \ldots, \varepsilon_s$ are all paths of length zero. Then $B = e_1B \oplus \ldots \oplus e_sB$, where $e_1B, \ldots, e_sB$ are indecomposable. Let $R$ be the Jacobson radical of $B$ and $AQ = \{\sigma_{ij}\}$ be the set of all arrows of $Q$. The elements $\pi(\sigma_{mp})$, where $\sigma_{mp} \in AQ$, form a basis of $e_mR$. Obviously, $e_mR^2 = 0$ for $m = 1, \ldots, s$. So, $e_mR$ is a semisimple module and $e_mR = \oplus_p U_p$ for all those $p$, where $\sigma_{mp} \in AQ$. Therefore $Q(B) = Q$ and $e_mR$ is a distributive module, by Theorem 3.27. Thus, B is a right semidistributive ring. The analogous arguments show that B is a left semidistributive ring.

So $B = kQ/J^2$ is an SPSD-algebra over a field $k$ and $Q(B) = Q$. □

**Corollary 3.5.** *The link graph $\mathcal{L}G(A)$ of an SPSD-ring A coincides with $Q(A)$.*

*Proof.* For any SPSD-ring A the following equalities hold: $\mathcal{L}G(A) = Q(A, R) = Q(A)$. □

**Theorem 3.6.** *For an Artinian ring A with $R^2 = 0$ the following conditions are equivalent:*

*(a) A is semidistributive;*

*(b) $Q(A)$ is simply laced and the left quiver $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows.*

*Proof.* $(a) \implies (b)$. By Theorem 3.4 it is sufficient to show that $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows. One can assume that $A$ is reduced. Write $A_A$ as a direct sum $A_A = P_1 \oplus \ldots \oplus P_s$, where the $P_i$ are indecomposable projective and let $1 = e_1 + \ldots + e_s$ be the corresponding decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents. If $A_{ij} = e_iAe_j \neq 0$, then, in view of Corollary 3.3,

$$A_{ij}R_j = R_iA_{ij} \text{ and } A_{ij} \subset R \text{ for } i \neq j.$$

Hence, $A_{ij}R_j = R_iA_{ij} = 0$ for $i \neq j$ and, in view of the $Q$-Lemma, it follows that there is a loop at the vertex $i$ both in $Q(A)$ and in $Q'(A)$. Thus the left quiver $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows.

$(b) \implies (a)$. By the Peirce decomposition for R we have: $R = \bigoplus_{i,j=1}^{s} e_iRe_j$, $e_iRe_i = R_i$ and $e_iRe_j = A$, $i \neq j$; $i, j = 1, \ldots, s$.

It follows that

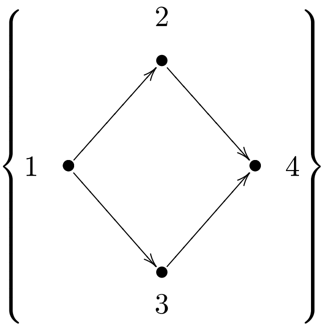
$$P_iR = (A_{i1}, \ldots, A_{ii-1}, R_i, A_{ii+1}, \ldots, A_{is})$$

for $i = 1, \ldots, s$. If $A_{ij} \neq 0$, for $i \neq j$, then, in view of the Q-Lemma, $A_{ij}$ is a simple right $A_{jj}$-module and a simple left $A_{ii}$-module. If $R_i \neq 0$, then $R_i$ is a simple

$A_{ii}$-module and a simple left $A_{ii}$-module. Thus, in view of Theorem 1.9, the ring $A$ is semidistributive. □

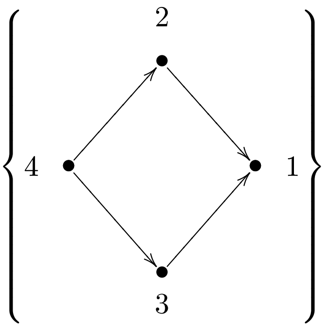
**Remark**. The implication $(b) \Longrightarrow (a)$ isn't true even in the case of finite dimensional algebras as is shown by the following example.

Let $A = kQ_4$ be the path k-algebra of the quiver $Q_4$

$Q_4 =$ 

.

The basis of $kQ_4$ is $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4,\ \sigma_{12}, \sigma_{13}, \sigma_{24}, \sigma_{34}, \sigma_{12}\sigma_{24}, \sigma_{13}\sigma_{34}$. The indecomposable projective $A$-modules are: $P_1 = \{\varepsilon_1, \sigma_{12}, \sigma_{13}, \sigma_{12}\sigma_{24}, \sigma_{13}\sigma_{34}\}$; $P_2 = \{\varepsilon_2, \sigma_{24}\}$; $P_3 = \{\varepsilon_3, \sigma_{34}\}$; $P_4 = \{\varepsilon_4\}$. Obviously, $socP_1 \simeq P_4 \oplus P_4$. By Theorem 1.7, $P_1$ is not distributive, but $Q(A) = Q_4$ and

$Q'(A) =$ 

,

i.e., A satisfies condition (b) of Theorem 1.5.

Note that if we identify routes $\sigma_{12}\sigma_{24}$ and $\sigma_{13}\sigma_{34}$ then obtain the distributive algebra, which is isomorphic to the matrix algebra $M_4(k)$ of the following form

$$\begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

A semiperfect ring $A$ such that $A/R^2$ is Artinian will be called **$Q$-symmetric** if the left quiver $Q'(A)$ can be obtained from the right quiver $Q(A)$ by reversing all arrows.

