BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(64), 2010, Pages 67–76
ISSN 1024–7696

Method for constructing one-point expansions of a topology* on a finite set and its applications

V. I. Arnautov, A. V. Kochina

Abstract. The article consists of two parts. In the first part we present an algorithm which allows to receive, for any topology τ which is given on a set X from n elements, all topologies on the set $X \bigcup \{y\}$ each of which induces the topology τ on the set X. In the second part (as an example) this algorithm is applied for calculation of the number of topologies on the set Y each of which induces the discrete topology on the set X.

Mathematics subject classification: 54A10.

Keywords and phrases: Finite set, topologies, one-point expansions, lattice isomorphic.

Introduction

The history of researches of the problem about the number of topologies on finite sets and some results received by different authors are given in [1].

The works [1] and [2] contain an extended list of articles, which are devoted to this problem.

At present the number of all topologies on sets having no more than 18 elements is known. These numbers are given in the following table, which can be find in [1] and [2].

*If $Y = X \bigcup \{y\}$ then a topology $\widetilde{\tau}$ on the set Y is called one-point expansion of the topology $\tau = \widetilde{\tau}|_X$.

The number of elements of the set X	The number of topologies on the set X
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241
8	642779354
9	63260289423
10	8977053873043
11	1816846038736192
12	519355571065774021
13	207881393656668953041
14	115617051977054267807460
15	88736269118586244492485121
16	93411113411710039565210494095
17	134137950093337880672321868725846
18	261492535743634374805066126901117203

This article adjoins the works in which this problem is studied. However, this question is investigated from other point of view.

Namely, we consider a topology on a set from $n+1$ elements as one-point expansion of a topology given on a set from n elements.

1 Justification of the algorithm

1.1. Theorem. *Let τ be a topology on a finite set X and let $\tilde{\tau}$ be such a topology on $Y = X\bigcup\{y\}$ that $\tilde{\tau}|_X = \tau$. Then there exist such $V_0 \in \tau$ and $U_0 \in \tau$ that the following statements are valid:*

1. $U_0 \subseteq \bigcap\limits_{V \nsubseteq V_0, V\in\tau} V$,

2. $\tilde{\tau} = \{V \in \tau | \, V \subseteq V_0\} \cup \{U \cup \{y\} | \, U \in \tau, U \supseteq U_0\}$.

Proof. We take $V_0 = \bigcup\limits_{\tilde{V}\in\tilde{\tau}, y\notin\tilde{V}} \tilde{V}$ and $U_0 = \bigcap\limits_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} (\tilde{V}\backslash\{y\})$.

As $y \notin V_0 = \bigcup\limits_{\tilde{V}\in\tilde{\tau}, y\notin\tilde{V}} \tilde{V} \in \tilde{\tau}$, then $V_0 = V_0 \cap X \in \tilde{\tau}|_X = \tau$.

Besides

$$U_0 = \bigcap_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} (\tilde{V}\backslash\{y\}) = (\bigcap_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} \tilde{V})\backslash\{y\} = (\bigcap_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} \tilde{V}) \cap X \in \tilde{\tau}|_X = \tau.$$

Prove the first statement.

Let $V' \in \tau$ and $V' \nsubseteq V_0$. Then there exists $\tilde{U}' \in \tilde{\tau}$ such that $V' = X \bigcap \tilde{U}'$. As $V' \nsubseteq V_0 = \bigcup\limits_{\tilde{V}\in\tilde{\tau}, y\notin\tilde{V}} \tilde{V}$, then $y \in \tilde{U}'$, and hence,

$$U_0 = \bigcap_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} (\tilde{V}\backslash\{y\}) \subseteq \tilde{U}' \setminus \{y\} = \tilde{U}' \bigcap X = V'.$$

From arbitrariness of the set V' it follows that $U_0 \subseteq \bigcap\limits_{V\nsubseteq V_0, V\in\tau} V$.

