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Convex Quadrics

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Abstract. We introduce and describe convex quadrics in $\mathbb{R}^n$ and characterize them as convex hypersurfaces with quadric sections by a continuous family of hyperplanes.

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1 Introduction and main results

Characterizations of ellipses and ellipsoids among convex bodies in the plane or in space became an established topic of convex geometry on the turn of 20th century. Comprehensive surveys on various characteristic properties of ellipsoids in the Euclidean space $\mathbb{R}^n$ are presented in [9] and [13] (see also [10]). Similar characterizations of unbounded convex quadrics, like paraboloids, sheets of elliptic hyperboloids or elliptic cones, are given by a short list of sporadic results (see, e. g., [1, 2, 15, 16]). Furthermore, even a classification of convex quadrics in $\mathbb{R}^n$ for $n \geq 4$ is not established (although it is used in [15, 16] without proof). Our goal here is to introduce and to describe convex quadrics in $\mathbb{R}^n$ and to provide a characteristic property of these hypersurfaces in terms of hyperplane sections.

In what follows, by a *convex solid* we mean an n-dimensional closed convex set in $\mathbb{R}^n$, distinct from the entire space (convex *bodies* are compact convex solids). As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and interior of a convex solid K. A *convex hypersurface* (a *surface* if $n = 3$ or a *curve* if $n = 2$) is the boundary of a convex solid. This definition includes a hyperplane or a pair of parallel hyperplanes.

In a standard way, a *quadric hypersurface* (or a *second degree hypersurface*) in $\mathbb{R}^n$, $n \geq 2$, is the locus of points $x = (\xi_1, \ldots, \xi_n)$ that satisfy a quadratic equation

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{1}$$

where not all a_{ik} are zero. We say that a convex hypersurface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric hypersurface $Q \subset \mathbb{R}^n$ and a convex component U of $\mathbb{R}^n \setminus Q$ such that S is the boundary of U. This definition allows us to include into considerations convex hypersurfaces like sheets of elliptic cones and sheets of elliptic hyperbolids, and not only ellipsoids and elliptic paraboloids.

The following theorem plays a key role in the description of convex quadrics.

Theorem 1. *The complement of a real quadric hypersurface $Q \subset \mathbb{R}^n$, $n \geq 2$, is the disjoint union of four or fewer open sets; at least one of these components is convex if and only if the canonical form of Q is given by one of the equations*

$$\begin{aligned}
&a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, && 1 \leq k \leq n,\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1, && 2 \leq k \leq n,\\
&a_1\xi_1^2 = 0, &&\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 2 \leq k \leq n,
\end{aligned}$$

where all scalars a_i involved are positive.

Corollary 1. *A convex hypersurface $S \subset \mathbb{R}^n$, $n \geq 2$, is a convex quadric if and only if S can be described in suitable Cartesian coordinates $\xi_1, \dots, \xi_n$ by one of the conditions*

$$\begin{aligned}
&a_1\xi_1^2 + \cdots + a_k\xi_k^2 = 1, && 1 \leq k \leq n,\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 1,\ \xi_1 \geq 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 = 0, &&\\
&a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 = 0,\ \xi_1 \geq 0, && 2 \leq k \leq n,\\
&a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 = \xi_k, && 2 \leq k \leq n,
\end{aligned}$$

where all scalars a_i involved are positive.

In what follows, a *plane* of dimension m in $\mathbb{R}^n$ is a translate of an m-dimensional subspace. We say that a plane L *properly* intersects a convex solid K provided L intersects both sets $\operatorname{bd} K$ and $\operatorname{int} K$.

A well-known result of convex geometry states that the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses (see [3, 12] for $n = 3$ and [7, pp. 91–92] for $n \geq 3$). This result is generalized in [15] by showing that the boundary of a convex solid $K \subset \mathbb{R}^n$ is a convex quadric if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are convex quadric curves. In this regard, we pose the following problem (solved in [6, 11] for the case of convex bodies).

Problem 1. *Given a convex solid $K \subset \mathbb{R}^n$, $n \geq 3$, and a point $p \in \mathbb{R}^n$, is it true that either* $\operatorname{bd} K$ *is a convex quadric or K is a convex cone with apex p provided all proper sections of* $\operatorname{bd} K$ *by 2-dimensional planes through p are convex quadric curves?*

Kubota [12] proved that, given a pair of bounded convex surfaces in $\mathbb{R}^3$, one being enclosed by the other, if all planar sections of the bigger surface by planes tangent to the second surface are ellipses, then the containing surface is an ellipsoid. Independently, Bianchi and Gruber [4] established the following far-reaching assertion: If K is a convex body in $\mathbb{R}^n$, $n \geq 3$, and $\delta(u)$ is a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ intersects $\operatorname{bd} K$ along an $(n-1)$-dimensional ellipsoid, then $\operatorname{bd} K$ is an ellipsoid. Our second theorem extends this assertion to the case of convex solids.

