BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(64), 2010, Pages 51–66
ISSN 1024–7696

Center problem for a class of cubic systems with a bundle of two invariant straight lines and one invariant conic

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Abstract. For a class of cubic differential systems with a bundle of two invariant straight lines and one invariant conic it is proved that a weak focus is a center if and only if the first four Liapunov quantities $L_j,\ j=\overline{1,4}$ vanish.

Mathematics subject classification: 34C05.
Keywords and phrases: Cubic differential system, center-focus problem, invariant algebraic curve, integrability.

1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y),\\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1}$$

in which all variables and coefficients are assumed to be real. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a weak focus. The purpose of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that the origin is a center for system (1) if and only if it has in some neighborhood of $O(0,0)$ a holomorphic integrating factor of the form

$$\mu = 1 + \sum \mu_j(x,y).$$

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

The quantities $L_j,\ j=\overline{1,\infty}$, are polynomials in the coefficients of system (1) called Liapunov quantities. The order of the weak focus $O(0,0)$ is r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ but $L_r \neq 0$.

The origin is a center for (1) if and only if $L_j = 0,\ j=\overline{1,\infty}$. By the Hilbert's basis theorem there exists a natural number N such that the infinite system $L_j = 0,\ j=\overline{1,\infty}$, is equivalent with a finite system $L_j = 0,\ j=\overline{1,N}$. The number N is known only for quadratic systems $N = 3$ [11] and for cubic systems with only homogeneous cubic nonlinearities $N = 5$ [16, 20]. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see for instance [1, 2, 4, 6–10, 13, 14, 17, 18]).

In this paper we solve the problem of the center for cubic differential system (1) assuming that (1) has two invariant straight lines and one invariant conic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. Results concerning the relation between integrability, invariant algebraic curves and Liapunov quantities are presented in Section 2. In Section 3 we find eight sufficient series of conditions for the existence of a bundle of two invariant straight lines and one invariant conic. In Section 4 we obtain sufficient conditions for the existence of a center and finally we give the proof of the main result: a weak focus $O(0,0)$ is a center for a class of cubic systems (1) with a bundle of two invariant straight lines and one invariant conic if and only if the first four Liapunov quantities vanish.

2 Invariant algebraic curves, Liapunov quantities, center

An algebraic curve $\Phi(x,y) = 0$ (real or complex) is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P\frac{\partial \Phi}{\partial x} + Q\frac{\partial \Phi}{\partial y} = \Phi K.$$

The polynomial K is called the cofactor of the invariant algebraic curve $\Phi = 0$. We shall consider only algebraic curves $\Phi = 0$ with Φ irreducible.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x,y) = 0,\ j = 1,\dots,q$, then in most cases an integrating factor can be constructed in the Darboux form

$$\mu = \Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}. \tag{2}$$

A function (2), with $\alpha_j \in \mathbb{C}$ not all zero, is an integrating factor for (1) if and only if

$$\sum_{j=1}^{q} \alpha_j K_j \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}.$$

System (1) is called Darboux integrable if the system has a first integral or an integrating factor of the form (2).

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These last years, interesting results which relate algebraic solutions, Liapunov quantities and Darboux integrability have been published (see, for example, [3, 5, 6, 9, 10, 15, 19]). The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

Definition 1. We shall say that $(\Phi_j,\ j = \overline{1,M};\ L = N)$ is *ILC* (I – invariant algebraic curves, L – Liapunov quantities, C – center) for (1), if the existence of M algebraic curves $\Phi_j(x,y) = 0$ and the vanishing of the focal values $L_\nu,\ \nu = \overline{1,N}$, implies the origin $O(0,0)$ to be a center for (1).

The works [6–9,17,18] are dedicated to investigation of the problem of the center for cubic differential systems with invariant straight lines. In these papers, the problem of the center was completely solved for cubic systems with at least three invariant straight lines. The principal results of these works are gathered in the following two theorems:

Theorem 1. *$(\Phi_j(x,y),\ \Phi_j(0,0)\neq 0,\ j=\overline{1,4};\ \ L=1)$ is ILC for system (1).*

Theorem 2. *$(a_jx+b_jy+c_j,\ j=\overline{1,4};\ \ L=2)$ and $(a_jx+b_jy+c_j,\ j=\overline{1,3};\ \ L=7)$ are ILC for cubic system (1).*

The problem of the center was solved for cubic systems (1) with two homogeneous invariant straight lines and one invariant conic; for cubic systems (1) with two parallel invariant straight lines and one invariant conic [10]:

Theorem 3. *$(x\pm iy,\ \Phi;\ \ L=2)$ and $(l_j=1+a_jx+b_jy,\ j=1,2,\ l_1||l_2,\ \Phi;\ \ L=3)$, where $\Phi=0$ is an irreducible invariant conic, are ILC for system (1).*

3 Conditions for the existence of a bundle of two invariant straight lines and one invariant conic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a point (x_0,y_0). The intersection point (x_0,y_0) is a singular point for (1) and has real coordinates. By rotating the system of coordinates ($x\to x\cos\varphi-y\sin\varphi$, $y\to x\sin\varphi+y\cos\varphi$) and rescaling the axes of coordinates ($x\to\alpha x$, $y\to\alpha y$), we obtain $l_1\cap l_2=(0,1)$. In this case the invariant straight lines can be written as

$$l_j=1+a_jx-y,\ a_j\in\mathbb{C},\ j=1,2;\ \Delta_{12}=a_2-a_1\neq 0. \tag{3}$$

The straight lines (3) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{array}{l} k=(a-1)(a_1+a_2)+g,\ \ l=-b,\ \ s=(1-a)a_1a_2,\\ m=-a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2,\ \ r=-f-1,\\ n=a_1a_2(-f-2)-(d+1),\ \ p=(f+2)(a_1+a_2)+b-c,\\ q=(a_1+a_2-c)a_1a_2-g,\ \ (a-1)^2+(f+2)^2\neq 0. \end{array} \tag{4}$$

If the conditions (4) are satisfied then the cubic system (1) looks:

$$\begin{array}{l} \dot{x}=y+ax^2+cxy+[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]x^2y-(f+1)y^3+\\ \quad fy^2+[(a-1)(a_1+a_2)+g]x^3+[(f+2)(a_1+a_2)+b-c]xy^2\equiv P(x,y),\\ \dot{y}=-x-gx^2-dxy-by^2+(a-1)a_1a_2x^3+[g+a_1a_2(c-a_1-a_2)]x^2y+\\ \quad [(f+2)a_1a_2+d+1]xy^2+by^3\equiv Q(x,y). \end{array} \tag{5}$$

Next for cubic system (5) we find conditions for the existence of one invariant conic passing through the same singular point $(0,1)$, i.e. forming a bundle. Let the conic curve be given by the equation

$$\Phi(x,y)\equiv a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0 \tag{6}$$

with $(a_{20}, a_{11}, a_{02}) \neq 0$ and $a_{20}, a_{11}, a_{02}, a_{10}, a_{01} \in \mathbb{R}$.