The first the statement is proved.

Now prove the second statement, i.e.

$$\tilde{\tau} = \{V \in \tau |\, V \subseteq V_0\} \cup \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}.$$

Let $\tilde{W} \in \tilde{\tau}$. If $y \notin \tilde{W}$, then from the definition of V_0 it follows that $\tilde{W} \subseteq V_0 = \bigcup\limits_{\tilde{V}\in\tilde{\tau}, y\notin\tilde{V}} \tilde{V}$, and as $\tilde{W} = \tilde{W} \cap X \in \tilde{\tau}|_X = \tau$, then

$$\tilde{W} \in \{V \in \tau |\, V \subseteq V_0\} \subseteq \{V \in \tau |\, V \subseteq V_0\} \cup \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}.$$

If $y \in \tilde{W}$, then $\tilde{W}\backslash\{y\} = \tilde{W} \cap X \in \tilde{\tau}|_X = \tau$. Besides, $\tilde{W}\backslash\{y\} \supseteq \bigcap\limits_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} (\tilde{V}\backslash\{y\}) = U_0$. Then

$$\tilde{W} = (\tilde{W}\backslash\{y\}) \cup \{y\} \in \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\} \subseteq$$

$$\{V \in \tau |\, V \subseteq V_0\} \cup \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}.$$

So, we have shown that $\tilde{\tau} \subseteq \{V \in \tau |\, V \subseteq V_0\} \cup \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}$.

Now let

$$W \in \{V \in \tau |\, V \subseteq V_0\} \cup \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}.$$

If $W \in \{V \in \tau |\, V \subseteq V_0\}$, then there exists $\tilde{W} \in \tilde{\tau}$ such that $W = \tilde{W} \cap X$.

As $y \notin V_0$, then $V_0 \subseteq X$, and hence $W = \tilde{W} \cap X \supseteq \tilde{W} \cap V_0$. As $W \subseteq \tilde{W}$ and $W \subseteq V_0$, then $W \subseteq \tilde{W} \cap V_0$, and hence $W = \tilde{W} \cap V_0$. Besides, $V_0 = \bigcup\limits_{\tilde{V}\in\tilde{\tau}, y\notin\tilde{V}} \tilde{V} \in \tilde{\tau}$, and hence, $W = \tilde{W} \cap V_0 \in \tilde{\tau}$.

If $W \in \{U \cup \{y\} |\, U \in \tau, U \supseteq U_0\}$ then there exists such $U' \in \tau$ that $U' \supseteq U_0$ and $U' \cup \{y\} = W$. As $U' \in \tau = \tilde{\tau}|_X$ then there exists $\tilde{W}' \in \tilde{\tau}$ such that $U' = W' \cap X$.

As $U_0 = \bigcap\limits_{\tilde{V}\in\tilde{\tau} y\in\tilde{V}} (\tilde{V}\backslash\{y\})$, then from finiteness of the set $\tilde{\tau}$ it follows that $U_0 \cup \{y\} = \bigcap\limits_{\tilde{V}\in\tilde{\tau}, y\in\tilde{V}} \tilde{V} \in \tilde{\tau}$. Then $y \in \tilde{W}' \cup U_0 \cup \{y\}$, and

$$W = U' \cup \{y\} = U' \cup U_0 \cup \{y\} =$$

$$((\tilde{W}' \cap X) \cup (U_0 \cup \{y\}) \cap X) \cup ((\tilde{W}' \cup (U_0 \cup \{y\})) \cap \{y\}) =$$

$$(\tilde{W}' \cup (U_0 \cup \{y\})) \cap (X \cup \{y\}) = \tilde{W}' \cup (U_0\{y\}) \in \tilde{\tau}.$$

Therefore, $\tilde{\tau} \supseteq \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, and hence, $\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$.

The theorem is completely proved.