**Corollary 3.7.** *Every SPSD-ring is Q-symmetric.*

**Remark.** Example 1.9 shows that an SPSDR-ring is not always $Q$-symmetric.

**Theorem 3.8.** *The intersection of all natural powers of Jacobson radical of SPSD-ring is equal to zero.*

*Proof.* Obviously we can consider the ring to be reducible. Denote

$$I_k = \begin{pmatrix} R_1^k & A_{12}R_2^k & \cdots & A_{1s}R_s^k \\ A_{21}R_1^k & R_2^k & \cdots & A_{2s}R_s^k \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1}R_1^k & A_{s2}R_2^k & \cdots & R_s^k \end{pmatrix}.$$

Obviously $I_k$ is two-sided ideal of the ring $A$. It is easy to check that

$$I_k I_l = I_{k+1} \text{ and } R^2 \subset I_1.$$

So, $R^{sk} \subset I_k$ whence

$$\bigcap_{n=0}^{\infty} R^n \subset \bigcap_{k=0}^{\infty} R^n I_k.$$

As all rings $A_{ii}$ are Noetherian chain rings then [12] it follows that they are either discrete valuation rings or uniserial Artinian rings Kiote rings. The intersection of all powers of the Jacobson radical of such rings is equal to zero [12]. According to Theorem 1.9 the ring $A_{ij}$ is a cyclic chain $A_{jj}$-module and a cyclic left chain $A_{ii}$-module. But in this case

$$\bigcap_{k=0}^{\infty} A_{ij}R_j^k = 0, \quad i, j = 1, \ldots, s.$$

This means that the intersection of $I_k$ for all natural $k$ is equal to zero. Whence, the intersection of all natural powers of Jacobson radical is equal to zero. □

## 3.2 Semiprime semiperfect rings

In this section we shall describe the minors of first and second order of right Noetherian semiprime SPSD-rings.

The endomorphism ring of an indecomposable projective module over a semiperfect ring is called a **principal endomorphism ring**.

**Proposition 3.9.** *An Artinian principal endomorphism ring of a semiprime semiperfect ring is a division ring.*

*Proof.* This ring is an Artinian prime local ring and, consequently, is a division ring. □

**Lemma 3.10.** *Let $A_A = P_1^{n_1} \oplus P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiprime semiperfect ring $A$ into principal modules and let $End_A(P_1) = D_1$ be a division ring. Then $A = M_{n_1}(D_1) \times End(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s})$.*

*Proof.* Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents, i.e., $f_iA = P_i^{n_i}$ for $i = 1, \ldots, s$. Let $f_1Af_1 = A_1$, $(1-f_1)A(1-f_1) = A_2$, $X = f_1A(1-f_1)$, $Y = (1-f_1)Af_1$. If either $X \neq 0$ or $Y \neq 0$, then $K = \begin{pmatrix} 0 & X \\ Y & YX \end{pmatrix}$ is a nilpotent ideal and we have the contradiction. So $X = 0$, $Y = 0$, proving the lemma. □

**Theorem 3.11. (Decomposition theorem for semiprime semiperfect rings)** *A semiprime semiperfect ring is a finite direct product of indecomposable rings. An indecomposable semiprime semiperfect ring is either a simple Artinian ring or an indecomposable semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian.*

A proof immediately follows from Lemma 3.10.

Let $1 = g_1 + g_2$ be a decomposition of the identity of $A$ into a sum of the mutually orthogonal idempotents, and let $A = (A_{ij})$ be the corresponding Peirce decomposition of $A$, i.e., $A_{ij} = g_iAg_j$, $i, j = 1, 2$. Similarly, if M is a two-sided ideal of $A$, then $M = (M_{ij})$ is the Peirce decomposition of $M$, where $M_{ij} = g_iMg_j$, $i, j = 1, 2$.

**Lemma 3.12.** *Let $M = (M_{ij})$ be a two-sided ideal of a semiprime ring $A$. If $M_{ij} \neq 0$ for $i \neq j$, then $M_{ji} \neq 0$. Moreover, if $M_{ij} \neq 0$ for $i \neq j$, then $M_{ij}M_{ji} \neq 0$ and $M_{ji}M_{ij} \neq 0$.*

*Proof.* Let $M_{ij}M_{ji} = 0$. Clearly, $Z = M_{ij}A_{ji} + A_{ij}M_{ji} + M_{ij} + M_{ji}$ is a two-sided ideal and $Z^8 = 0$. The remaining cases are treated analogously. □

**Corollary 3.13.** *Let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of $A$ into a sum of the mutually orthogonal idempotents, $A_{ij} = e_iAe_j$, $i, j = 1, \ldots, n$, and let $M$ be a two sided ideal in $A$, $M_{ij} = e_iMe_j$, $i, j = 1, \ldots, n$. If $M_{ij} \neq 0$ for $i \neq j$, then $M_{ji} \neq 0$ and $M_{ij}M_{ji} \neq 0$, $M_{ji}M_{ij} \neq 0$. Moreover, from the equality $A_{ij}A_{ji} = 0$ it follows that $A_{ij} = 0$ and $A_{ji} = 0$.*

**Proposition 3.14.** *Let $A$ be a prime (resp. semiprime) ring, $e^2 = e \in A$. Then the ring $eAe$ is prime (resp. semiprime).*

**Theorem 3.15.** *For a semiprime semiperfect ring $A$ the following conditions are equivalent:*

*(1) $A$ is a finite direct product of prime rings;*

*(2) all principal endomorphism rings of $A$ are prime.*

*Proof.* (1) $\Longrightarrow$ (2) follows from Proposition 3.14.

(2) $\Longrightarrow$ (1) Obviously, we can assume that $A$ is indecomposable and reduced. Let $1 = e_1 + \ldots + e_n$ be a decomposition of $1 \in A$ into the sum of pairwise orthogonal local idempotents. We shall prove the theorem by induction on $n$. The case $n = 1$ is obvious. Suppose that $A$ is not prime. Then there exist two-sided nonzero ideals $M, N$ such that $MN = 0$. Let $h_1 = e_1 + \ldots + e_{n-1}$ and $h_2 = e_n$. We have

the equality $h_1Mh_1Nh_1 = 0$. By the induction hypothesis either $h_1Mh_1 = 0$ or $h_1Nh_1 = 0$. Let $h_1Mh_1 = 0$, then by Corollary 3.13 $h_1Mh_2 = 0$ and $h_2Mh_1 = 0$. If $h_2Mh_2 = 0$, then the theorem is proved, so $h_2Mh_2 \neq 0$ and $h_2Nh_2 = 0$. We have again $h_2Nh_1 = 0$ and $h_1Nh_2 = 0$. One can assume that $e_iNe_i \neq 0$ for $i = 1, \ldots, t$ and $e_jNe_j = 0$ for $j = t+1, \ldots, n$. So $N_{ii}A_{ij} = 0$ for $i = 1, \ldots, t$ and $j = t+1, \ldots, n$. Consequently, $N_{ii}A_{ij}A_{ji} = 0$ for the same $i$ and $j$. Since the $A_{ii}$ are prime, it follows that $A_{ij}A_{ji} = 0$. By Corollary 3.13, we obtain $A_{ij} = 0$ and $A_{ji} = 0$ for $i = 1, \ldots, t$ and $j = t+1, \ldots, n$. Hence, the ring $A$ is decomposable and we obtain a contradiction, which proves the theorem. □

Let $A$ be a ring, $P$ a finitely generated projective $A$-module which can be decomposed into a direct sum of $n$ indecomposable modules. The endomorphism ring $B = End_A(P)$ of the module P is called a **minor** of order $n$ of the ring $A$.