For every conic curve (6) the following quantities [12]:

$$I_1 = a_{02} + a_{20}, \quad I_2 = (4a_{02}a_{20} - a_{11}^2)/4,$$
$$I_3 = (4a_{02}a_{20} - a_{01}^2 a_{20} + a_{01}a_{10}a_{11} - a_{02}a_{10}^2 - a_{11}^2)/4$$

are invariants with respect to the translation and rotation of axes. These invariants will be taken into account classifying conics. A conic (6) is reducible into two straight lines if and only if $I_3 = 0$. If $I_2 > 0$, then (6) is an ellipse, if $I_2 < 0$ – a hyperbola and if $I_2 = 0$ – a parabola.

In order the conic (6) pass through a singular point $(0, 1)$ and form a bundle with the invariant straight lines (3), we shall assume $a_{01} = -a_{02} - 1$. In this case

$$\Phi(x, y) \equiv a_{20}x^2 + a_{11}xy + a_{10}x + (a_{02}y - 1)(y - 1) = 0. \tag{7}$$

The conic (7) is an invariant conic for (5) if and only if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$, where $c_{10} = -a_{01}$, $c_{01} = a_{10}$, such that

$$P(x, y)\frac{\partial \Phi}{\partial x} + Q(x, y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x, y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + (a_{02} + 1)x + a_{10}y). \tag{8}$$

Identifying the coefficients of $x^i y^j$ in (8), we reduce this identity to three systems of equations $\{F_{ij} = 0\}$ for the unknowns a_{20}, a_{11}, a_{02}, a_{10}, c_{20}, c_{11}, c_{02} :

$$\begin{aligned}
F_{40} &\equiv (a - 1)(a_1a_2a_{11} + 2a_1a_{20} + 2a_2a_{20}) + a_{20}(2g - c_{20}) = 0, \\
F_{31} &\equiv (a - 1)(2a_1a_2a_{02} + a_1a_{11} + a_2a_{11}) - (a_2a_{11} + 2a_{20})a_1^2 - \\
&\quad - (a_1a_{11} + 2a_{20})a_2^2 + (ca_{11} - 2a_{20})a_1a_2 + (2ca_1 + 2ca_2 - 2a - \\
&\quad - c_{11} + 2d + 4)a_{20} + (2g - c_{20})a_{11} = 0, \\
F_{22} &\equiv 2(c - a_1 - a_2)a_1a_2a_{02} + (2g - c_{20})a_{02} + [c(a_1 + a_2) - a_1^2 - \\
&\quad - a_2^2 + (f + 1)a_1a_2 - a - c_{11} + 2d + 3]a_{11} + \\
&\quad + [2(f + 2)(a_1 + a_2) + 2b - 2c - c_{02}]a_{20} = 0, \\
F_{13} &\equiv (f + 2)[2a_1a_2a_{02} + (a_1 + a_2)a_{11}] + (2 + 2d - c_{11})a_{02} + \\
&\quad + (2b - c - c_{02})a_{11} - 2(f + 1)a_{20} = 0, \\
F_{04} &\equiv (2b - c_{02})a_{02} - (f + 1)a_{11} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
F_{30} &\equiv (a - 1)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - ga_{11} + \\
&\quad + (2a - 1 - a_{02})a_{20} + (g - c_{20})a_{10} = 0, \\
F_{21} &\equiv [g - c_{20} + ca_1a_2 - (a_1 + a_2)a_1a_2](-a_{02} - 1) + \\
&\quad + [c(a_1 + a_2) - a_1^2 - a_1a_2 - a_2^2 - a + d + 2 - c_{11}]a_{10} + \\
&\quad + (2c - a_{10})a_{20} + (a - d + 1 + a_{02})a_{11} - 2ga_{02} = 0, \\
F_{12} &\equiv (f + 2)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - (d + 1 - c_{11})(a_{02} + 1) - \\
&\quad - (a_{02} + 2d + 1)a_{02} + (b - c - c_{02})a_{10} + (c - b - a_{10})a_{11} + 2fa_{20} = 0, \\
F_{03} &\equiv (b - c_{02})(a_{02} + 1) + (a_{10} + 2b)a_{02} + (f + 1)a_{10} - fa_{11} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
F_{20} &\equiv (a - a_{02} - 1)a_{10} + g(a_{02} + 1) - a_{11} - c_{20} = 0, \\
F_{11} &\equiv (a_{02} + d + 1)a_{01} + (a_{10} - c)a_{10} + 2a_{02} - 2a_{20} + c_{11} = 0, \\
F_{02} &\equiv c_{02} - (a_{10} + b)(a_{02} + 1) - fa_{10} - a_{11} = 0.
\end{aligned} \tag{11}$$

Let us denote

$$j_1 = (a_1 + a_2 - c)a_{02} + (f + 1)a_{11}, \quad j_2 = a_{02}a_1^2 + a_{11}a_1 + a_{20},$$
$$j_3 = a_{02}a_2^2 + a_{11}a_2 + a_{20}, \quad j_4 = 4a_{02}a_{20} - a_{11}^2.$$

We shall study the compatibility of the system of equations $\{(9), (10), (11)\}$ when $f + 2 \neq 0$, $I_3 \neq 0$ and split the investigation into five subcases: $\{j_1 = 0\}$, $\{j_1 \neq 0, j_2 = 0\}$, $\{j_1 j_2 \neq 0,\ j_3 = 0\}$, $\{j_1 j_2 j_3 \neq 0, j_4 = 0\}$, $\{j_1 j_2 j_3 j_4 \neq 0\}$.

Remark 1. If $a_{02} = 1$, then the system $\{(9), (10), (11)\}$ is not compatible.

Indeed, we express c_{02} from $F_{04} = 0$ of (9) and substituting in (11) we obtain

$$F_{02} \equiv (a_{10} + a_{11})(f + 2) = 0.$$

If $a_{11} + a_{10} = 0$, then $I_3 = 0$. Next we shall assume that $a_{02} - 1 \neq 0$.