Theorem 2. *Let K be a convex solid in $\mathbb{R}^n$, $n \geq 3$, and $\delta(u)$ be a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ either lies in K or intersects* $\operatorname{bd} K$ *along an $(n-1)$-dimensional convex quadric. Then* $\operatorname{bd} K$ *is a convex quadric.*

2 Proof of Theorem 1

Let $Q \subset \mathbb{R}^n$ be a real quadric hypersurface. Choosing a suitable orthogonal basis, we may suppose that Q has one of the following canonical forms:

$$\begin{array}{lll}
A_k & : \xi_1^2 + \cdots + \xi_k^2 = 1, & 1 \leq k \leq n, \\
B_{k,r} & : \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_r^2 = 1, & 1 \leq k < r \leq n, \\
C_k & : \xi_1^2 + \cdots + \xi_k^2 = 0, & 1 \leq k \leq n, \\
D_{k,r} & : \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_r^2 = 0, & 1 \leq k < r \leq n, \\
E_{k,r} & : \xi_1^2 + \cdots + \xi_k^2 - \xi_{k+1}^2 - \cdots - \xi_{r-1}^2 = \xi_r, & 1 \leq k < r \leq n.
\end{array}$$

First, we exclude the trivial cases $Q = A_1$ (when Q is a pair of parallel hyperplanes) and $Q = C_k$ (when Q is an $(n-k)$-dimensional subspace). Furthermore, the proof can be reduced to the case when Q has one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$, since otherwise Q is a cylinder generated by a lower-dimensional quadric of the same type.

We are going to express each of the hypersurfaces A_n, $B_{k,n}$, $D_{k,n}$, $E_{k,n}$ as the set of revolution of a respective lower-dimensional surface. To describe these revolutions, choose any subspaces L_1, L_2, and L_3 of $\mathbb{R}^n$ such that $L_1 \subset L_2 \subset L_3$ and

$$\dim L_1 = m - 1, \quad \dim L_2 = m, \quad \dim L_3 = m + 1, \quad 2 \leq m \leq n - 1.$$

Let M be the 2-dimensional subspace of L_3 orthogonal to L_1. Given a point $y \in L_2$, put $M_y = y + M$ and denote by z the point of intersection of L_1 and M_y (z is the orthogonal projection of y on L_1). Let C_y be the circumference in M_y with center z and radius $\|y - z\|$. We say that a set $X \subset L_3$ is the *set of revolution* of a set $Y \subset L_2$ about L_1 within L_3 provided $X = \cup_{y \in Y} C_y$. A set $Z \subset \mathbb{R}^n$ is called *symmetric* about a subspace $N \subset \mathbb{R}^n$ if for any point $x \in Z$ and its orthogonal projection u on N, the point $2u - x$ lies in Z.

In these terms, we formulate three lemmas (the first one being obvious). In what follows, $\langle e_1, \ldots, e_k \rangle$ means the span of vectors $e_1, \ldots, e_k$.

Lemma 1. *If Y is a subset of L_2 and X is the set of revolution of Y about L_1 within L_3, then X is symmetric about L_2 and any component of X is the set of revolution of a suitable component of Y about L_1 within L_3.* □

Lemma 2. *If a set $Y \subset L_2$ is symmetric about L_1 and X is the set of revolution of Y about L_1 within L_3, then X is a convex set if and only if Y is a convex set.*

Proof. Without loss of generality, we may put $L_3 = \mathbb{R}^n$. Choose an orthonormal basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ such that

$$L_1 = \langle e_1, \dots, e_{n-2} \rangle \quad \text{and} \quad L_2 = \langle e_1, \dots, e_{n-1} \rangle.$$

Clearly, $x = (\xi_1, \dots, \xi_n)$ belongs to X if and only if there is a point

$$y = (\xi_1, \dots, \xi_{n-2}, \xi'_{n-1}, 0) \in Y \quad \text{where} \quad \xi'_{n-1} = \sqrt{\xi_{n-1}^2 + \xi_n^2}.$$

If X is convex, then Y is convex due to $Y = X \cap L_2$. Conversely, let Y be convex. Choose any points $a = (\alpha_1, \dots, \alpha_n)$ and $b = (\beta_1, \dots, \beta_n)$ in X and a scalar $\lambda \in [0, 1]$. We intend to show that $c = (1 - \lambda)a + \lambda b \in X$. Let

$$a' = (\alpha_1, \dots, \alpha_{n-2}, \alpha'_{n-1}, 0), \quad b' = (\beta_1, \dots, \beta_{n-2}, \beta'_{n-1}, 0),$$

and

$$c' = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}, 0)$$

be points in Y, where

$$\alpha'_{n-1} = \sqrt{\alpha_{n-1}^2 + \alpha_n^2}, \quad \text{and} \quad \beta'_{n-1} = \sqrt{\beta_{n-1}^2 + \beta_n^2}.$$

Then $a', b' \in Y$ and $c' = (1 - \lambda)a' + \lambda b' \in Y$ due to convexity of Y. Because Y is symmetric about L_1, we have

$$((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \mu, 0) \in Y$$

for any scalar μ with $|\mu| \le (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$. Let

$$y = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \rho, 0),$$

where

$$\rho = \sqrt{\big((1 - \lambda)\alpha_{n-1} + \lambda\beta_{n-1}\big)^2 + \big((1 - \lambda)\alpha_n + \lambda\beta_n\big)^2}.$$