1.2. Theorem. *Let τ be a topology on a set X and $V_0 \in \tau$. Consider a set $U_0 \in \tau$ such that $U_0 \subseteq \bigcap\limits_{V \in \tau, V \not\subseteq V_0} V$ (we assume that $\bigcap\limits_{V \in \emptyset} V = X$). Then*

$$\tilde{\tau}(V_0, U_0) = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$$

is a topology on the set $Y = X \bigcup \{y\}$, and $\tilde{\tau}(V_0, U_0)|_X = \tau$.

Proof. Prove first that $\tilde{\tau}(V_0, U_0)$ is a topology on the set Y.

As $\emptyset \subseteq V_0$, then $\emptyset \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}$. Besides, as $X \in \tau$ and $U_0 \subseteq X$, then $X \in \{U | U \in \tau, U \supseteq U_0\}$, and hence, $Y = X \cup \{y\} \in \tilde{\tau}$.

Now let $A, B \in \tilde{\tau}$, then:

– If $A, B \in \{V \in \tau | V \subseteq V_0\}$, then $A \cap B \in \tau$ and $A \cap B \subseteq V_0$, and hence, $A \cap B \in \tilde{\tau}$.

– If $A \in \{V \in \tau | V \subseteq V_0\}$ and $B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, then $A \in \tau$ and $B \backslash \{y\} \in \tau$, and as $A \subseteq V_0$ and $B \backslash \{y\} \supseteq U_0$, then $A \cap (B \backslash \{y\}) \in \tau$. As $y \notin A$, then $A \cap B = A \cap (B \backslash \{y\}) \subseteq V_0$, and hence, $A \cap B \in \tilde{\tau}$.

It is similarly proved that $A \cap B \in \tilde{\tau}$ if $B \in \{V \in \tau | V \subseteq V_0\}$ and

$$A \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}.$$

– If $A, B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, then $A \backslash \{y\} \in \tau$, $B \backslash \{y\} \in \tau$ and $A \backslash \{y\} \supseteq U_0$, $B \backslash \{y\} \supseteq U_0$. As τ is a topology on the set X, then $(A \backslash \{y\}) \cap (B \backslash \{y\}) \in \tau$. Besides, as $(A \backslash \{y\}) \cap (B \backslash \{y\}) \supseteq U_0$, then $A \cap B \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} \subseteq \tilde{\tau}$, and hence $A \cap B \in \tilde{\tau}$.

So, we have checked that $A \cap B \in \tilde{\tau}$, for any $A, B \in \tilde{\tau}$.

Now let $\{A_\gamma | \gamma \in \Gamma\} \subseteq \tilde{\tau}$. If $A_\gamma \in \{V \in \tau | V \subseteq V_0\}$ for any $\gamma \in \Gamma$, then $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \in \tau$ and $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \subseteq V_0$, and hence $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \in \tilde{\tau}$.

If there exists $\gamma_0 \in \Gamma$ such that $A_{\gamma_0} \notin \{V \in \tau | V \subseteq V_0\}$, then $A_{\gamma_0} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$, and hence, $A_{\gamma_0} = U_{\gamma_0} \cup \{y\}$, where $U_{\gamma_0} \in \tau$. Then $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \supseteq A_{\gamma_0} \supseteq U_0$ and $(\bigcup\limits_{\gamma \in \Gamma} A_\gamma) \backslash \{y\} = \bigcup\limits_{\gamma \in \Gamma} (A_\gamma \backslash \{y\})$.

Let $\gamma \in \Gamma$. If $A_\gamma \in \{V \in \tau | V \subseteq V_0\} \subseteq \tau$, then $A_\gamma \backslash \{y\} = A_\gamma \in \tau$ and if $A_\gamma \notin \{V \in \tau | V \subseteq V_0\}$, then there exists $V_\gamma \in \tau$ such that $V_\gamma \supseteq U_0$ and $A_\gamma = V_\gamma \cup \{y\}$. But then $A_\gamma \backslash \{y\} = V_\gamma \in \tau$.