3.1 Case $\mathbf{j_1 = 0}$

3.1.1. $a_{02} = a_{11} = 0$. In this case $F_{04} = 0$ and the equation $F_{13} = 0$ yields $f = -1$. We express c_{02}, c_{11} and c_{20} from (9), a_{20} from $F_{11} = 0$, g from $F_{20} = 0$ and replace in (10). Reduce the equations of (10) by b from $F_{02} = 0$, then we get

$$F_{12} \equiv (a_1 - a_{10})(a_2 - a_{10}) = 0.$$

If $a_{10} = a_1$ or $a_{10} = a_2$, then we obtain the following series of conditions

1) $$a = 1/2,\ f = -1,\ g = (4c - 3b)/6,\ a_1 = (2c)/3,\ a_2 = (2c - 3b)/6$$

for the existence of an invariant parabola for system (5):

$$(9b^2 - 6bc - 4c^2 - 18d - 36)x^2 - 24cx + 36(y - 1) = 0.$$

3.1.2. $a_{02} = 0,\ a_{11} \neq 0$. In this case the equation $j_1 = 0$ yields $f = -1$ and $F_{04} \equiv 0$. We express c_{02}, c_{11}, c_{20} from (9) and obtain $F_{40} \equiv f_1 f_2 f_3 = 0$, where

$$f_1 = a_1 a_{11} + a_{20},\ f_2 = a_2 a_{11} + a_{20},\ f_3 = (a_1 + a_2 - c)a_{20} + (a - 1)a_{11}.$$

Let $f_1 = 0$ and reduce the equations of (10) and (11) by b from $F_{02} = 0$, d from $F_{11} = 0$ and g from $F_{20} = 0$, then we get $F_{12} \equiv (a_{11} + a_{10} - a_2)I_3 = 0$.

If $a_{11} = a_2 - a_{10}$, then we obtain the following series of conditions

2) $$a = 0,\ d = (g^2 - 2cg - 2bg - 8)/4,\ f = -1,\ a_1 = g/2,\ a_2 = b + g$$

for the existence of an invariant conic for (5):

$$g(4b - 2c + 3g)x^2 + 2(2c - 4b - 3g)xy + 2(2b - 2c + g)x + 4y - 4 = 0.$$

The case $f_2 = 0$ can be reduced to $f_1 = 0$ if we replace a_1 with a_2.

Assume now $f_1 f_2 \neq 0$ and $f_3 = 0$. We express $a_{11} = a_1 + a_2 + b - c$ from $F_{02} \equiv F_{03} = 0$ and reduce the equations of (10) by d from $F_{11} = 0$ and g from $F_{20} = 0$, then we get

$$F_{12} \equiv (a_{10} + a_1 + b - c)(a_{10} + a_2 + b - c) = 0.$$

If $a_{10} = c - b - a_1$ or $a_{10} = c - b - a_2$, then we obtain

3)

$$c = (2b^2 - 6a - 3bp + p^2 + 3)/(b - p),\ g = (ab^2 - 2abp + ap^2 - 4a + 2)/(b - p),$$
$$d = (6ab^2 - 8a^2 - 10abp + 4ap^2 + 8a - 5b^2 + 8bp - 3p^2 - 2)/(b - p)^2,$$
$$f = -1,\ a_1 = (c - 2b + p)/3,\ a_2 = (2c - b + 2p)/3.$$

The invariant conic is

$$(a-1)px^2+(2-4a+(b-p)^2)x+(b-p)(1-y+pxy)=0.$$

3.1.3. $a_{02}\neq 0$. In this case we express c from $j_1=0$, c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and b, d, g from (11). Then we get $F_{12}\equiv e_1e_2(f+2)$, where $e_1=a_1a_{02}-a_1+a_{10}+a_{11},\ e_2=a_2a_{02}-a_2+a_{10}+a_{11}$.

3.1.3.1. If $e_1=0$, then $a_1=(a_{10}+a_{11})/(1-a_{02})$ and (9) becomes:

$$F_{40}\equiv h_1[a_{20}(2a_{02}^2a_2-2a_{02}a_{10}-a_{02}a_{11}-2a_{02}a_2-a_{11})-a_{02}a_{11}a_2(a_{10}+a_{11})]=0,$$
$$F_{31}\equiv h_1[2a_{20}a_{02}(a_{02}-1)+2a_{02}^2a_{10}a_2+a_{02}^2a_{11}a_2+a_{02}a_{10}a_{11}+a_{02}a_{11}a_2+a_{11}^2]=0,$$

where $h_1=(a-1)a_{02}-(f+1)a_{20}$.

Let $h_1=0$ and reduce the equations of (10) by a from $h_1=0$. Express a_{20} from $F_{30}=0$, a_{11} from $F_{21}=0$, a_{02} from $h_1=0$ and obtain the following series of conditions

4)

$$\begin{aligned}
&b=[(v+1+h)(2a_2-a_{10})(v+1)]/(hv),\\
&c=[(hv-h-v-1)a_{10}+(2-2hv^2+hv+2h+2v)a_2)/(hv),\\
&d=[(h+2v^2+3v+1)a_{10}a_2-2(hv+h+2v^2+3v+1)a_2^2-\\
&\quad -h(hv+h+3v+1)]/(hv),\quad h=2a+a_{10}a_2-2fa_2^2-4a_2^2-2,\\
&g=[a_{10}^2a_2-2(v+2)a_{10}a_2^2+ha_{10}+4(v+1)a_2^3+2ha_2]/(2h),\\
&a_1=[2hva_2+(h+v+1)(2a_2-a_{10})]/(hv),\quad v=f+1
\end{aligned}$$

for the existence of an invariant conic $(h+1)[2vy^2+(2a_2^2+2va_2^2-a_2a_{10}+h)x^2+2(a_{10}-2va_2-2a_2)xy]+2v[a_{10}x-(h+2)y+1]=0$.

Assume now $h_1\neq 0$, then from $F_{31}=0$ we find a_{20}, and the equation $F_{40}=0$ becomes $F_{40}\equiv(2a_{02}a_2+a_{11})I_3=0$.

If $a_{11}=-2a_{02}a_2$, then $F_{31}\equiv(a_{02}-1)^2a_{02}^2\neq 0$.

3.1.3.2. The case $e_2=0$ can be reduced to $e_1=0$, if we replace a_1 with a_2.

3.2 Case $\mathbf{j_1\neq 0,\ j_2=0}$

In this case $a_{20}=-a_1(a_{11}+a_1a_{02}),\ I_2<0$ and the conic is a hyperbola. If $a_{02}=0$, then $F_{04}\equiv j_1\neq 0$. Next assuming $a_{02}\neq 0$ we express c_{02}, c_{11}, c_{20} from the equations $\{F_{04}=0,\ F_{13}=0,\ F_{22}=0\}$ of (9) and b, d, g from the equations of (11). Then we get $F_{12}\equiv e_1e_2(f+2)$, where

$$e_1=a_1a_{02}-a_1+a_{10}+a_{11},\ e_2=a_2a_{02}-a_2+a_{10}+a_{11}.$$

3.2.1. Let $e_1=0$, then $I_3=0$ and the conic is reducible.

3.2.2. Assume $e_1\neq 0$ and $e_2=0$. In this case we express a_2 from $e_2=0$ and a_{10} from $F_{21}=0$. If $a_{11}=-2a_{02}a_1$, then $F_{30}=I_3\neq 0$.