**Proposition 3.16.** *Every minor of an SPSD-ring is an SPSD-ring.*

The proof follows from Theorem 1.9 and Corollary 3.1.

**Corollary 3.17.** *Every minor of a right Noetherian semiprime SPSD-ring is a right Noetherian semiprime SPSD-ring.*

The proof follows from Theorem 2.1 and Proposition 3.14.

From Theorems 1.9 and 2.1 we obtain the following statement.

**Corollary 3.18.** *Every minor of a Noetherian SPSD-ring is a Noetherian SPSD-ring.*

**Proposition 3.19.** *A minor of the first order of a right Noetherian SPSD-ring is uniserial and it is either a discrete valuation ring or an Artinian uniserial ring.*

Let $\mathcal{O}$ be right local uniserial Noetherian ring with the unique maximal ideal $\mathfrak{M}$.

Consider the following descending chain of two-sided ideals.

$$\mathcal{O} \supset \mathfrak{M} \supset \mathfrak{M}^2 \supset \ldots \supset \mathfrak{M}^n \supset \ldots$$

By Nakayama Lemma $\mathfrak{M}^k$ strictly contains $\mathfrak{M}^{k+1}$ for any $k \in \mathbb{N}$. As $\mathcal{O}$ is serial ring then right factor module $\mathfrak{M}^k/\mathfrak{M}^{k+1}$ is simple if $\mathfrak{M}^k \neq 0$.

Assume that $\mathfrak{M} \neq 0$. In this case if $\pi \in \mathfrak{M}\backslash\mathfrak{M}^2$ then $\pi\mathcal{O} + \mathfrak{M}^2 = \mathfrak{M}$ and according to Nakayama Lemma $\mathfrak{M} = \pi\mathcal{O}$.

Consider left ideals $\mathcal{O}\pi$ and $\mathfrak{M}$. The local property of the ring $\mathcal{O}$ gives that $\mathfrak{M} \supseteq \mathcal{O}\pi$.

The strong inclusion $\mathcal{O}\pi \supset \mathfrak{M}^2$ follows from that $\mathcal{O}$ is serial. Factor module $\mathfrak{M}/\mathfrak{M}^2$ is semisimple right $\mathcal{O}$-module and is left $\mathcal{O}$-module. As the ring $\mathcal{O}$ is serial then $\mathfrak{M}/\mathfrak{M}^2$ is simple from both sides. Whence $\mathcal{O}\pi = \pi\mathcal{O} = \mathfrak{M}$.

The next proposition immediately follows from this fact.

**Proposition 3.20.** *Let $\mathcal{O}$ be a right local Noetherian serial ring with the unique maximal ideal $\mathfrak{M} \neq 0$. Then $\mathfrak{M} = \pi\mathcal{O} = \mathcal{O}\pi$ and the ring $\mathcal{O}$ is both sides Artinian if and only if the element $\pi$ is nilpotent.*

That is why in future we will assume that the element $\pi$ is not nilpotent.

Consider the endomorphism $\pi$ of the right $\mathcal{O}$-module $\mathcal{O}_{\mathcal{O}}$ which multiplies $\alpha \in \mathcal{O}$ with the element $\pi$ from the left, i.e. $\pi(\alpha) = \pi\alpha$.

**Step** 1. $\ker \pi \subset \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$.

*Proof.* Let $ker\pi = \{\alpha \in \mathcal{O} |\ \pi\alpha = 0\}$. It is obvious that $ker\pi$ is two-sided ideal. Really, if $\alpha \in ker\pi$ then $\pi(\alpha\alpha_1) = (\pi\alpha)\alpha_1 = 0$, i.e. $\alpha\alpha_1 \in ker\pi$.

Let $\alpha \in ker\pi$ and $\beta \in \mathcal{O}$. Consider $\pi(\beta\alpha) = (\pi\beta)\alpha = (\beta_1\pi)\alpha = \beta_1(\pi\alpha) = 0$. If $ker\pi = \mathfrak{M}^n$ for some $n$ then $\pi\mathfrak{M}^n = \pi\mathcal{O}\mathfrak{M}^n = \mathfrak{M}^{n+1} = 0$, whence $\pi^{n+1} = 0$. So, $ker\pi \subset \mathfrak{M}^n$ for any natural $n$, whence $ker\pi \subset Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$. □

**Step** 2. $\ker \pi = 0$.

*Proof.* Let $X = ker\pi \neq 0$. Then there exists ascending chain of ideals

$$ker\pi \subset ker\pi^2 \subset \ldots \subset ker\pi^n \subset \ldots.$$

Let us show that $\ker \pi^k \neq ker\pi^{k+1}$ for all $k$. Let $\ker \pi^k = ker\pi^{k+1}$ for some $k$ and $x \in X$, $x \neq 0$. So, $\pi x = 0$ and $x \in \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$. This is followed by $x = \pi^k\alpha_k = \pi^{k+1}\alpha_{k+1}$. The equality $\pi x = 0$ implies $\pi^{k+1}\alpha_{k+1} = 0$ i.e. $\alpha_k \in ker\pi^{k+1}$ and this means that $\alpha_k \in ker\pi^k$ and $\pi^k\alpha_k = 0 = x$. That is why there exists strongly ascending chain of two-sided ideals

$$ker\pi \subset ker\pi^2 \subset \ldots \subset ker\pi^n \subset \ldots.$$

and this is a contradiction with the property of the ring $\mathcal{O}$ to be right Noetherian. The proposition is proved. □

**Step** 3. $\bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n = 0$.

*Proof.* Let $Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n \neq 0$. Consider two-sided ideal $Y\mathfrak{M}$ of the ring $\mathcal{O}$ which is the unique maximal submodule of $\mathfrak{M}$ as the ring $\mathcal{O}$ is right Noetherian.