From $\alpha_{n-1}\beta_{n-1} + \alpha_n\beta_n \le \alpha'_{n-1}\beta'_{n-1}$, we obtain $\rho \le (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$, which gives $y \in Y$. Clearly, the point

$$z = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, 0, 0)$$

is the orthogonal projection of y on L_1. The equalities $\|c - z\| = \|y - z\| = \rho$ imply that $c \in C_y \subset X$. Hence X is convex. □

Lemma 3. *Within $\mathbb{R}^n$, $n \ge 3$, we have*

1) *A_n is the set of revolution of $A_{n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ about $\langle e_1, \dots, e_{n-2} \rangle$,*

2) *$B_{k,n}$ is the set of revolution of $B_{k,n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ about $\langle e_1, \dots, e_{n-2}\rangle$, $1 \le k \le n-2$,*

3) *$D_{k,n}$ is the set of revolution of $D_{k,n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ about $\langle e_1, \dots, e_{n-2}\rangle$, $1 \le k \le n-2$,*

4) *$B_{k,n}$ is the set of revolution of $B_{k-1,n-1} \subset \langle e_2, \dots, e_n\rangle$ about $\langle e_3, \dots, e_n\rangle$, $2 \le k \le n-1$,*

5) *$D_{k,n}$ is the set of revolution of $D_{k-1,n-1} \subset \langle e_2, \dots, e_n\rangle$ about $\langle e_3, \dots, e_n\rangle$, $2 \le k \le n-1$.* □

Proof. 1) Given a point $x = (\xi_1, \dots, \xi_n) \in A_n$, put

$$y = (\xi_1, \dots, \xi_{n-2}, \sqrt{\xi_{n-1}^2 + \xi_n^2}, 0), \quad z = (\xi_1, \dots, \xi_{n-2}, 0, 0). \tag{2}$$

Then $y \in A_{n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ and z is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2}\rangle$. From

$$\|x - z\| = \|y - z\| = \sqrt{\xi_{n-1}^2 + \xi_n^2}$$

we see that $x \in C_y$. So, A_n lies in the revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2}\rangle$. Conversely, if $y = (\eta_1, \dots, \eta_{n-1}, 0)$ is a point in $A_{n-1} \subset \langle e_1, \dots, e_{n-1}\rangle$ and $z = (\eta_1, \dots, \eta_{n-2}, 0, 0)$ is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2}\rangle$, then any point u from the circle $C_y \subset y + \langle e_{n-1}, e_n\rangle$ can be written as

$$u = (\eta_1, \dots, \eta_{n-2}, \gamma_{n-1}, \gamma_n), \quad \text{where} \quad \gamma_{n-1}^2 + \gamma_n^2 = \eta_{n-1}^2.$$

Clearly, $u \in A_n$, which shows that A_n contains the set of revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2}\rangle$.

Cases 2)–5) are considered similarly, where the points y and z are defined, respectively, by (2) in cases 2) and 3), and by

$$y = (0, \sqrt{\xi_1^2 + \xi_2^2}, \xi_3, \dots, \xi_n), \quad z = (0, 0, \xi_3, \dots, \xi_n)$$

in cases 4) and 5). □

Proof of Theorem 1. Our further consideration is organized by induction on n. The cases $n = 2$ and $n = 3$ follow immediately from the well-known properties of quadric curves and surfaces. Suppose that $n \ge 4$. Assuming that the conclusion of Theorem 1 holds for all $m < n$, let the quadric hypersurface $Q \subset \mathbb{R}^n$ have one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$. We consider these forms separately.

Case 1. Let $Q = A_n$. By Lemma 3, A_n can be obtained from

$$A_2 = \{(\xi_1, \xi_2) \mid \xi_1^2 + \xi_2^2 = 1\} \subset \langle e_1, e_2\rangle$$

by consecutive revolutions of $A_i \subset \langle e_1, \dots, e_i\rangle$ about $\langle e_1, \dots, e_{i-1}\rangle$ within the subspace $\langle e_1, \dots, e_{i+1}\rangle$, $i = 2, \dots, n-1$. Since both components of $\langle e_1, e_2\rangle \setminus A_2$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus A_n$ consists of two components; one of them, given by $\xi_1^2 + \cdots + \xi_n^2 < 1$, is convex.

Case 2. Let $Q = B_{k,n}$, $1 \le k \le n-1$. If $k = 1$, then Lemma 3 implies that $B_{1,n}$ can be obtained from

$$B_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 1\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $B_{1,i} \subset \langle e_1, \ldots, e_i \rangle$ about $\langle e_1, \ldots, e_{i-1} \rangle$ within the subspace $\langle e_1, \ldots, e_{i+1} \rangle$, $i = 2, \ldots, n-1$. Since all three components of $\langle e_1, e_2 \rangle \setminus B_{1,2}$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus B_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \cdots + \xi_n^2 + 1} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \cdots + \xi_n^2 + 1},$$

are convex. If $k \ge 2$, then $B_{k,n}$ can be obtained from

$$B_{1,2} = \{(\xi_k, \xi_{k+1}) \mid \xi_k^2 - \xi_{k+1}^2 = 1\} \subset \langle e_k, e_{k+1} \rangle$$

in two steps. First, we obtain $B_{k,k+1} \subset \mathbb{R}^{k+1} = \langle e_1, \ldots, e_{k+1} \rangle$ by consecutive revolutions of $B_{i,i+1} \subset \langle e_{k+1-i}, e_{k+2-i}, \ldots, e_{k+1} \rangle$ about $\langle e_{k+2-i}, \ldots, e_{k+1} \rangle$ within $\langle e_{k-i}, e_{k+1-i}, \ldots, e_{k+1} \rangle$, $i = 1, 2, \ldots, k-1$. The complement of

$$B_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 1\}$$

in $\langle e_{k-1}, e_k, e_{k+1} \rangle$, consists of two components, both symmetric about $\langle e_k, e_{k+1} \rangle$. Since none of these components is convex, Lemmas 1 and 2 imply that $\mathbb{R}^{k+1} \setminus B_{k,k+1}$ consists of two components, both symmetric about any k-dimensional coordinate subspace of $\mathbb{R}^{k+1}$, but none of them convex.