Let $2a_{02}a_1+a_{11}\neq 0$, then express c from $F_{30}=0$ and

$$F_{40}\equiv F_{31}=2aa_{02}a_1+aa_{11}-a_{02}^2a_1-a_{02}a_1-a_{02}a_{11}=0.$$

If $a_{02}=a$, then $F_{40}=0$ yields $a_1=0$ and we obtain

5)

$$\begin{aligned}
&d=-a-f-3,\quad b=[(f+2)(f+a+1)a_{11}]/[a(1-a)],\\
&g=0,\quad c=[(af-2f-2)a_{11}^2+a^2(a-1)^2]/[(a^2-a)a_{11}],\\
&a_1=0,\quad a_2=[(a+f+1)a_{11}]/(a-a^2).
\end{aligned}$$

The invariant hyperbola is $(1+f+ay)a_{11}x + a(ay-1)(y-1) = 0$.

If $a_{02} \neq a$, then express a_{11} from $F_{40} = 0$ and obtain the following series of conditions for the existence of a hyperbola

6)
$$\begin{aligned}
&b = -[(f+1+a_{02})(f+2)a_1]/h, \quad a_2 = [a_1(h-a-f-1)]/h,\\
&c = [(af-2f+2a_{02}-2a-2)a_1^2 + h^2]/(ha_1), \quad g = (f+3)a_1,\\
&d = [(2a+f^2+5f+4)a_1^2 - (a_{02}+f+3)h]/h, \quad h = a_{02} - a.
\end{aligned}$$

The invariant hyperbola is $a_{02}a_1^2(a-1)x^2 + a_{02}a_1(h-a+1)xy + a_1(2a - fa_{02} - 3a_{02} + f + 1)x - h(a_{02}y-1)(y-1) = 0$.

3.3 Case $\mathbf{j_1 \cdot j_2 \neq 0}$, $\mathbf{j_3 = 0}$

In this case we also obtain the series of conditions 5) and 6).

3.4 Case $\mathbf{j_1 \cdot j_2 \cdot j_3 \neq 0}$, $\mathbf{j_4 = 0}$

If $a_{02} = 0$, then $j_4 = 0$ yields $a_{11} = 0$ and $j_1 = 0$. Next assume $a_{02} \neq 0$ and from $j_4 = 0$ we find $a_{20} = a_{11}^2/(4a_{02})$. In this case $I_2 = 0$ and the conic is a parabola. We express c_{02}, c_{11}, c_{20} from the equations $\{F_{04} = 0,\ F_{13} = 0,\ F_{22} = 0\}$ of (9) and b, d, g from the equations of (11), then we obtain $F_{12} \equiv e_1 e_2 (f+2) = 0$, where

$$e_1 = a_{02}a_1 - a_1 + a_{10} + a_{11}, \quad e_2 = a_{02}a_2 - a_2 + a_{10} + a_{11}.$$

3.4.1. Assume $e_1 = 0$, i.e. $a_{10} = a_1 - a_{02}a_1 - a_{11}$. Reduce the equations $\{F_{31} = 0, F_{30} = 0\}$ by f from $F_{21} = 0$, the equation $F_{30} = 0$ by a from $F_{31} = 0$ and express c from $F_{30} = 0$, then we obtain

$$\begin{aligned}
&F_{21} \equiv a_1 a_{02}(a_{02}-1) + a_{11}(a_{02}+f+1) + 2(f+2)a_2 a_{02} = 0,\\
&F_{31} \equiv a_{11}(2a - a_{02} - 1) + 4(a-1)a_2 a_{02} = 0.
\end{aligned} \tag{12}$$

If $f = -2a$, then $a \neq 1$. Solving (12) for a_1 and a_2 we get

7)
$$\begin{aligned}
&b = [a_{11}(a_{02}-2h-1)]/a_{02}, \quad g = [a_{11}(1-2a_{02}+2h)]/(2a_{02}),\\
&d = [(4ha_{02}+a_{02}-4h^2-4h-1)a_{11}^2 - 4ha_{02}^2(a_{02}-2h+1)]/(4ha_{02}^2),\\
&f = -2a, \quad c = [a_{11}^2(a_{02}-4h^2-4h-1) + 8a_{02}^2h^2]/(4ha_{11}a_{02}),\\
&a_1 = 0, \quad a_2 = [a_{11}(a_{02}-2h-1)]/(4ha_{02}), \quad h = a-1.
\end{aligned}$$

The invariant parabola is $a_{11}^2x^2 + 4a_{02}(y-1)(a_{11}x + a_{02}y - 1) = 0$.

If $f + 2a \neq 0$, then express a_{11} from $F_{31} + F_{21} = 0$ and a_2 from $F_{21} = 0$. We obtain

8)
$$\begin{aligned}
&b = [a_1(f+2)(a_{02}+f+1)]/v, \quad a_2 = [a_1(2a-a_{02}-1)]/(2v),\\
&g = [a_1(2aa_{02} - 2a^2 + av + 3a - a_{02}v - 2a_{02} - 1)]/v, \quad v = 2a+f,\\
&c = [a_1^2(4a^2 - 2av - 4a - a_{02} + 4v + 1) - 2v^2]/(2va_1),\\
&d = [a_1^2(8(a-1)^2(a_{02}+v-a) + (a-1)(2a_{02} - 8va_{02} - 2v^2 + 6v - 2)+\\
&\quad + (2v-1)(a_{02}-1)v) + 2v^2(2a - v - a_{02} - 3)]/(2v^2).
\end{aligned}$$

The invariant parabola is $(a-1)a_1a_{02}[(a-1)a_1x - 2vy]x + v(2aa_{02} - va_{02} - 2a_{02} + v)a_1x + v^2(a_{02}y-1)(y-1) = 0$.

3.4.2. The case $e_2 = 0$ can be reduced to $e_1 = 0$, if we replace a_1 with a_2.

3.5 Case $\mathbf{j_1 \cdot j_2 \cdot j_3 \cdot j_4 \neq 0}$

We express c_{02} from $F_{04} = 0$, c_{11} from $F_{13} = 0$, c_{20} from $F_{22} = 0$ and substitute into the equations $\{F_{40} = 0,\ F_{31} = 0\}$ of (9). Calculating the resultant of the equations $\{F_{40} = 0,\ F_{31} = 0\}$ by a we obtain

$$Res(F_{40}, F_{31}, a) = j_1 j_2 j_3 j_4.$$

In this case $Res(F_{40}, F_{31}, a) \neq 0$ and therefore the system of algebraic equations $\{(9), (10), (11)\}$ is not compatible.

Remark 2. For cubic differential system (1) we obtained 8 series of conditions for the existence of two invariant straight lines and one invariant conic passing through the same singular point (0, 1).