Considering the factor ring $\mathcal{O}/Y\mathfrak{M}$ one may assume that the intersection $Y = \bigcap\limits_{n=1}^{\infty} \mathfrak{M}^n$ is a simple right $\mathcal{O}$-module in the former ring $\mathcal{O}$. The property of $Y$ to be a two-sided ideal and the equality $ker\pi = 0$ imply that $\pi Y = Y$.

Let $W = \{\alpha \in \mathcal{O} |\ \alpha\pi \in Y\}$. Obviously $W \neq 0$ because $y \in \mathfrak{M}$, $y \in Y$, $y \neq 0$ and $y = \alpha\pi$.

Let us show that $W$ is a two-sided ideal of the ring $\mathcal{O}$. Obviously $\alpha + \alpha_1 \in W$ if $\alpha$, $\alpha_1 \in W$.

Let $\alpha \in W$, i.e. $\alpha\pi \in Y$. Then $(\beta\alpha)\pi = \beta y_1 \in Y$, i.e. $\beta\alpha \in W$ for any $\beta \in \mathcal{O}$. Moreover, $(\alpha\beta)\pi = \alpha(\beta\pi) = \alpha(\pi\beta_1) = (\alpha\pi)\beta_1 \in Y$. If $W \not\subset Y$ then

$W = \pi^n \mathcal{O} = \mathcal{O}\pi^n$ and $W\pi \in Y$, i.e. $W\pi = \mathfrak{M}^{n+1} \subset Y$. The obtained contradiction shows that $W \subset Y$ and so $W = Y$. Let $\in Y$ and $y \neq 0$. Then $y = y_1\pi$ for some $y_1 \in Y$. But $Y\pi = 0$, whence $y = 0$. The obtained contradiction proves that $\mathcal{O}$ is a discrete valuation ring with the unique maximal ideal $\mathfrak{M} = \pi\mathcal{O} = \mathcal{O}\pi$. For more details see Warfield [1975]. □

**Corollary 3.21.** *A minor of the first order of a right Noetherian semiprime SPSD-ring is either a discrete valuation ring or a division ring.*

A ring $A$ is called **semimaximal** if it is a semiperfect semiprime right Noetherian ring such that for each local idempotent $e \in A$ the ring $eAe$ is a discrete valuation ring (not necessarily commutative), i.e., all principal endomorphism rings of $A$ are discrete valuation rings.

**Proposition 3.22.** *A semimaximal ring is a finite direct product of prime semimaximal rings.*

A proof follows from Theorem 3.15.

So, a semimaximal ring $A$ is indecomposable if and only if $A$ is prime.

**Proposition 3.23.** *A semiperfect reduced indecomposable ring $B$ is a second order minor of a right Noetherian semiprime SPSD-ring if and only if $B$ is semimaximal.*

*Proof.* Let $1 = e_1 + e_2$ be a decomposition of $1 \in B$ into a sum of local idempotents, let $B = \bigoplus_{i,j=1}^{2} e_iBe_j$ be the corresponding two-sided Peirce decomposition, and let $B_{ij} = e_iBe_j$ $(i, j = 1, 2)$. The Jacobson radical $R$ of $B$ has the form: $R = \begin{pmatrix} R_1 & B_{12} \\ B_{21} & R_2 \end{pmatrix}$, where $R_i$ is the Jacobson radical of $B_{ii}$ $(i = 1, 2)$. Obviously,

$$R^2 = \begin{pmatrix} R_1^2 + B_{12}B_{21} & R_1B_{12} + B_{12}R_2 \\ R_2B_{21} + B_{21}R_1 & R_2^2 + B_{21}B_{12} \end{pmatrix}$$

By Corollary 3.19, $B_{ii}$ is either a discrete valuation ring or a division ring. If $B_{11} = D$ is a division ring, then $R = \begin{pmatrix} 0 & B_{12} \\ B_{21} & R_2 \end{pmatrix}$. Obviously, $J = \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{21}B_{12} \end{pmatrix}$ is a nonzero ideal in $B$ and $J_2 = 0$. So $B$ is semimaximal.

Let's now show that a semimaximal ring $B$ is semidistributive. We can assume that $B$ is prime. Let $R_i = \pi_iB_{ii} = B_{ii}\pi_i$ $(i = 1, 2)$. Now $b_{12}b_2 \neq 0$ for any $b12 \neq 0$ and $b_2 \neq 0$ $(b_{12} \in B_{12}, b_2 \in B_{22})$. Indeed, we can suppose that $b_2 = \pi_2^m$. Then $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \neq 0$ and, consequently, $b_{12}B_{22}p_2^m = b_{12}p_2^mB_{22} \neq 0$. So, $b_{12}p_2^m \neq 0$. Analogously, $b_{ij}b_j \neq 0$ and $b_ib_{ij} \neq 0$ for $i, j = 1, 2$. Further $b_{ij}b_{ji} \neq 0$ for $i \neq j$ and both factors are nonzero. We shall prove that $b_{21}b_{12} \neq 0$ for $b_{12} \neq 0$ and $b_{21} \neq 0$. Indeed, $\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix} \neq$

0. So, $b_{12}B_{22}b_{21} \neq 0$ and thus there exists $b_2 \in B_{22}$ such that $b_{12}b_{22}b_{21} \neq 0$. If $b_{21}b_{12} = 0$, then $b_{21}b_{12}b_{22}b_{21} = 0$ and we obtain a contradiction.

Next $B_{12}$ is a uniserial right $B_{22}$-module and a uniserial left $B_{11}$-module. By Theorem 2.1, $B_{12}$ is a finitely generated $B_{22}$-module. Consequently, if $B_{12}$ isn't uniserial, then $B_{12} = B_{12}^{(1)} \oplus B_{12}^{(2)}$, where $B_{12}^{(1)}$ and $B_{12}^{(2)}$ are nonzero $B_{22}$-submodules of $B_{12}$. Let $b_{21} \neq 0$. Then $b_{21}B_{12} = b_{21}B_{12}^{(1)} \oplus b_{21}B_{12}^{(2)}$, where $b_{21}B_{12}^{(1)}$ and $b_{21}B_{12}^{(1)}$ are nonzero right ideals in $\mathcal{O}_2$. This is a contradiction. Consequently, $B_{12}$ is a uniserial right $B_{22}$-module.