Second, we obtain $B_{k,n}$ from $B_{k,k+1}$ by consecutive revolutions of $B_{k,j} \subset \langle e_1, \ldots, e_j \rangle$ about $\langle e_1, \ldots, e_{j-1} \rangle$ within $\langle e_1, \ldots, e_{j+1} \rangle$, $j = k+1, \ldots, n-1$. As above, $\mathbb{R}^n \setminus B_{k,n}$ consists of two components, none of them convex.

Case 3. Let $Q = D_{k,n}$, $1 \le k \le n-1$. If $k = 1$, then $D_{1,n}$ can be obtained from

$$D_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 0\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $D_{1,i} \subset \langle e_1, \ldots, e_i \rangle$ about $\langle e_1, \ldots, e_{i-1} \rangle$ within the subspace $\langle e_1, \ldots, e_{i+1} \rangle$, $i = 2, \ldots, n-1$. The complement of

$$D_{1,3} = \{(\xi_1, \xi_2, \xi_3) \mid \xi_1^2 - \xi_2^2 + \xi_3^2 = 0\}$$

in $\langle e_1, e_2, e_3 \rangle$ consists of tree components, all symmetric about $\langle e_1, e_2 \rangle$. Since two of these components are convex, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus D_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \cdots + \xi_n^2} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \cdots + \xi_n^2},$$

are convex.

Since the case $k = n-1$ is reducible to that of $k = 1$ (by reordering $e_1, e_2, \dots, e_n$ as $e_n, e_{n-1}, \dots, e_1$), we may assume that $2 \le k \le n-2$. Then $D_{k,n}$ can be obtained from

$$D_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 0\} \subset \langle e_{k-1}, e_k, e_{k+1}\rangle$$

in two steps. First, we obtain $D_{2,n-k+2} \subset \langle e_{k-1}, e_k, \dots, e_n\rangle$ by consecutive revolutions of $D_{2,i} \subset \langle e_{k-1}, e_k, \dots, e_i\rangle$ about $\langle e_{k-1}, e_k, \dots, e_{i-1}\rangle$ within $\langle e_{k-1}, e_k, \dots, e_{i+1}\rangle$, $i = k+1, \dots, n-1$. Clearly, $\langle e_{k-1}, e_k, e_{k+1}\rangle \setminus D_{2,3}$ consists of three components; two of them,

$$\xi_{k+1} > \sqrt{\xi_{k-1}^2 + \xi_k^2} \quad \text{and} \quad -\xi_{k+1} < \sqrt{\xi_{k-1}^2 + \xi_k^2},$$

are convex and symmetric to each other about $\langle e_{k-1}, e_k\rangle$. Hence $\langle e_{k-1}, e_k, e_{k+1}, e_{k+2}\rangle \setminus D_{3,4}$ consists of two components, none of them convex. Lemmas 1 and 2 imply that $\mathbb{R}^{n-k+2} \setminus D_{2,n-k+2}$ consists of two components, none of them convex.

Next, we obtain $D_{k,n}$ from $D_{2,n-k+2}$ by consecutive revolutions of the surface $D_{i,n-k+i} \subset \langle e_{k-i+1}, \dots, e_n\rangle$ about $\langle e_{k-i+2}, \dots, e_n\rangle$ within $\langle e_{k-i}, \dots, e_n\rangle$, $i = 2, \dots, k-1$. As above, $\mathbb{R}^n \setminus D_{k,n}$ consists of two components, none of them convex.

Case 4. Let $Q = E_{k,n}$, $1 \le k \le n-1$. Clearly, $E_{k,n}$ is the graph of a real-valued function φ on $\mathbb{R}^{n-1} = \langle e_1, \dots, e_{n-1}\rangle$, given by

$$\xi_n = \varphi(\xi_1, \dots, \xi_{n-1}) = \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_{n-1}^2.$$

Hence $\mathbb{R}^n \setminus E_{k,n}$ has two components. The Hessian $\left(\frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}\right)$ is a diagonal $n \times n$-matrix, with 2's on its first k diagonal entries and -2's on the other $n-k-1$ diagonal entries. Therefore, φ is not concave, being convex if and only if $k = n-1$. So, $\mathbb{R}^n \setminus E_{k,n}$ has a convex component if and only if $k = n-1$; this component is given by $\xi_1^2 + \dots + \xi_{n-1}^2 < \xi_n$. □

3 Proof of Theorem 2

In what follows, the origin of $\mathbb{R}^n$ is denoted by o. We say that a plane L *supports* a convex solid K provided L intersects K such that $L \cap \operatorname{int} K = \varnothing$. The *recession cone* of K is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n \mid x + \alpha e \in K \text{ for all } x \in K \text{ and } \alpha \ge 0\}.$$

It is well-known that $\operatorname{rec} K \ne \{o\}$ if and only if K is unbounded; K is called *line-free* if it contains no line. Finally, $\operatorname{rint} M$ and $\operatorname{rbd} M$ denote the relative interior and the relative boundary of a convex set $M \subset \mathbb{R}^n$.