4 Sufficient conditions for the existence of a center

Lemma 1. *The following ten series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a = 1/2,\ \ f = -1,\ \ d = (-5)/2,\ \ g = (4c - 3b)/6,\ \ a_1 = (2c)/3,$$
$$3b^2c - 2bc^2 + 9b + 12c = 0,\ \ a_2 = (2c - 3b)/6;$$

ii)

$$a = d = 0,\ \ b = -g - 2/g,\ \ c = (3g^2 - 4)/(2g),\ \ f = -1,\ \ a_1 = g/2,\ \ a_2 = -2/g;$$

iii)

$$a = (f^2 + f + 1)/(1 - f),\ \ b = (f + 2)a_1,\ \ g = -fa_1,\ \ c = (1 - 2f)a_1,$$
$$d = (2f^2 + 3f + 4)/(f - 1),\ \ f(f - 1)a_1^2 + f^2 + 3f + 2 = 0,\ \ a_2 = 0;$$

iv)

$$b = [(f^3 + (a + 5)f^2 + (7a + 5)f + 4a^2 + 2a + 2)(f + 2)u]/[(f + 1)v^2a_2],$$
$$c = [((3 - 2a)f^3 - 2(a^2 + a - 5)f^2 - (a^2 + a - 14)f - a^2 + 4a + 5)u]/$$
$$[(f + 1)v^2a_2],\ \ d = -2[f^3 + (a + 5)f^2 + 6(a + 1)f + 3a^2 + a + 4]/v,$$
$$g = [(f^3 + (a + 2)f^2 - 5a + 1)(a - 1)u]/[(f + 1)v^2a_2],$$
$$a_1 = [(2a + f)u]/[(f + 1)va_2],\ \ (f + 1)v^2a_2^2 - (a - 1)u^2 = 0,$$
$$u = f^2 + (a + 1)f + 1 - a,\ \ v = f^2 + (f + 1)(a + 3);$$

v)

$$a = (-h)/(f + 3),\ \ d = -a - f - 3,\ \ b = [(f + 2)(f + a + 1)a_{11}]/[a(1 - a)],$$
$$g = 0,\ \ c = [2(3f^2 + 2f + 3)(f + 2)h]/[(f + 3)^2(f + 1)a_{11}],\ \ h = 2f^2 + 3f - 3,$$
$$(f + 1)(f + 3)^3a_{11}^2 + 4f(f + 2)h^2 = 0,\ \ a_1 = 0,\ \ a_2 = [(a + f + 1)a_{11}]/(a - a^2);$$

vi)

$$a = -(f^2 + 6f + 3)/3,\ \ c = -(f^4 + 14f^3 + 60f^2 + 87f + 48)/[3(f + 5)a_1],$$
$$b = -[(2f^3 + 15f^2 + 27f + 6)(f + 2)a_1]/[(f^2 + 6f + 6)(f + 1)],$$
$$d = (3f^3 + 23f^2 + 42f + 6)/[3(f + 5)],\ \ 3(f + 5)a_1^2 + (f + 1)(f^2 + 6f + 6) = 0,$$
$$g = (f + 3)a_1,\ \ a_2 = -[(f^2 + 3f - 6)a_1]/[(f + 1)(f^2 + 6f + 6)];$$

vii)

$$
\begin{aligned}
&b=-[(a+\alpha t^2+f+1)(f+2)\beta]/(\alpha t),\ \ a_2=[(\alpha t^2-a-f-1)\beta]/(\alpha t),\\
&c=[\alpha^2t^2+2\alpha\beta^2t^2+\beta^2(af-2f-2)]/(\alpha\beta t),\ \ g=(f+3)\beta t,\\
&d=[(2a+f^2+5f+4)\beta^2-\alpha^2t^2-(a+f+3)\alpha]/\alpha,\\
&[(f+4)(f+1)+2a](f+3)(f+2)\beta^4+(af^2+10af+15a+3f^3+\\
&\quad +18f^2+28f+9)\alpha\beta^2+(af+3a+2f^2+3f-3)\alpha^2=0,\\
&t^2=[(1-a-2f-f^2)\alpha\beta^2-(3af+5a+f^3+7f^2+13f+7)\beta^4]/\\
&\quad [\alpha^3+(f+2)\alpha^2\beta^2-(f+3)\alpha\beta^4];
\end{aligned}
$$

viii)

$$
\begin{aligned}
&a=(18t-t^2-24)/[2t(t+2)],\ \ b=[a_{11}(3t^2-14t+24)]/[t(3t-10)],\\
&c=(9t^6+144t^5-2416t^4+12912t^3-33872t^2+44352t-23040)/\\
&\qquad [8a_{11}t(5t^3+14t^2-4t-24)],\ \ g=[3a_{11}(4t-t^2-4)]/[t(3t-10)],\\
&d=(9t^3-74t^2+168t-192)/[4t(t+2)],\ \ f=(t^2-18t+24)/[t(t+2)],\\
&4(5t-6)(t+2)^2a_{11}^2+(3t^2-14t+24)(3t-10)^2(t-2)=0,\\
&a_1=0,\ \ a_2=[a_{11}(t+2)]/[2(10-3t)];
\end{aligned}
$$

ix)

$$
\begin{aligned}
&a=(t^3-3t-1)/[(t^2-1)t],\ \ b=[(t^2+t+3)(2t+1)a_1]/[(t^2+3t-1)(1-t^2)],\\
&c=[-(t^3+6t^2+4t-4)(2t+1)(t+2)]/[(t^2+3t-1)(t+1)a_1t],\\
&f=(2t^2+2t-1)/(1-t^2),\ \ g=[(t^3-t^2+5t+4)a_1]/[(t^2+3t-1)(t^2-1)],\\
&d=(5t^4+23t^3+21t^2+16t+7)/[(t^2+3t-1)(t^2-1)t],\\
&ta_1^2+(2t+1)(t+2)=0,\ \ a_2=[a_1(t-2)]/(t^2+3t-1);
\end{aligned}
$$

x)

$$
\begin{aligned}
&b=[a_1(f+2)(a_{02}+f+1)]/v,\ \ a_2=[a_1(2a-a_{02}-1)]/(2v),\\
&g=[a_1(2aa_{02}-2a^2+av+3a-a_{02}v-2a_{02}-1)]/v,\\
&c=[a_1^2(4a^2-2av-4a-a_{02}+4v+1)-2v^2]/(2va_1),\\
&d=[a_1^2(8(a-1)^2(a_{02}+v-a)+(a-1)(2a_{02}-8va_{02}-2v^2+6v-2)\\
&\quad +(2v-1)(a_{02}-1)v)+2v^2(2a-v-a_{02}-3)]/(2v^2),\\
&(2a-f^2-2f-2)(a-1)(f+2)(f+1)a_1^6-v(2a^2f^2-6a^2-2af^3-\\
&\quad -28af^2-66af-40a-f^4-12f^3-30f^2-22f-2)a_1^4+\\
&\quad +v^3(3af+3a-5f-7)a_1^2-v^5=0,\ \ v=2a+f,\\
&a_{02}=[(8a^3f+16a^3+2a^2f^3+22a^2f^2+42a^2f+10a^2+2af^4+16af^3+\\
&\quad +30af^2+10af-3f^3-10f^2-8f-2)a_1^4+(2a^2f+2a^2+2a-2f^3-\\
&\quad -6f^2-5f-2)v^2a_1^2-2v^4(a+f+1)]/[a_1^2((2a^2f+2a^2+2af^3+\\
&\quad +14af^2+26af+16a+2f^4+13f^3+28f^2+24f+6)a_1^2+\\
&\quad +(2af+2a+2f^2+3f)v^2)].
\end{aligned}
$$