Finally $B_{12}$ is a uniserial left $B_{11}$-module. If this isn't true, then there exists a left $B_{11}$-submodule $N_{12}$ with two noncyclic generators in $B_{12}$. Consequently, $N_{12} = N_{12}^{(1)} \oplus N_{12}^{(2)}$ is a direct sum of two nonzero left $B_{11}$-submodules and so $N_{12}b_{21} = N_{12}^{(1)}b21 \oplus N_{12}^{(2)}b_{21}$ is a direct sum of two nonzero left ideals in $B_{11}$ for any nonzero $b_{21}$. This is a contradiction and so B12 is a uniserial left $B_{11}$-module. Analogously, $B_{21}$ is a uniserial right $B_{11}$-module and a uniserial left $B_{22}$-module. Thus, by Theorem 1.9 $B$ is semidistributive. The proposition is proved. □

**Corollary 3.24.** *An intersection of a finite number of nonzero submodules of an indecomposable projective module over a Noetherian prime SPSD- ring is nonzero.*

**Lemma 3.25.** *A local idempotent of a Noetherian prime SPSD-ring $A$ is a local idempotent of its classical ring of fractions.*

Note that an example of semimaximal rings is non-Artinian both sides Noetherian semiprime hereditary rings. They are exactly semimaximal hereditary rings. The article [19] contains a condition for the prime semimaximal ring $\Lambda$ to be of finite type. This condition is as follows. As an arbitrary prime semimaximal ring $\Lambda$ can be included into the prime ring of fractions $Q$, let $\mathfrak{M}(\Lambda)$ be the partially ordered set (in the sense of inclusion) of all projective $\Lambda$-modules which belong to some prime $Q$-module. So, the equivalent condition for the prime semimaximal ring to be of finite type is the nonexistence of critical subsets in the set $\mathfrak{M}(\Lambda)$. Here a subset of a partially ordered set is called critical if it is one of the following sets: $(1,1,1,1)$, $(2,2,2)$, $(1,3,3)$, $(1,2,5)$, $R = \{a_1 < a_2 > b_1 < b_2; c_1 < c_2 < c_3 < c_4\}$. Here we denote by $(l_1, \ldots, l_m)$ the cardinal sum of $m$ linearly ordered sets which contain $l_1, \ldots, l_m$ elements correspondingly.

For proving Lemma 3.25 we need the following proposition [6, Prop. 9.3.10].

**Proposition 3.26.** *Let $Q$ be a semisimple ring and $A$ be a right order in $Q$. Then $Q$ is a simple ring if and only if $A$ is prime.*

*Proof of Lemma 3.25.* By Proposition 3.26 $A$ is a right order in the simple Artinian ring $Q = M_n(D)$. One can assume that the local idempotent $e \in A$ is a sum of matrix idempotents $e = e_{i_1i_1} + \ldots + e_{i_ki_k}$. Let $k \geq 2$. Then there exist $q_1, \ldots, q_k \in Q$ such that $e_{i_1i_1}q_1, \ldots, e_{i_ki_k}q_k \in A$ and, consequently, $e_{i_1i_1}q_1A, \ldots, e_{i_ki_k}q_kA$ are nonzero right submodules of the right indecomposable projective module $eA$ and $e_{i_mi_m}q_mA \cap e_{i_pi_p}q_pA = 0$ for $m \neq p$. We obtain a contradiction with Corollary 3.24. □

### 3.3 Right Noetherian semiprime SPSD-rings

The following is a **decomposition theorem** for semiprime right Noetherian SPSD-rings.

**Theorem 3.27.** *The following conditions for a semiperfect semiprime right Noetherian ring $A$ are equivalent:*

*(a) the ring $A$ is semidistributive;*

*(b) the ring $A$ is a direct product of a semisimple Artinian ring and a semimaximal ring.*

*Proof.* $(a) \Longrightarrow (b)$. From Theorem 3.11 it follows that $A$ is a finite direct product of indecomposable semiprime rings. Every indecomposable ring is either a simple Artinian ring or a semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian. In the second case, by Corollary 3.21, such a ring is semimaximal.

$(b) \Longrightarrow (a)$. Obviously, a semiprime Artinian ring is a semiprime SPSD-ring. A semimaximal ring is an SPSD-ring, by Proposition 3.11 and the reduction theorem for SPSD-rings. □

**Lemma 3.28.** *The right uniserial modules over the ring $H_m(\mathcal{O})$ are exhausted by the $D^m$, all principal $H_m(\mathcal{O})$-modules and quotient modules of these modules.*

**Theorem 3.29.** *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \cdots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \cdots & \pi^{\alpha_{2n}}\mathcal{O} \\ \cdots & \cdots & \cdots & \cdots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \cdots & \mathcal{O} \end{pmatrix} \qquad (1)$$

*where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element $\pi$, and the $\alpha_{ij}$ are integers such that $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all $i, j, k$ ($\alpha_{ii} = 0$ for any $i$).*

*Proof.* By Proposition 3.10 a semimaximal ring is a finite direct product of prime semimaximal rings. We shall show that a prime semimaximal ring is isomorphic to a ring of the form (1).

Let $1 = e_1 + \ldots + e_m$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal local idempotents, $A_{ij} = e_i A e_j$ for $i, j = 1, \ldots, m$. Denote by $B_{ij}$ $(i \neq j)$ the following second order minor: $B_{ij} = \begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}$. If $B_{ij}$ isn't reduced, then $B_{ij} \simeq M_2(A_{ii})$ and $B_{ij}$ is left Noetherian. If $B_{ij}$ is reduced, then $A_{ij}a_{ji} \subset A_{ij}$, $\varphi_{ji} : A_{ij} \to A_{ii}$ being the monomorphism of left $A_{ii}$-modules (for any nonzero $a_{ji}$) such that $\varphi_{ji}(a_{ij}) = a_{ij}a_{ji}$. If $A_{ij}$ isn't finitely generated, then $A_{ii}$ contains a non-finitely generated left $A_{ii}$-submodule $A_{ij}a_{ji}$, where $a_{ji} \neq 0$. This gives a contradiction. So, by Lemma 3.28, $A_{ij} \simeq A_{ii}$ and $B_{ij}$ is left Noetherian, by Theorem 2.1. Applying induction on $m$ and Theorem 2.1, we see that $A$ is left Noetherian. Consequently, $A$

is a prime Noetherian SPSD-ring. By Proposition 3.26, $A$ is a right order in a simple Artinian ring $Q = M_n(D)$. Suppose that every local idempotent $e_i$ from the above decomposition $1 = e_1 + \ldots + e_m$ is local in $M_n(D)$. Hence, the two decompositions: $1 = e_1 + \ldots + e_m$ and $1 = e_{11} + \ldots + e_{nn}$ are conjugate. Consequently, $m = n$ and we can assume that the matrix idempotents are the local idempotents of $A$.