So, we have proved that $A_\gamma \backslash \{y\} \in \tau$ for any $\gamma \in \Gamma$. Having put $V_\gamma = A_\gamma \backslash \{y\}$ for those $\gamma \in \Gamma$, receive

$$\bigcup_{\gamma \in \Gamma} A_\gamma = A_{\gamma_0} \cup (\bigcup_{\gamma \notin \Gamma, \gamma \neq \gamma_0} A_\gamma) = (V_{\gamma_0} \cup \{y\}) \cup (\bigcup_{\gamma \notin \Gamma \gamma \neq \gamma_0} A_\gamma \backslash \{y\}) =$$

$$V_{\gamma_0} \cup (\bigcup_{\gamma\in\Gamma\gamma\neq\gamma_0} V_\gamma)) \cup \{y\} = (\bigcup_{\gamma\in\Gamma} V_\gamma) \cup \{y\}.$$

As $\bigcup\limits_{\gamma\in\Gamma} V_\gamma \in \tau$ and $\bigcup\limits_{\gamma\in\Gamma} V_\gamma \supseteq U_0$, then, $\bigcup\limits_{\gamma\in\Gamma} A_\gamma \in \tilde{\tau}$.

So, we have proved that $\tilde{\tau}(V_0, U_0)$ is a topology on the set Y.

Now prove that $\tilde{\tau}(V_0, U_0)|_X = \tau$.

Let $U \in \tilde{\tau}(V_0, U_0)|_X$. Then there exists $\tilde{U} \in \tilde{\tau}(V_0, U_0)$ such that $U = \tilde{U} \cap X$. If $\tilde{U} \in \{V \in \tau | V \subseteq V_0\}$, then $\tilde{U} \in \tau$ and $y \notin \tilde{U}$. Then $U = \tilde{U} \cap X = \tilde{U} \in \tau$.

Now let $\tilde{U} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$. Then $\tilde{U}\backslash\{y\} \in \tau$, and hence, $U = \tilde{U} \cap X = \tilde{U}\backslash\{y\} \in \tau$. From arbitrariness of U it follows that we have the inclusion $\tilde{\tau}(V_0, U_0)|_X \subseteq \tau$. Now show the inverse inclusion.

Let $V' \in \tau$. Two cases are possible:

1) $V' \subseteq V_0$;

2) $V' \nsubseteq V_0$.

If $V' \subseteq V_0$, then $V' \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}(V_0, U_0)$, and $y \notin V'$. Then $V' = V' \cap X \in \tilde{\tau}(V_0, U_0)|_X$.

If $V' \nsubseteq V_0$, then $V' \notin \{V \in \tau | V \subseteq V_0\}$, and according to the condition of the theorem we have that $V' \supseteq \bigcap\limits_{V\in\tau, V\nsubseteq V_0} V \supseteq U_0$. Then, from the definition of the topology $\tilde{\tau}(V_0, U_0)$ it follows that $V' \cup \{y\} \in \tilde{\tau}(V_0, U_0)$.

Besides, as $V' \subseteq X$, then $V' = (V' \cup \{y\}) \cap X \in \tilde{\tau}(V_0, U_0)|_X$. From arbitrariness of V' it follows that $\tilde{\tau}(V_0, U_0)|_X \supseteq \tau$, and hence, $\tilde{\tau}(V_0, U_0)|_X = \tau$.

The theorem is completely proved.

1.3. Theorem. *Let X be a finite set, $\tilde{\tau}$ and $\tilde{\tau}'$ be such topologies on the set $Y = X \bigcup \{y\}$ that $\tilde{\tau}|_X = \tilde{\tau}'|_X = \tau$. If $V_0, U_0, V_0', U_0' \in \tau$, $\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$ and $\tilde{\tau}' = \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\}$, then $\tilde{\tau} \neq \tilde{\tau}'$ if and only if $(V_0, U_0) \neq (V_0', U_0')$.*

Proof. Necessity. We assume the contrary, i.e. $\tilde{\tau} \neq \tilde{\tau}'$, but $(V_0, U_0) = (V_0', U_0')$.