Proof. In each of the cases **i)–x)** the system (5) has two invariant straight lines of the form (3) and one invariant conic $\Phi=0$. The system (5) has a Darboux integrating factor of the form

$$\mu=l_1^{\alpha_1}l_2^{\alpha_2}\Phi^{\alpha_3}.$$

In the case i): $\Phi = (27b^3 - 24bc^2 - 27b - 16c^3 - 36c)x^2 - 12(2cx - 3y + 3)(3b + 4c)$ and $\alpha_1 = (3b + 4c)^2/[2(8c^2 - 9b^2 + 3bc)],\ \alpha_2 = -4,\ \alpha_3 = (18b^2 + 21bc + 8c^2)/[2(9b^2 - 3bc - 8c^2)]$.

In the case ii): $\Phi = (g^2 + 1)(gx - 2y)x - g(y - 1 - 2gx)$ and $\alpha_1 = 3,\ \alpha_2 = (2 - g^2)/[2(g^2 + 1)],\ \alpha_3 = (-5g^2 - 8)/[2(g^2 + 1)]$.

In the case iii): $\Phi = (2f + 1)[f(f + 2)x^2 + 2f(f - 1)a_1xy - (f^2 - 1)y^2] - 2f(f - 1)^2a_1x + 2(f^3 - 1)y + (f - 1)^2$ and $\alpha_1 = [3(f + 1)]/(2f + 1),\ \alpha_2 = -2,\ \alpha_3 = -(8f + 7)/[2(2f + 1)]$.

In the case iv): $\Phi = (a - 1)[vwa_2x^2 - 2uwxy - 2(v + 2a - 2)ux] + (f + 1)[vwa_2y^2 - 2(2a^2 + 3af + a + f^2 + f + 1)va_2y + v^2a_2]$ and $\alpha_1 = [(2a - 2 + v)(2a + 2f + 1)]/w,\ \alpha_2 = -2,\ \alpha_3 = -(4a^2f + 16a^2 + 8af^2 + 31af + 11a + 4f^3 + 15f^2 + 9f + 1)/(2w)$, where $w = 4a^2 + 5af + a + f^2 - f - 1$.

In the case v): $\Phi = (f + 3)ha_{11}xy - (f + 3)^2(f + 1)a_{11}x - h^2y^2 + 2(f^2 + f - 3)hy + h(f + 3)$ and $\alpha_1 = 3,\ \alpha_2 = -(f + 3)^2/h,\ \alpha_3 = (18 - 3f - 5f^2)/h$.

In the case vi): $\Phi = (f^2 + 6f + 6)(f + 1)hx^2 + 12ha_1xy - 18(f + 5)(f + 4)(f + 1)a_1x + 3(f + 1)hy^2 - 6(f^3 + 9f^2 + 21f + 3)(f + 1)y - 9(f + 5)(f + 1)$ and $\alpha_1 = 3,\ \alpha_2 = -(f + 3)^3/h,\ \alpha_3 = -(5f^3 + 45f^2 + 108f + 36)/h$, where $h = 2f^3 + 18f^2 + 45f + 21$.

In the case vii): $\Phi = (ay + \alpha t^2y - 1)(1 - y)\alpha t - x[(a - \alpha t^2 - 1)y - (a - 1)\beta tx](a + \alpha t^2)\beta + \beta(1 - af - a - \alpha ft^2 - 3\alpha t^2 + f)x$ and $\alpha_1 = 3,\ \alpha_2 = [(f + 3)\alpha^3 - (a - 1)(f + 3)\alpha^2\beta^2 - ((f + 4)(f + 1) + 2a)(f + 3)(f + 1)\beta^6 - ((f + 3)a + 2(f + 2)(f + 1))(f + 3)\alpha\beta^4]/[a\alpha^3 + (af + a - f^2 - 2f + 1)\alpha^2\beta^2 - (4af + 8a + f^3 + 7f^2 + 13f + 7)\alpha\beta^4]$, $\alpha_3 = [(8af + 18a + 2f^3 + 15f^2 + f\alpha_2 + 36f + 3\alpha_2 + 33)\alpha\beta^4 - (3a + f + \alpha_2 + 6)\alpha^3 + (f^2 - 3af - 3a - f\alpha_2 - f - 2\alpha_2 - 9)\alpha^2\beta^2 - (2af + 6a + f^3 + 8f^2 + 19f + 12)\beta^6]/[(a + 1)\alpha^3 + (af + a - f^2 - f + 3)\alpha^2\beta^2 - (4af + 8a + f^3 + 7f^2 + 14f + 10)\alpha\beta^4]$.

In the case viii): $\Phi = (3t^2 - 14t + 24)(3t - 10)x^2 + (5t - 6)[8(t + 2)a_{11}xy - 8(t + 2)a_{11}x - 4(3t - 10)(t - 2)y^2 + 4(3t^2 - 18t + 16)y + 8(t + 2)]$ and $\alpha_1 = [6(t - 2)^2]/[t(3t - 10)],\ \alpha_2 = -4,\ \alpha_3 = (38t - 9t^2 - 48)/[2t(3t - 10)]$.

In the case ix): $\Phi = (t^2 - t - 3)(t + 1)[(2t + 1)x^2 + 2ta_1xy - t(t + 2)y^2] - 2t^2(t^2 + 2)a_1x + 2t(t^4 + 2t^3 - 4t^2 - 5t - 3)y - t^2(t^2 + 3t - 1)(t - 1)$ and $\alpha_1 = (t + 2)^2/(3 + t - t^2),\ \alpha_2 = -4,\ \alpha_3 = (3t^2 + 7t + 5)/[2(t^2 - t - 3)]$.