Denote $A_{ii}$ by $A_i$. We have $Q = \sum\limits_{i,j=1}^{n} e_{ij}\mathcal{D}$ ($\mathcal{D}$ is a division ring, the $e_{ij}$, are matrix units commuting with the elements from $\mathcal{D}$) and $A = \sum\limits_{i,j=1}^{n} e_{ij}A_{ij}$, where $A_{ij} \subset \mathcal{D}$. All $A_i$ are discrete valuations rings, $A_{ij}A_{jk} \subset A_{ik}$ and $A_{ij} \neq 0$ for $i, j = 1, \ldots, n$ ($A$ is prime and $e_{ii}Ae_{jj} = A_{ij} \neq 0$).

We shall prove that $A_{ij} = d_{ij}A_j = A_i d_{ij}$, where $d_{ij} \in A_{ij} \subset \mathcal{D}$. Indeed, let $R_i$ be the Jacobson radical of $A_i$ and let $\pi_i A_i = A_i \pi_i = R_i$. By corollary 3.3, $R_i A_{ij} = A_{ij} R_j$. Take an element $0 \neq d_{ij} \in A_{ij}$ so that $A_i d_{ij} + R_i A_{ij} = A_{ij}$. By Nakayama's Lemma $A_{ij} = d_{ij}A_j = A_i d_{ij}$. Let $T = diag(d_{12}^{-1}, d_{23}^{1}, \ldots, d_{n-1n}^{-1}, 1)$. Consider $TAT{-}1$. One can assume that the following equalities $d_{12} = \ldots = d_{n-1n}$ hold in $A$, hence $A_1 = A_2 = \ldots = A_n$. Write $A_1 = \mathcal{O}$, where $\mathcal{O}$ is a discrete valuation ring (non-necessarily commutative). Consequently, $A_{ij} \supset \mathcal{O}$ for $i \leq j$. From $A_{ij}A_{ji} \subset \mathcal{O}$ we have $A_{ij}A_{ji} \supset A_{ji}$ and $A_{ji} \subset \mathcal{O}$ for $j \leq i$. So, one can assume that $d_{ji} = \pi^{\alpha_{ij}}$, where $\mathcal{M} = \pi\mathcal{O} = \mathcal{O}\pi$ is the unique maximal ideal of $\mathcal{O}$, $\alpha_{ji} \geq 0$ for $j \geq i$. Obviously, $d_{ij} = \pi^{\alpha_{ij}}$, where $\alpha_{ij} \geq -\alpha_{ji}$. Hence, we obtain a ring of the form 3.27. The converse assertion follows from the definition of a semimaximal ring. □

A ring A is called **a tiled order** if it is a prime Noetherian SPSD-ring with nonzero Jacobson radical.

**Remark**. Let $\mathcal{O}$ be a discrete valuation ring. Then from Theorem 3.29 it follows that each tiled order is of the form (1).

The ring $\mathcal{O}$ is embedded into a classical ring of fractions $\mathcal{D}$, which is a division ring. Therefore (14.5.1) denotes the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that $a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}Ae_{jj}$, where the $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $M_n(\mathcal{D})$ is the classical ring of fractions of $A$.

According to the terminology of V. A. Jategaonkar and R. B. Tarsy, a ring $A \subset M_n(K)$, where $K$ is the quotient field of a commutative discrete valuation ring $\mathcal{O}$, is called a tiled order over $\mathcal{O}$ if $M_n(K)$ is the classical ring of fractions of $A$, $e_{ii} \in A$ and $e_{ii}Ae_{ii} = \mathcal{O}$ for $i = 1, \ldots, n$, where the $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(K)$ (see [8]).

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in Mn(\mathbb{Z})$. We shall call a matrix $\mathcal{E} = (\alpha_{ij})$ an **exponent matrix** if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$. A matrix $\mathcal{E}$ is called a **reduced exponent matrix** if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.

We shall use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of a ring $A$, i.e., $A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the $e_{ij}$ are the matrix units. If a tiled order is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$, $i \neq j$, i.e., $\mathcal{E}(A)$ is reduced.

Let $\mathcal{O}$ be a discrete valuation ring. A right (resp. left) $A$-module $M$ (resp. $N$) is called a **right** (resp. **left**) $A$**-lattice** if $M$ (resp. $N$) is a finitely generated free $\mathcal{O}$-module.

For example, all finitely generated projective $A$-modules are $A$-lattices.

Given a tiled order $A$ we denote by $Lat_r(A)$ (resp. $Lat_l(A)$) the category of right (resp. left) $A$-lattices. We denote by $S_r(A)$ (resp. $S_l(A)$) the partially ordered set (by inclusion), formed by all $A$-lattices contained in a fixed simple $M_n(\mathcal{D})$-module $U$ (resp. in a left simple $M_n(\mathcal{D})$-module $V$). Such $A$-lattices are called **irreducible**.

Note that every simple right $M_n(\mathcal{D})$-module is isomorphic to a simple $M_n(\mathcal{D})$-module $U$ with $\mathcal{D}$-basis $e_1, \ldots, e_n$ such that $e_i e_{jk} = \delta_{ij} e_k$, where $e_{jk} \in M_n(\mathcal{D})$ are the matrix units. Respectively, every simple left $M_n(\mathcal{D})$-module is isomorphic to a left simple $M_n(\mathcal{D})$-module $V$ with $D$-basis $e_1, \ldots, e_n$ such that $e_{ij} e_k = \delta_{jk} e_i$.

Let $A = \{\mathcal{O}, E(A)\}$ be a tiled order, and let $U$ (resp. $V$) be a simple right (resp. left) $M_n(\mathcal{D})$-module as above.