Then $V_0 = V_0'$ and $U_0 = U_0'$, hence,

$$\tilde{\tau} = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} =$$

$$= \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\} = \tilde{\tau}'.$$

Receive a contradiction with the assumption that $\tilde{\tau} \neq \tilde{\tau}'$.

Hence $(V_0, U_0) \neq (V_0', U_0')$.

Sufficiency. We assume the contrary, i.e. $\tilde{\tau} = \tilde{\tau}'$ and $(V_0, U_0) \neq (V_0', U_0')$.

If $V_0 \neq V_0'$, then $V_0 \nsubseteq V_0'$, or $V_0' \nsubseteq V_0$.

We assume, for definiteness, that $V_0 \nsubseteq V_0'$. Then $V_0 \in \{V \in \tau | V \subseteq V_0\} \subseteq \tilde{\tau}$ and $V_0 \notin \{V \in \tau | V \subseteq V_0'\}$.

As any set from $\{U \cup \{y\} | U \supseteq U_0'\}$ contains y and $y \notin V_0$, then

$$V_0 \notin \{V \in \tau | V \subseteq V_0'\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} = \tilde{\tau}',$$

and hence, in this case $\tilde{\tau} \neq \tilde{\tau}'$.

If $U_0 \neq U_0'$, then $U_0 \nsubseteq U_0'$, or $U_0' \subseteq U_0$.

We assume, for definiteness, that $U_0 \nsubseteq U_0'$. Then $U_0' \cup \{y\} \in \{U \cup \{y\} | U \in \tau, U \supseteq U_0'\} \subseteq \tilde{\tau}'$, and $U_0' \cup \{y\} \notin \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}$. As any set from $\{V \in \tau | V \subseteq V_0\}$ does not contain y and $y \in U_0' \cup \{y\}$, then

$$U_0' \cup \{y\} \notin \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\} = \tilde{\tau},$$

and hence, $\tilde{\tau} \neq \tilde{\tau}'$ in this case, too.

Therefore $\tilde{\tau} \neq \tilde{\tau}'$.

The theorem is completely proved.

1.4. Remark. We notice that if $\tilde{\tau}$ and $\tilde{\tau}'$ are such topologies on the set $Y = X \bigcup \{y\}$ that $\tilde{\tau}|_X \neq \tilde{\tau}'|_X$, then $\tilde{\tau} \neq \tilde{\tau}'$. Therefore any extensions on the set Y of various topologies set on the set X will be various.

So, from Theorems 1.2 and 1.3 the following algorithm for the construction of all topologies on the set $Y = X \bigcup \{y\}$ follows, knowing all topologies on the finite set X.

1.5. Algorithm.

1. We choose any topology τ_0 set on the set X;
2. We choose arbitrarily a subset $V_0 \in \tau_0$;
3. We choose arbitrarily such subset $U_0 \in \tau_0$ that $U_0 \subseteq \bigcap\limits_{V \in \tau_0, V \nsubseteq V_0} V$ (consider that $\bigcap\limits_{V \in \emptyset} V = X$);
4. We determine the topology

$$\tilde{\tau}(V_0, U_0) = \{V \in \tau | V \subseteq V_0\} \cup \{U \cup \{y\} | U \in \tau, U \supseteq U_0\}.$$

2 Application of the algorithm for calculation of the number of some topologies

2.1. Definition. As it is usual, a partially ordered set $(X, \leq)$ is called a *lattice* if for any elements $a, b \in X$ there exists $inf\{a, b\}$ and $sup\{a, b\}$.