Under the assumptions of Theorem 2, we divide the proof into a sequence of lemmas.

Lemma 4. *If K contains a line, then* $\operatorname{bd} K$ *is a convex quadric cylinder.*

Proof. If l is a line in K, then K is the direct sum $\langle u_0 \rangle \oplus (K \cap H(u_0))$, where $\langle u_0 \rangle$ is the 1-dimensional subspace spanned by a unit vector u_0 parallel to l. By the assumption, $\operatorname{bd} K \cap H(u_0)$ is an $(n-1)$-dimensional convex quadric. Hence $\operatorname{bd} K = \langle u_0 \rangle \oplus (\operatorname{bd} K \cap H(u_0))$ is a convex quadric cylinder. □

Due to Lemma 4, we may further assume that K is line-free. Then no hyperplane lies in K; so, every hyperplane $H(u)$, $u \in S^{n-1}$, properly intersects K.

Lemma 5. *For any $(n-2)$-dimensional plane L supporting K, there is a hyperplane $H(u)$, $u \in S^{n-1}$, that contains L.*

Proof. Let P be the 2-dimensional subspace orthogonal to L and π be the orthogonal projection of $\mathbb{R}^n$ on P. Clearly, the intersection $L \cap P$ is a singleton, say $\{v\}$. The set $M = \pi(K)$ is convex, $\operatorname{rint} M = \pi(\operatorname{int} K)$, and $v \in \operatorname{rbd} M$. Choose an orientation in P and denote by l a line in P that supports M at v. Let u_0 be the unit vector in P orthogonal to l such that u_0 is an outward unit normal to M at v. Let m denote the line through v orthogonal to l, and T be the open halfplane of P bounded by l and disjoint from M.

Assume, for contradiction, that no line $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, contains v. In particular, the line $l(u_0)$ is distinct from l. Continuously rotating the unit vector u from the initial position u_0 in a positive direction along $P \cap S^{n-1}$, we obtain a continuous family of lines each of them missing v. This is possible only if the parallel lines $l(u_0)$ and $l(-u_0)$ intersect m at points that belong to the opposite open halflines with common apex v. Hence one of the lines $l(u_0), l(-u_0)$ entirely lies in T, thus missing M, which is impossible due to $K \cap H(u_0) \neq \varnothing$ and $K \cap H(-u_0) \neq \varnothing$. □

We recall that a convex solid $K \subset \mathbb{R}^n$ is called *strictly convex* if $\operatorname{bd} K$ contains no line segments. Furthermore, K is called *regular* provided any point $x \in \operatorname{bd} K$ belongs to a unique hyperplane supporting K.

Lemma 6. *If K is neither strictly convex nor regular, then* $\operatorname{bd} K$ *is a sheet of an elliptic cone.*

Proof. First, we are going to show that if K is not regular, then K is not strictly convex. Indeed, suppose that K is not regular and choose a singular point $x \in \operatorname{bd} K$. Then there are distinct hyperplanes G_1 and G_2 both supporting K at x. Choose a hyperplane G through $G_1 \cap G_2$ supporting K and different from each of G_1 and G_2. Let $L \subset G$ be an $(n-2)$-dimensional plane through x which is distinct from $G_1 \cap G_2$. By Lemma 5, there is a hyperplane $H(u)$ containing L. Because $H(u)$ meets $\operatorname{int} K$, the point x is singular for the $(n-1)$-dimensional convex quadric $E(u) = \operatorname{bd} K \cap H(u)$. According to Corollary 1, $E(u)$ must be a sheet of an $(n-1)$-dimensional elliptic cone. Choosing a line segment in $E(u)$, we conclude that K is not strictly convex.

Now, assume that K is not strictly convex and choose a line segment $[x, z] \subset \operatorname{bd} K$. By Lemma 5, there is a hyperplane $H(u_0)$ containing the line through x and z. Since the $(n-1)$-dimensional convex quadric $E(u_0) = \operatorname{bd} K \cap H(u_0)$ is line-free and contains a line segment, it should be a sheet of an $(n-1)$-dimensional elliptic cone. Let v be the apex of $E(u_0)$. Denote by h_1 the halfline $[v, x)$ and choose another halfline $h_2 = [v, w) \subset E(u_0)$ such that the 2-dimensional plane through $h_1 \cup h_2$ intersects $\operatorname{int} K$ (this is possible since $H(u_0)$ meets $\operatorname{int} K$). Let P_2 be a hyperplane supporting K with the property $h_2 \subset P_2$. By the above, $h_1 \not\subset P_2$.