Then any right (resp. left) irreducible $A$-lattice $M$ (resp. $N$) lying in $U$ (resp. in $V$) is an $A$-module with $\mathcal{O}$-basis $(\pi_1^{\alpha} e_1, \ldots, \pi^{\alpha_n} e_n)$, while

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{for the right case;} \\ \alpha_{ij} + \alpha_j \geq \alpha_i, & \text{for the left case.} \end{cases} \tag{2}$$

Thus, irreducible $A$-lattices $M$ can be identified with an integer-valued vector $(\alpha_1, \ldots, \alpha_n)$ satisfying (3.29). We shall write $[M] = (\alpha_1, \ldots, \alpha_n)$ or $M = (\alpha_1, \ldots, \alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

**Remark.** Obviously, two irreducible $A$-lattices $M_1 = (\alpha_1, \ldots, \alpha_n)$ and $M_2 = (\beta_1, \ldots, \beta_n)$ are isomorphic if and only if $\alpha_i = \beta_i + z$ for $i = 1, \ldots, n$ and (a fixed) $z \in \mathbb{Z}$. We shall denote by $(\alpha_1, \ldots, \alpha_n)^T$ the column vector with coordinates $\alpha_1, \ldots, \alpha_n$.

Note that the posets $S_r(A)$ and $S_l(A)$ do not depend on the choice of simple $M_n(\mathcal{D})$-modules $U$ and $V$.

**Proposition 3.30.** *The posets $S_r(A)$ and $S_l(A)$ are anti-isomorphic distributive lattices.*

*Proof.* Since A is a semidistributive ring, $S_r(A)$ (resp. $S_l(A)$) is a distributive lattice with respect to the sum and intersection of submodules.

Let $M = (\alpha_1, \ldots, \alpha_n) \in S_r(A)$. We put $M^* = (-\alpha_1, \ldots, -\alpha_n)^T \in S_l(A)$. If $N = (\beta_1, \ldots, \beta_n)^T \in S_l(A)$, then $N^* = (-\beta_1, \ldots, -\beta_n) \in S_r(A)$.

Obviously, the operation * satisfies the following conditions:

1. $M^{**} = M$; 2. $(M_1+M_2)^* = M_1^* \cap M_2^*$; 3. $(M_1 \cap M_2)* = M_1^* + M_2^*$ in the right case and there are analogous rules in the left case. Thus, the map $*: S_r(A) \to S_l(A)$ is the anti-isomorphism. □

**Remark**. The map * defines a duality for irreducible A-lattices.

If $M_1 \subset M_2$, $(M_1, M_2 \in S_r(A))$, then $M_2^* \subset M_1^*$. In this case, the $A$-lattice $M_2$ is called an **overmodule** of the $A$-lattice $M_1$ (resp. $M_1^*$ is an overmodule of $M_2^*$).

### 3.4 Quivers of tiled orders

Recall that a quiver is called **strongly connected** if there is a path between any two vertices. By convention, a one-point graph without arrows will be considered a strongly connected quiver. A quiver $Q$ without multiple arrows and multiple loops is called **simply laced**, i.e., $Q$ is a simply laced quiver if and only if its adjacency matrix $[Q]$ is a $(0,1)$-matrix.

**Theorem 3.31.** *Let $A$ be a semiperfect two-sided Noetherian ring with the quiver $Q(A)$. Suppose the matrix $[Q]$ is block upper triangular with permutationally irreducible matrices $B_1, \ldots, B_t$ on the main diagonal of the Peirce quiver of $A$. Then there exists a decomposition of $1 \in A$ into a sum of mutually orthogonal idempotents: $1 = g_1 + \ldots g_t$ such that*

$$A = \bigoplus_{i,j=1}^{t} giAg_j$$

*is the two-sided Peirce decomposition with $g_iAg_j = 0$ for $j < i$, moreover, the adjacency matrices of the quivers $Q(A_i)$ of the rings $A_i = g_iAg_i$ coincide with $B_i$, $i = 1, \ldots, t$.*

**Theorem 3.32.** *The quiver $Q(A)$ of a right and left Noetherian indecomposable semiprime semiperfect ring $A$ is strongly connected.*

A proof follows from Theorem 3.31 and Proposition 3.14. We use notations from Theorem 3.31. If $Q(A)$ isn't strongly connected, then the ring $(g_1+g_2)A(g_1+g_2)$ isn't semiprime. Indeed, for the nonzero ideal $J = \begin{pmatrix} 0 & g_1Ag_2 \\ 0 & 0 \end{pmatrix}$ we have $J^2 = 0$.

Let I be a two-sided ideal of a tiled order A. Obviously,

$$U = \sum_{i,j=1}^{n} e_{ij}\pi^{\mu_{ij}}\mathcal{O},$$

where the $e_{ij}$ are matrix units. Denote by $E(I) = (\mu_{ij})$ the exponent matrix of the ideal $I$. Suppose that $I$ and $J$ are two-sided ideals of the ring $A$, $\mathcal{E}(I) = (\mu_{ij})$, and $\mathcal{E}(J) = (\nu_{ij})$. It follows easily that $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = min_k\{\mu_{ik} + \nu_{kj}\}$.

**Theorem 3.33.** *The quiver $Q(A)$ of a tiled order $A$ over a discrete valuation ring $\mathcal{O}$ is strongly connected and simply laced. If $A$ is reduced, then $Q(A) = \mathcal{E}(R^2) - \mathcal{E}(R)$.*

*Proof.* Taking into account that $A$ is a prime Noetherian semiperfect ring, it follows from Theorem 3.32 that $Q(A)$ is a strongly connected quiver. Let A be a reduced order. Then $[Q(A)]$ is a reduced matrix. We shall use the following notation: $\mathcal{E}(A) = (\alpha_{ij})$; $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ii} = 1$ for $i = 1, \dots, n$ and $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ $(i, j = 1, \dots, n)$; $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = \min_{1 \leqslant k \leqslant n}\{\alpha_{ik} + \beta_{kj}\}$ for $i, j = 1, \dots, n$. Since, $\mathcal{E}(A)$ is reduced, we have $\alpha_{ij} + \alpha_{ji} \geqslant 1$ for $i, j = 1, \dots, n$, i.e., $\gamma_{ii} = \min_{1 \leqslant k \leqslant n;\, k \neq i,j}\{\alpha_{ik} + \alpha_{ki}\} = min_{1 \leqslant k \leqslant n,\, k \neq = i}\{\alpha_{ik} + \alpha_{ki}\}$. Hence $\gamma_{ii}$ is equal to 1 or 2. If $i \neq j$, then $\beta_{ij} = \alpha_{ij}$ and $\gamma_{ij} = \min\{\min_{1 \leqslant k \leqslant n, k \neq i,j}\{\alpha_{ik} + \alpha_{kj}\}, \alpha_{ij} + 1\}$, i.e., $\gamma_{ij}$ equals $\alpha_{ij}$ or $\alpha_{ij} + 1$.