2.2. Definition. Lattices $(X, \leq)$ and $(Y, \leq)$ are called:

– *isomorphic* if there exists such a bijection $f : X \to Y$ that $f(inf\{a, b\}) = inf\{f(a), f(b)\}$ and $f(sup\{a, b\}) = sup\{f(a), f(b)\}$, for any elements $a, b \in X$;

– *antiisomorphic* if there exists such a bijection $f : X \to Y$ that $f(inf\{a, b\}) = sup\{f(a), f(b)\}$ and $f(sup\{a, b\}) = inf\{f(a), f(b)\}$, for any elements $a, b \in X$.

2.3. Definition. If (X, τ_1) and (Y, τ_2) are topological spaces then the topologies τ_1 and τ_2 are called:

– *lattice isomorphic* if the lattices $(\tau_1, \subseteq)$ and $(\tau_2, \subseteq)$ are isomorphic;
– *lattice antiisomorphic* if the lattices $(\tau_1, \subseteq)$ and $(\tau_2, \subseteq)$ are antiisomorphic.

2.4. Remark. If X is a finite set, (X, τ) is a topological space and $\tau' = \{X \setminus V \big| V \in \tau\}$, it is easy to notice that τ' is a topology on the set X which is lattice antiisomorphic with the topology τ.

2.5. Proposition. *Let X be a finite set and $Y = X \bigcup \{y\}$. If τ is a topology on the set X and $\tau' = \{X \setminus V \big| V \in \tau\}$, then τ' is a topology on the set X and τ and τ' have the same number of expansions on the set Y.*

Proof. Let Ω and Ω' be sets of all expansions of topologies τ and τ' on the set Y, accordingly. Define the following mapping $\psi : \Omega \to \Omega'$:

map each topology $\widehat{\tau} \in \Omega$ onto the topology $\psi(\widehat{\tau}) = \widehat{\tau}' = \{Y \setminus \widehat{V} \big| \widehat{V} \in \widehat{\tau}\}$. As

$$\psi(\widehat{\tau})\big|_X = \{(Y \setminus \widehat{V}) \bigcap X \big| \widehat{V} \in \widehat{\tau}\} = \{X \setminus (\widehat{V} \bigcap X) \big| \widehat{V} \in \widehat{\tau}\} = \{X \setminus V) \big| V \in \tau\} = \tau',$$

then $\psi(\widehat{\tau}) \in \Omega'$.

If $\widehat{\tau}' \in \Omega'$, then $\widehat{\tau} = \{Y \setminus V \big| V \in \widehat{\tau}'\} \in \Omega$ and $\psi(\widehat{\tau}) = \widehat{\tau}'$, and hence, $\psi : \Omega \to \Omega'$ is a surjective mapping.

Besides if $\widehat{\tau}_1 \neq \widehat{\tau}_2$, then

$$\psi(\widehat{\tau}_1) = \{Y \setminus V \big| V \in \widehat{\tau}_1\} \neq \{Y \setminus U \big| U \in \widehat{\tau}_1\} = \psi(\widehat{\tau}_2),$$

and hence, $\psi : \Omega \to \Omega'$ is injective mapping, i.e. $\psi : \Omega \to \Omega'$ is an bijective mapping.

The proposition is completely proved.

2.6. Theorem. *Let τ' and τ'' be such topologies on finite sets X and Z, accordingly, that they are lattice isomorphic or lattice antiisomorphic. If $\widetilde{X} = X \bigcup \{y\}$ and $\widetilde{Z} = Z \bigcup \{y\}$, then the topologies τ' and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly.*

Proof. First we consider the case when the topologies τ' and τ'' are lattice isomorphic. Let $f : (\tau', \subseteq) \to (\tau'', \subseteq)$ be a corresponding lattice isomorphism.