In the case x): $\Phi = (a - 1)^2a_{02}a_1^2x^2 - 2v(a - 1)a_{02}a_1xy + (v - (f + 2)a_{02})va_1x + v^2(a_{02}y - 1)(y - 1)$ and $\alpha_1 = [a_1^2(2a_{02}f^2 - 4a^2 - 2af^2 - 10af - 4a + 7fa_{02} + 6a_{02} + 3f + 2) - 2(v + a_{02}\alpha_3 + a_{02} + \alpha_3 - 1)v^2]/(2v^2),\ \alpha_2 = -4,\ \alpha_3 = [a_1^4(2f^2a_{02} - 4a^2 - 2af^2 - 10af - 4a + 7fa_{02} + 6a_{02} + 3f + 2) - a_1^2(8a^2 + 4aa_{02} + 10af + 6fa_{02} + 5a_{02} + 6f^2 + 6f + 3)v - 2v^3]/[4v(a + f + 1)a_{02}a_1^2]$. □

Lemma 2. *The following nine series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)
$$a = 1/2,\ \ a_1 = c = 0,\ \ d = (-3)/2,\ \ f = -1,\ \ a_2 = g = (-b)/2;$$

ii)
$$a = 0,\ \ b = -g - 2g^{-1},\ \ a_1 = c = g/2,\ \ d = (g^2 - 2)/2,\ \ f = -1,\ \ a_2 = -2g^{-1};$$

iii)

$$a = (2b^2 - 3bp + p^2 + 2)/4, \quad d = [((b-p)^2 - 2)(2b-p)]/[2(b-p)],$$
$$c = (p-2b)/2, \quad g = (2b^3 - 5b^2p + 4bp^2 - 6b - p^3 + 2p)/4,$$
$$f = -1, \quad a_1 = -b + p/2, \quad a_2 = p - b;$$

iv)

$$a = 1/2, \quad c = 3b, \quad d = (-3)/2, \quad f = -1, \quad g = b, \quad a_1 = 0, \quad a_2 = b;$$

v)

$$b = (a-1)a_1, \quad c = (2a^2 + a - 1)/a_1, \quad d = 4a - 2a^2 - 3, \quad f = -1,$$
$$g = (5a - 2a^2 - 2)/a_1, \quad a_1^2 - 2a + 1 = 0, \quad a_2 = a(2a^2 + a - 1);$$

vi)

$$b = [(3-4a)a_2]/[4(a-1)], \quad c = [(2a-3)a_2]/[2(a-1)], \quad d = 2a - 3,$$
$$f = (-3)/2, \quad g = (3a_2)/2, \quad a_1 = a_2/(2-2a);$$

vii)

$$a = -[(4f^2 + 10f + h^2 + 6)a_2 + (2f+1)h]/(2h), \quad h = a_{10} - 2(f+2)a_2,$$
$$b = -[(4fa_2 + 6a_2 + h)(f+2)]/(2f+3), \quad c = a_2 + 2h(f+1)/(2f+3),$$
$$d = [(4f^2 + 14f - h^2 + 12)a_2 + 2h(f+1)]/h, \quad a_1 = -h/(2f+3),$$
$$g = [(4f^2 + 18f - h^2 + 18)a_2 + h(2f+3)]/[2(2f+3)];$$

viii)

$$a = ((f+2)a_2^2 - f)/2, \quad b = [(f+2)(1-a_2^2)(fa_2^2 - a_2^2 + f + 1)]/(2za_2),$$
$$c = [(2 - 3f - 3f^2)a_2^4 + 2(1 - 3f - f^2)a_2^2 + f^2 + 5f + 4]/(2za_2),$$
$$d = [(f^2 - 3)a_2^4 + 2(1 - 2f - f^2)a_2^2 - 3(f^2 + 4f + 5)]/(2z),$$
$$g = [(2f^2 + 5f + 1)a_2^4 + 2(f^2 + 5f + 7)a_2^2 + f + 1]/(2za_2),$$
$$a_1 = (3fa_2^4 + 5a_2^4 + 4fa_2^2 + 10a_2^2 + f + 1)/(2za_2), \quad z = (f+1)a_2^2 + f + 3;$$

ix)

$$a = ((f+2)a_1^2 - f)/2, \quad b = [((f+2)a_1^2 + f + 1)(1 - a_1^2)(f+2)]/(wa_1),$$
$$c = [(-3f^2 - 9f - 4)a_1^4 - (2f^2 + 3f + 1)a_1^2 + (f+1)^2]/(wa_1),$$
$$d = [(2f^2 + 6f + 3)a_1^4 - 2(2f^2 + 13f + 19)a_1^2 - 3(2f^2 + 8f + 7)]/(2w),$$
$$g = [(4f^2 + 19f + 23)a_1^4 + 2(2f^2 + 7f + 5)a_1^2 - f - 1]/(2a_1w),$$
$$a_2 = [(2a_1^2 - 1)(f+1) + (3f+7)a_1^4]/(2a_1w), \quad w = (2f+5)a_1^2 + 2f + 3.$$

Proof. In each of the cases **i)–ix)** the first Liapunov quantity vanishes $L_1 = 0$. The system (5) along with invariant straight lines (3) has also one more invariant straight line $l_3 = 0$ and one invariant conic $\Phi = 0$.

In the case i): $l_3 = bx - 2, \quad \Phi = (b^2 - 1)x^2 + 4y - 4.$

In the case ii): $l_3 = (g^2 + 2)(2x + gy) + 2g, \quad \Phi = (g^2 + 4)(gx^2 - 2xy) + 2(g^2 + 2)x - 2g(y - 1).$

In the case iii): $l_3 = (b - p)(bx - 1) + by, \quad \Phi = p(2b^2 - 3bp + p^2 - 2)x^2 + 4(b - p)[(px - 1)y - bx + 1].$

In the case iv): $l_3 = 1 + bx, \quad \Phi = x^2 - 4bxy + 8bx - 4y + 4.$

In the case v): $l_3 = (1-a)a_1x + (2a-1)y + 1$, $\Phi = a(a-1)a_1x^2 - (2a-1)(ay + 1)x + a_1(y-1)$.

In the case vi): $l_3 = a_2x + 2(a-1)y + 1$, $\Phi = (2a-1)[2(a-1)x^2 - y^2] - a_2x + 2ay - 1$.

In the case vii): $l_3 = [(2f+3)y + hx]a_{10} + h$, $\Phi = [(4f^2a_2 + 10fa_2 + h^2a_2 + 6a_2 + 2fh + 3h)x^2 - 2h^2xy - 2h(f+1)y^2](a_{10} - a_2) + h^2a_{10}x + h(2fa_{10} + a_{10} + 2a_2)y + h^2$.

In the case viii): $l_3 = ((f+3)a_2^2 + f + 1)(a_2^2x - 2a_2y - x) - 2za_2$, $\Phi = (f+2)(a_2^4 - 1)a_2x^2 - ((3f+5)a_2^2 - f - 1)(a_2^2+1)xy - 2x((f+3)a_2^2 + f + 1) + 2[(z-2)y^2 - (z-4)y - 2]a_2$.

In the case ix): $l_3 = ((f+3)a_1^2 + f + 1)[(a_1^2 - 1)x - 2a_1y] + 2a_1w$, $\Phi = (a - 1)a_1a_{02}[(a-1)a_1x - 2vy]x + v(2aa_{02} - va_{02} - 2a_{02} + v)a_1x + v^2(a_{02}y - 1)(y-1)$, where $a_{02} = -[(f+2)^2a_1^4 + (2f^2 + 6f + 3)a_1^2 + (f+1)^2]/w$, $v = 2a + f$.