Choose a halfline h with apex v tangent to K and so close to h_1 that $h \not\subset P_2$. Let P be a hyperplane through h which supports K. By Lemma 5, there is a hyperplane $H(u)$ that meets $\operatorname{int} K$ and contains h. Since the section $E(u) = \operatorname{bd} K \cap H(u)$ is bounded by both P and P_2, the point v is singular for $E(u)$. As above, $E(u)$ is a sheet of an $(n-1)$-dimensional elliptic cone. Hence $h \subset \operatorname{bd} K$. Varying h and h_2, we obtain by the argument above that every tangent halfline of K at v lies in $\operatorname{bd} K$. This shows that K is a convex cone with apex v. Finally, choose a hyperplane $H(u_1)$ that properly intersects K along a bounded set (this is possible since K is line-free). By the assumption, $\operatorname{bd} K \cap H(u_1)$ is an $(n-1)$-dimensional ellipsoid. So, $\operatorname{bd} K$ is a sheet of an elliptic cone with apex v generated by $\operatorname{bd} K \cap H(u_1)$. ☐

Lemma 7. *Let K be strictly convex and regular. There are hyperplanes $H(u_1)$ and $H(u_2)$, $u_1, u_2 \in S^{n-1}$, such that both sections $\operatorname{bd} K \cap H(u_1)$ and $\operatorname{bd} K \cap H(u_2)$ are $(n-1)$-dimensional ellipsoids whose intersection is an $(n-2)$-dimensional ellipsoid.*

Proof. Since K is line-free, there is a 2-dimensional subspace P such that the orthogonal projection, M, of K on P is a line-free closed convex set (see, e.g., [14]). Choose any orientation in P. Denote by $\mathcal{F}$ the family of lines $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, such that $M \cap l(u)$ is bounded. Let $l(u_0)$ be one of these lines. Put $[v, w] = M \cap l(u_0)$. The line $l(u_0)$ cuts M into 2-dimensional closed convex subsets, M' and M'', at least one of them, say M', being compact. If there is a line $l(u) \in \mathcal{F}_0 = \mathcal{F} \setminus \{l(u_0)\}$ which intersects the open line segment $]v, w[$, then the respective hyperplanes $H(u)$ and $H(u_0)$ have the desired property.

Assume that no line $l(u) \in \mathcal{F}_0$ intersects $]v, w[$. We state that no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M'$. Indeed, if a line $l(u_1) \in \mathcal{F}_0$ intersected $\operatorname{rint} M'$, then, rotating u about $P \cap S^{n-1}$ from the initial position u_1, we would find a line $l(u_2)$ supporting M at v or at w (which is impossible since $\operatorname{int} K \cap H(u_2) \neq \varnothing$). In a similar way, no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M''$ if M'' is bounded.

This argument shows that M'' should be unbounded, since otherwise no line $l(u) \in \mathcal{F}_0$ intersects $\operatorname{rint} M = \operatorname{rint} M' \cup \operatorname{rint} M'' \cup]v, w[$, which is impossible due to $\operatorname{int} K \cap H(u_2) \neq \varnothing$. Rotating u about $P \cap S^{n-1}$ in a positive direction from the initial position u_0, we observe that the lines $l(u) \in \mathcal{F}_0$ cover the whole unbounded branch of $\operatorname{rbd} M''$ with endpoint v. Rotating u about $P \cap S^{n-1}$ in a negative direction from the initial position u_0, we see that the lines $l(u) \in \mathcal{F}_0$ cover the second unbounded branch of $\operatorname{rbd} M''$, with endpoint w. This implies the existence of lines $l(u_3), l(u_4) \in \mathcal{F}_0$ such that the line segments $M \cap l(u_3)$ and $M \cap l(u_4)$ have a common interior point. The respective $(n-1)$-dimensional ellipsoids $\operatorname{bd} K \cap H(u_3)$ and $\operatorname{bd} K \cap H(u_4)$ satisfy the conclusion of the lemma. ☐

Lemma 8. *Let K be strictly convex and regular. If* $\operatorname{bd} K$ *contains an open piece of a real quadric hypersurface, then* $\operatorname{bd} K$ *is a convex quadric.*

Proof. Let A be an open piece of a real quadric hypersurface $Q \subset \mathbb{R}^n$ which lies in $\operatorname{bd} K$. We state that $\operatorname{bd} K \subset Q$. Assume, for contradiction, that $\operatorname{bd} K \not\subset Q$, and choose a maximal (under inclusion) open piece B of $\operatorname{bd} K \cap Q$ that contains A. Let $U_r(x) \subset \mathbb{R}^n$ be an open ball with center $x \in B$ and radius $r > 0$ such that $\operatorname{bd} K \cap U_r(x) \subset B$. Continuously moving x towards $\operatorname{bd} K \setminus B$, we find points $x_0 \in B$ and $z_0 \in \operatorname{bd} K \setminus B$ with the property $\operatorname{bd} K \cap U_r(x_0) \subset B$ and $\|x_0 - z_0\| = r$.