If $\Omega_1 = \{(V', U') \big| V' \in \tau', U' \in \tau', \text{ and } U' \subseteq \bigcap\limits_{V \in \tau', V \nsubseteq V'} V\}$ and $\Omega_2 = \{(V'', U'') \big| V'' \in \tau'', U'' \in \tau'', \text{ and } U'' \subseteq \bigcap\limits_{W \in \tau'', W \nsubseteq V''} W\}$, then we define the mapping $\Psi : \Omega_1 \to \Omega_2$ as follows: $\Psi((V', U')) = (f(V'), f(U'))$.

As $f : (\tau', \subseteq) \to (\tau'', \subseteq)$ is a lattice isomorphism, then $U \subseteq V$ if and only if $f(U) \subseteq f(V)$ for any $U, V \in \tau_1$.

If $(V', U') \in \Omega_1$, then $U' \subseteq \bigcap\limits_{V \in \tau', V \nsubseteq V'} V$, and hence,

$$f(U') \subseteq \bigcap_{V \in \tau', V \nsubseteq V'} f(V) = \bigcap_{W \in \tau'', W \nsubseteq f(V')} W,$$

i.e. $\Psi((V', U')) = (f(V'), f(U')) \in \Omega_2$.

The injectivity of the mapping $\Psi : \Omega_1 \to \Omega_2$ follows from the injectivity of the mapping $f : \tau' \to \tau''$.

If $(V'', U'') \in \Omega_2$, then $U'' \subseteq \bigcap\limits_{W \in \tau',\, W \not\subseteq V''} W$. Then

$$f^{-1}(U'') \subseteq \bigcap_{W \in \tau'',\, W \not\subseteq V''} f^{-1}(W) = \bigcap_{V \in \tau',\, V \not\subseteq f^{-1}(V')} V,$$

and hence, $(f^{-1}(V''), f^{-1}(U'')) \in \Omega_1$, and

$$\Psi((f^{-1}(V''), f^{-1}(U''))) = (f(f^{-1}(V'')), f(f^{-1}(U''))) = (V'', U'').$$

Therefore, $\Psi : \Omega_1 \to \Omega_2$ is a bijection.

So, we have proved that the sets Ω_1 and Ω_2 have the same number of elements.

From Theorems 1.1, 1.2 and 1.3 it follows that the number of expansions of the topology τ' on the set $\widetilde{X}$ is equal to the number of elements of the set Ω_1, and the number of expansions of the topology τ'' on the set $\widetilde{Z}$ is equal to the number of elements of the set Ω_2. Hence the number of expansions of the topology τ' on the set $\widetilde{X}$ is equal to the number of expansions of the topology τ'' on the set $\widetilde{Z}$.

The theorem is proved for the case when topologies τ' and τ'' are lattice isomorphic.

If the topologies τ' and τ'' are lattice antiisomorphic, then it is easy to notice that the topology $\tau'_1 = \{X \setminus V \big| V \in \tau'\}$ will be lattice isomorphic to topology τ''. Then, according to proved above, the topologies τ'_1 and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly. According to Proposition 2.5, the topologies τ'_1 and τ' have the same number of expansions on the sets $\widetilde{X}$, and hence, the topologies τ'_1 and τ'' have the same number of expansions on the sets $\widetilde{X}$ and $\widetilde{Z}$, accordingly.

The theorem is completely proved.

2.7. Theorem. [1] *If X is a set from n elements and $Y = X \bigcup \{y\}$, then on the set Y precisely $2^{n+1} + n - 1$ topologies are present, each of which induces the discrete topology on the set X.*

Proof. If τ is the discrete topology on the set X, then $\tau = \{V \big| V \subseteq X\}$. For any subset $V_0 \in \tau$ we consider the sets $\widetilde{U}(V_0) = \{U \in \tau \big| U \subseteq \bigcap\limits_{V \in \tau, V \not\subseteq V_0} V\}$ and $\Omega(V_0) = \{(V_0, U) \big| U \in \widetilde{U}(V_0)\}$.