To any irreducible $A$-lattice $M$ with $\mathcal{O}$-basis $(\pi_{\alpha_1} e_1, \dots, \pi^{\alpha_n} e_n)$ associate the $n$-tuple $[M] = (\alpha_1, \dots, \alpha_n)$. Let us consider

$$[P_i] = (\alpha_{i1}, \dots, 0, \dots, \alpha_{in}),$$

$$[P_i R] = (\alpha_{i1}, \dots, 1, \dots, \alpha_{in}) = (\beta_{i1}, \dots, \beta_{in}).$$

Set $[P_i R^2] = (\gamma_{i1}, \dots, \gamma_{in})$. Then $\overrightarrow{q}_i = [P_i R^2] - [P_i R]$ is a (0, 1)-vector. Suppose that the positions of the units of $\overrightarrow{q}_j$ are $j_1, \dots, j_m$. In view of the annihilation lemma, this means that $P_i R / P_i R^2 = U_{j_1} \oplus \ldots \oplus U_{j_m}$. By the definition of $Q(A)$ we have exactly one arrow from the vertex $i$ to each of $j_1, \dots, j_m$. Thus, the adjacency matrix $[Q(A)]$ is:

$$[Q(A)] = \mathcal{E}(R^2) - \mathcal{E}(R).$$

The theorem is proved. □

A tiled order $A = \{\mathcal{O}, \mathcal{E}(A)\}$ is called a $(0,1)$**-order** if $\mathcal{E}(A)$ is a $(0,1)$-matrix.

Henceforth a (0, 1)-order will always mean a tiled (0, 1)-order over a discrete valuation ring $\mathcal{O}$.

With a reduced $(0,1)$-order $A$ we associate the partially ordered set

$$P_A = \{1, \dots, n\}$$

with the relation $\leqslant$ defined by $i \leqslant j \Leftrightarrow \alpha_{ij} = 0$.

Obviously, $(P, \leqslant)$ is a partially ordered set (poset).

Conversely, to any finite poset $P = \{1, \dots, n\}$ assign a reduced $(0,1)$-matrix $\mathcal{E}_p = (A_{ij})$ in the following way: $A_{ij} = 0 \Leftrightarrow i \leqslant j$, otherwise $A_{ij} = 1$. Then $A(P) = \{\mathcal{O}, \mathcal{E}_P\}$ is a reduced (0, 1)-order.

We give a construction which for a given finite partially ordered set $P = \{p_1, \dots, p_n\}$ yields a strongly connected quiver without multiple arrows and multiple loops.

Denote by $P_{max}$ (respectively $P_{min}$) the set of the maximal (respectively minimal) elements of P and by $P_{max} \times P_{min}$ their Cartesian product.

The quiver $\widetilde{Q}(P)$ obtained from the diagram $Q(P)$ by adding the arrows $\sigma_{ij}$ : $i \to j$ for all $(p_i, p_j) \in P_{max} \otimes P_{min}$ is called **the quiver associated with the partially ordered set** $P$.

Obviously, $\widetilde{Q}(P)$ is a strongly connected simply laced quiver.

**Theorem 3.34.** *The quiver $Q(A(P))$ coincides with the quiver $\widetilde{Q}(P)$.*

*Proof.* Recall that $[Q(A(P))] = \mathcal{E}(R^2) - \mathcal{E}(R)$. Suppose that in $Q(P)$ there is an arrow from $s$ to $t$. This means that $\alpha_{st} = 0$ and there is no positive integer $k$ ($k \neq s, t$) such that $\alpha_{sk} = 0$ and $\alpha_{kt} = 0$. The elements $\beta_{ss}$ and $\beta_{tt}$ of the exponent matrix $\mathcal{E}(R) = (\beta_{ij})$ are equal to 1. We have that $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = min_{1 \leqslant k \leqslant n}(\beta_{sk} + \beta_{kt}) = 1$. Thus, in $[Q(A(P))]$ at the $(s,t)$-th position we have $\gamma_{st} - \beta_{st} = 1 - \alpha_{st} = 1 - 0 = 1$. Consequently, $Q(A(P))$ has an arrow from $s$ to $t$.

Suppose that $p \in P_{max}$. This means that $\alpha_{pk} = 1$ for $k \neq p$. Therefore the entries of the $p$-th row of $\mathcal{E}(R)$ are all 1, i.e., $(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pn}) = (1, \ldots, 1, \ldots, 1)$.

Similarly, if $q \in P_{min}$, then the $q$-th column $(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T$ of $\mathcal{E}(R)$ is $(1, \ldots, 1, \ldots, 1)^T$. Hence, $\gamma_{pq} = 2$, $\beta_{pq} = 1$, and $Q(A(P))$ has an arrow from $p$ to $q$. Consequently, we proved that $\widetilde{Q}(P)$ is a subquiver of $Q(A(P))$.

We show now the converse inclusion. Suppose that $\gamma_{pq} = 2$. Then obviously

$$(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pq}) = (1, \ldots, 1, \ldots, 1)$$

and

$$(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T = (1, \ldots, 1, \ldots, 1)^T.$$

Therefore $p \in P_{max}, q \in P_{min}$ and there is an arrow, which goes from $p$ to $q$.

Suppose $\gamma_{pq} = 1$ and $\beta_{pq} = 0$. Consequently, $p \neq q$, $\beta_{pq} = \alpha_{pq} = 0$ and $p < q$. Since $\gamma_{pq} = min_{1 \leqslant k \leqslant n}(\beta_{pk} + \beta_{kp})$, then $\beta_{pk} + \beta_{kq} \geqslant 1$ for $k = 1, \ldots, n$. Thus, for $k \neq p, q$ we have $\beta_{pk} + \beta_{kq} \geqslant 1$, whence we obtain $\alpha_{pk} + \alpha_{kp} \geqslant 1$. Therefore, there is no positive integer $k$ ($k \neq p, q$) such that $\alpha_{pk} = \alpha_{kq} = 0$. This means that there is an arrow from $p$ to $q$ in $\widetilde{Q}(P)$, and this proves the opposite inclusion. □

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*Received December 6, 2010*