Let G be the hyperplane through z_0 which supports K (G is unique since K is regular). Denote by $\mathcal{G}$ the family of $(n-2)$-dimensional planes $L \subset G$ that contain z_0 and are distinct from the $(n-2)$-dimensional plane $L_0 \subset G$ tangent to $U_r(x_0) \cap G$ at z_0. Due to Lemma 5, any plane $L \in \mathcal{G}$ lies in a respective hyperplane $H_L(u)$. By continuity, there is a scalar $t > 0$ so small that the union of $(n-1)$-dimensional convex quadrics $E_L(u) = \operatorname{bd} K \cap H_L(u)$, $L \in \mathcal{G}$, is dense in the hypersurface t-neighborhood $\operatorname{bd} K \cap U_t(z_0)$ of z_0. Each $E_L(u)$ has a nontrivial strictly convex intersection with B. Since $E_L(u)$ is a unique convex quadric containing $E_L(u) \cap B$, we conclude that $E_L(u) \subset Q$. By continuity,

$$\operatorname{bd} K \cap U_t(z_0) \subset \operatorname{cl}\big(\bigcup_{L \in \mathcal{G}} E_L(u)\big) \subset Q.$$

Hence $\operatorname{bd} K \cap U_t(z_0) \subset B$, contrary to the choice of $z_0 \in \operatorname{bd} K \setminus B$. Thus $\operatorname{bd} K \subset Q$. Because $\operatorname{int} K$ is a convex component of $\mathbb{R}^n \setminus Q$, the hypersurface $\operatorname{bd} K$ is a convex quadric. □

Lemma 9. *Let E_1 and E_2 be $(n-1)$-dimensional ellipsoids in $\mathbb{R}^n$, $n \geq 3$, which lie, respectively, in hyperplanes H_1 and H_2 of $\mathbb{R}^n$ such that $E = E_1 \cap E_2$ is an $(n-2)$-dimensional ellipsoid. For any point $v \in \mathbb{R}^n \setminus (H_1 \cup H_2)$, there is a quadric hypersurface Q that contains $\{v\} \cup E_1 \cup E_2$.*

Proof. Choose an orthonormal basis for $\mathbb{R}^n$ such that

$$\begin{aligned} E &= \{(0,0,\xi_3,\dots,\xi_n) \mid \xi_3^2 + \cdots + \xi_n^2 = 1\}, \\ E_1 &= \{(\xi_1,0,\xi_3,\dots,\xi_n) \mid (\xi_1 - \rho_1)^2 + \xi_3^2 + \cdots + \xi_n^2 = \rho_1^2 + 1\}, \\ E_2 &= \{(0,\xi_2,\xi_3,\dots,\xi_n) \mid (\xi_2 - \rho_2)^2 + \xi_3^2 + \cdots + \xi_n^2 = \rho_2^2 + 1\}, \end{aligned}$$

where $\rho_1 > 0$ and $\rho_2 > 0$. Then H_1 and H_2 are described by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively. Consider the family of quadric hypersurfaces $Q(\mu) \subset \mathbb{R}^n$ given by

$$\xi_1^2 + \cdots + \xi_n^2 + 2\mu\xi_1\xi_2 - 2\rho_1\xi_1 - 2\rho_2\xi_2 - 1 = 0,$$

where $\mu \in \mathbb{R}$. We have $E_i = H_i \cap Q(\mu)$, $i = 1, 2$. The point $v = (\nu_1, \dots, \nu_n)$ belongs to $\mathbb{R}^n \setminus (H_1 \cup H_2)$ if and only if $\nu_1\nu_2 \neq 0$. Then $v \in Q(\mu_0)$ provided

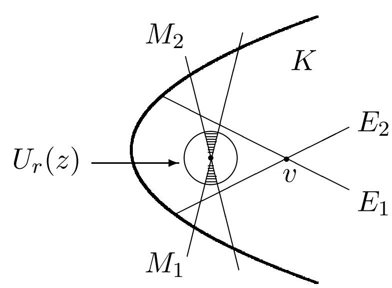
$$\mu_0 = (1 + 2\rho_1\nu_1 + 2\rho_2\nu_2 - \nu_1^2 - \cdots - \nu_n^2)/(2\nu_1\nu_2). \qquad \square$$

Lemma 10. *If K is strictly convex and regular, then* $\operatorname{bd} K$ *contains an open piece of a quadric hypersurface.*

Proof. We proceed by induction on $n\,(\geq 3)$. Let $n = 3$. By Lemma 7, there are planes $H(u_1)$ and $H(u_2)$ such that both sections $E_1 = \operatorname{bd} K \cap H(u_1)$ and $E_2 = \operatorname{bd} K \cap H(u_2)$ are ellipses, with precisely two points, say v and w, in common. The set $\operatorname{bd} K \setminus (E_1 \cup E_2)$ consists of four open pieces, at least three of them being bounded because K is line-free. We choose any of these pieces if K is bounded, and choose the piece opposite to the unbounded one if K is unbounded. Denote by Γ the chosen piece. Let L be a plane through $[v, w]$ that misses Γ and is distinct from both $H(u_1)$ and $H(u_2)$. There is a neighborhood $\Omega \subset \operatorname{bd} K$ of v such that for any point $z \in \Gamma \cap \Omega$, the plane L_z through z parallel to L intersects each of the ellipses E_1 and E_2 at two distinct points.