The following 3 cases are possible:

1. $V_0 = X$;
2. $V_0 \in \{X \setminus \{x\} \big| x \in X\}$;

[1] The proof of this theorem given below shows the way of using the mentioned above algorithm for calculation of one-point expansions for some topologies. Though, other and probably shorter proofs of this theorem can be. The referee kindly informed authors of this work about one of such proofs.

3. $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$.

Consider each of these cases separately.

1. Let $V_0 = X$. As $\{V \in \tau \big| V \nsubseteq V_0 = X\} = \emptyset$, then the set

$$\widetilde{U}(X) = \{U \in \tau \big| U \subseteq \bigcap_{V \in \emptyset} V\} = \{U \in \tau \big| U \subseteq X\} = \tau,$$

contains precisely 2^n subsets of the set X. Then the set $\Omega(X) = \{(X, U) \big| U \in \widetilde{U}(X)\}$ contains precisely 2^n elements.

2. Let $V_0 \in \{X \setminus \{x\} \big| x \in X\}$. As $\{V \in \tau \big| V \nsubseteq X \setminus \{x\}\} = \{A \subseteq X \big| x \in A\}$, then the set

$$\widetilde{U}(X \setminus \{x\}) = \{U \in \tau \big| U \subseteq \bigcap_{V \in \{A \subseteq X \big| x \in A\}} V\} = \{U \in \tau \big| U \subseteq \{x\}\} = \{\emptyset, \{x\}\}$$

contains precisely 2 subsets of the set X. Then the set

$$\Omega(X \setminus \{x\}) = \{(X \setminus \{x\}, U) \big| U \in \widetilde{U}(X \setminus \{x\})$$

contains precisely 2 elements for any $x \in X$, and hence the set $\bigcup_{x \in X} \Omega(X \setminus \{x\})$ contains precisely $2 \cdot n$ elements.

3. Now let $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$. Then $\{x_1\} \nsubseteq V_0$ and $\{x_2\} \nsubseteq V_0$ for the some elements $x_1, x_2 \in X$, and hence,

$$\widetilde{U}(V_0) = \{U \in \tau \big| U \subseteq \bigcap_{V \nsubseteq V_0} V\} \subseteq \{x_1\} \bigcap \{x_2\} = \{\emptyset\}$$

contains only $\emptyset$. Therefore the set $\Omega(V_0) = \{(V_0, \emptyset)\}$ contains precisely 1 element for any $V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})$. Then the set $\bigcup_{V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})} \Omega(V_0)$ contains precisely $2^n - 1 - n$ elements.

From Theorems 1.1, 1.2 and 1.3 it follows that the number of topologies on the set $Y = X \bigcup \{y\}$ each of which induces the topology τ on the set X is equal to the number of elements of the set

$$\{(V, U) \big| V, U \in \tau, U \subseteq \bigcap_{W \in \tau, W \nsubseteq V} W\} =$$

$$\Omega(X) \bigcup \Omega(X \setminus \{x\}) \bigcup (\bigcup_{V_0 \in \tau \setminus (\{X\} \bigcup \{X \setminus \{x\} \big| x \in X\})} \Omega(V_0)),$$

i.e. it is equal to $2^n + 2 \cdot n + 2^n - 1 - n = 2^{n+1} + n - 1$.

The theorem is completely proved.

Acknowledgement. The authors are grateful to the referee for comments and corrections.

References

[1] Gustavo N., Rubisno O. *Sobre el numero de topologias en un conjunto finito*, Boletin de Matematicas, Nueva Serie, 2006, **XIII**, No. 2, 136–158.

[2] http://www.rescarch.att.com/, The on-Line Encyclopedia of Integer Sequences!, number of topologies.

[3] Birkhoff G. *The Theory of lattices*, Moscow, Nauka, 1984 (in Russian).

V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *arnautov@math.md*

Received September 16, 2010