Choose a point $z \in \Gamma \cap \Omega$ and denote by P_z the plane through z that supports K (P_z is unique since K is regular), and by l_z the line through z parallel to $[v, w]$. Let $\mathcal{F}_\alpha$, $\alpha > 0$, be the family of planes through l_z forming with L_z an angle of size α or less. By continuity, the neighborhood Ω and the scalar α can be chosen so small that for any given plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects each of the ellipses E_1 and E_2 at two distinct points. Furthermore, we can find a scalar $r > 0$ such that for any plane $H(u)$ trough z, the convex quadric curve $\operatorname{bd} K \cap H(u)$ intersects the closed curve $\operatorname{bd} K \cap S_r(z)$ at two points, where $S_r(z) \subset \mathbb{R}^3$ is the sphere of radius r centered at z.

Due to Lemma 9, there is a quadric surface Q containing $\{z\} \cup E_1 \cup E_2$. By the above, given a plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects $\operatorname{bd} K$ along an ellipse, which has five points in Q (namely, z and two on each ellipse E_i, $i = 1, 2$). Since an ellipse is uniquely defined by five points in general position, the ellipse $E(u) = \operatorname{bd} K \cap H(u)$ lies in Q for any choice of a plane $H(u)$ through the line $M \cap P_z$, where $M \in \mathcal{F}_\alpha$. This argument shows the existence of two open "triangular" regions in $\operatorname{bd} K \cap Q \cap U_r(z)$ which have a common vertex z and are bounded by a pair of planes $M_1, M_2 \in \mathcal{F}_\alpha$ (see the shaded sectors of $\operatorname{bd} K \cap U_r(z)$ in the figure below). Hence the case $n = 3$ is proved.



Suppose that the inductive statement holds for all $m \leq n - 1$, $n \geq 4$, and let $K \subset \mathbb{R}^n$ be a line-free, strictly convex and regular solid that satisfies the hypothesis of Theorem 2. Since the case when K is compact is proved in [4], we may assume that K is unbounded. Then the recession cone $\operatorname{rec} K$ contains halflines and is line-free. Choose a halfline $h \subset \operatorname{rec} K$ with endpoint o such that the $(n-1)$-dimensional subspace $L \subset \mathbb{R}^n$ orthogonal to h satisfies the condition $L \cap \operatorname{rec} K = \{o\}$. Then any proper section of K by a hyperplane parallel to L is bounded (see, e.g., [14]).

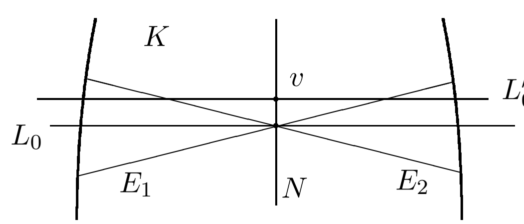
Because the set $\Delta = \{\delta(u)u \mid u \in L \cap S^{n-1}\}$ is compact, we can choose a hyperplane L_0 parallel to L and properly intersecting K so far from Δ that every hyperplane $H(u)$, $u \in L \cap S^{n-1}$, intersects $\operatorname{rint}(K \cap L_0)$. Since any section

$$\operatorname{bd} K \cap H(u) \cap L_0, \quad u \in L \cap S^{n-1},$$

is an $(n-2)$-dimensional convex quadric, $K \cap L_0$ satisfies the hypothesis of Theorem 2 (with L_0 instead of $\mathbb{R}^n$). By the inductive assumption, $\operatorname{rbd}(K \cap L_0)$ contains a relatively open piece of an $(n-1)$-dimensional quadric, and Lemma 8 implies that $\operatorname{bd} K \cap L_0$ is an $(n-1)$-dimensional ellipsoid. Let $G \subset L_0$ be an $(n-2)$-dimensional plane through the center of $K \cap L_0$. By continuity and the argument above, there is an $\varepsilon > 0$ such that the hyperplanes L_1 and L_2 through G forming with L_0 an angle of size ε also intersect $\operatorname{bd} K$ along $(n-1)$-dimensional ellipsoids E_1 and E_2, respectively. Denote by N the hyperplane through G parallel to h, and choose a point $v \in (\operatorname{bd} K \cap N) \setminus (L_1 \cup L_2)$ so close to L_0 that the hyperplane L_0' through v parallel to L_0 satisfies the following conditions (see the figure below):

a) $\operatorname{bd} K \cap L_0'$ is an $(n-1)$-dimensional ellipsoid,

b) L_0' intersects the relative interior of each of the $(n-1)$-dimensional solid ellipsoids $K \cap L_1$ and $K \cap L_2$.



By Lemma 9, there is a real quadric hypersurface Q that contains $\{v\} \cup E_1 \cup E_2$. Since the $(n-1)$-dimensional ellipsoid $E_0' = \operatorname{bd} K \cap L_0'$ is uniquely determined by the set $\{v\} \cup (E_1 \cap L_0') \cup (E_2 \cap L_0')$, we have $E_0' \subset \operatorname{bd} K \cap Q$. By continuity, there is a $\beta > 0$ such that any hyperplane L' through G that forms with L_0' an angle of size β or less satisfies conditions a) and b) above; whence $\operatorname{bd} K \cap L'$ is an $(n-1)$-dimensional ellipsoid that lies in $\operatorname{bd} K \cap Q$. The union of such ellipsoids $\operatorname{bd} K \cap L'$ covers an open piece of Q that lies in $\operatorname{bd} K$. $\square$

Summing up the statements of Lemmas 4–10, we conclude that $\operatorname{bd} K$ is a convex quadric.

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