By Theorem 1 in each of these cases the origin is a center. $\square$

Lemma 3. *The following two series of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a = (-2f)/3,\ \ b = (f+2)a_2,\ \ c = [3(-22f - 41)]/[a_2(13f + 24)],\ \ g = 0,$$
$$d = (-f-9)/3,\ \ (9f^2 + 12f - 9)a_2^2 + (f+3)^2 = 0,\ \ f^2 - 3f - 9 = 0,\ \ a_1 = 0;$$

ii)

$$a = [-f(f^2 + 7f + 9)]/v,\ \ b = [-f(f+3)(f+2)a_1]/u,$$
$$c = [(2f^3 + 13f^2 + 48f + 54)u]/[(f+3)^2a_1v],\ \ g = (f+3)a_1,$$
$$d = [-(2f^2 + 17f + 24)f^2]/[v(f+3)],\ \ v(f+3)^2a_1^2 + u^2 = 0,$$
$$a_2 = [(f^2 - 9f - 18)a_1]/u,\ \ u = f^2 - 3f - 9,\ \ v = f^2 + 12f + 18.$$

Proof. In each of the cases **i)** and **ii)** the first Liapunov quantity vanishes $L_1 = 0$. The system (5) along with invariant straight lines (3) has also two more invariant straight lines $l_3 = 0$, $l_4 = 0$ and one invariant conic $\Phi = 0$.

In the case i): $l_{3,4} = a_2b_j(55f + 102)x + (b_jy + 1)(87 + 47f - (8f + 15)b_j)$, where b_j, $j = 3,4$ are the solutions of the equation $3(48f + 89)b_j^2 - 3(185f + 343)b_j + 521f + 966 = 0$ and $\Phi = a_2(10fy - 18f + 18y - 33)x + (6fy + 2f + 12y + 3)(y-1)$.

In the case ii): $l_3 = 3(f+3)^2(f+2)a_1x - u(3fy + f + 3y + 3)$, $l_4 = (f+3)^2(2f + 3)a_1x - u(2fy + f + 3y + 3)$ and $\Phi = 2(2f+3)^2u^2x^2 + a_1v[2f(2f+3)^3y - (f + 3)(7f + 12)v]x - uv(y-1)(8f^2y + 24fy + 18y + v)$.

By Theorem 1 in each of these cases the origin is a center. $\square$

Theorem 4. $(l_j = 1 + a_jx - y,\ j = 1,2,\ \Phi;\ L = 4)$, *where $f + 2 \neq 0$ and $\Phi = 0$ is an invariant conic of the form (7), is ILC for system (1), i.e. the order of a weak focus is at most four.*

Proof. To prove the theorem, we compute the first four Liapunov quantities L_j, $j = \overline{1,4}$ in each series of conditions **1)–8)** using the algorithm described in [19]. In the expressions for L_j we will neglect denominators and non-zero factors.

In the case **1)** the first Liapunov quantity is $L_1 = 6(3b + 4c)d - (6b^2c - 4bc^2 - 27b - 36c)$. From $L_1 = 0$ we find d and replacing into the expression for L_2, we obtain $L_2 = f_1 f_2$, where $f_1 = c,\ f_2 = 3b^2c - 2bc^2 + 9b + 12c$. If $f_1 = 0$, then we are in the conditions of Lemma 2, i), if $f_2 = 0$, then Lemma 1, i).

In the case **2)** the vanishing of the first Liapunov quantity gives $b = -g - 2/g$. Then $L_2 = f_1 f_2$, where $f_1 = 2c - g,\ f_2 = 2cg - 3g^2 + 4$. If $f_1 = 0$, then we are in the conditions of Lemma 2, ii), if $f_2 = 0$, then Lemma 1, ii).

In the case **3)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = 4a - 2b^2 + 3bp - p^2 - 2,\ g_2 = a^2(b^2 - 2bp + p^2 - 4) + a(b^2 - bp + 4) - b^2 + bp - 1$.

If $g_1 = 0$, then Lemma 2, iii). Assume $g_1 \neq 0$ and calculate L_2. The resultant of the polynomials g_2 and L_2 by b is

$$Res(g_2, L_2, b) = 144a^6(2a - 1)^{15}(2a^3 - a^2 - p^2).$$

If $a = 0$, then $g_2 = 0$ yields $p = (b^2 + 1)/b$ and $I_3 = 0$. If $a = 1/2$, then $g_2 = 0$ yields $p = -b$ and we are in the conditions of Lemma 2, iv). If $p^2 = a^2(2a - 1)$ and $g_2 = 0$, then Lemma 2, v).

In the case **4)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = 4(f+2)^2 a_2^3 - 4(f+2)a_{10}a_2^2 - (4af + 8a - 2 - a_{10}^2)a_2 + (2a + 2f + 1)a_{10},\ g_2 = a_2^2(-af^2 - 2af + a - f^3 - 4f^2 - 6f - 5) + a_2 a_{10}(af + f^2 + 2f + 2) + (af - a + f^2 + f + 1)(a - 1)$.

Assume $g_1 = 0$. If $a_{10} = 2(f+2)a_2$ and $f = (-3)/2$, then Lemma 2, vi). If $g_1 = 0$ and $a_{10} \neq 2(f+2)a_2$, then Lemma 2, vii).

Let $g_1 \neq 0$ and $g_2 = 0$. If $a_2 = 0$, then $g_2 = 0$ yields $a = (f^2 + f + 1)/(1 - f)$ and $L_2 = (f - 1)a_{10}^2 + 4f(f+1)(f+2)$. If $L_2 = 0$, then Lemma 1, iii).

If $a_2 \neq 0$ and $a = (-f^2 - 2f - 2)/f$, then $g_2 = 0$ yields $f = (-2)/(a_2^2 + 1)$. In this case $L_2 = f_1 f_2$, where $f_1 = (a_2^2 + 1)a_{10} - 6a_2^3 + 2a_2,\ f_2 = 2a_{10}a_2 - 3a_2^2 + 1$. If $f_1 = 0$, then Lemma 1, iv) and if $f_2 = 0$, then Lemma 2, viii) ($f = (-2)/(a_2^2 + 1)$).

Assume $a_2 \neq 0$ and $a \neq (-f^2 - 2f - 2)/f$. From $g_2 = 0$ we find a_{10} and replace into the expression for L_2. We obtain $L_2 = h_1 h_2$, where $h_1 = 2a + f - (f+2)a_2^2$, $h_2 = (f+1)[f^2 + (f+1)(a+3)]^2 a_2^2 - (a-1)[f^2 + (a+1)f + 1 - a]^2$.

If $h_1 = 0$, then Lemma 2, viii) and if $h_2 = 0$, then Lemma 1, iv).

In the case **5)** the vanishing of the first Liapunov quantity gives $a_{11} = [a^2(a-1)^2]/(1 - 2f - f^2 - a)$. In this case $L_2 = f_1 f_2$, where $f_1 = (f+3)a + 2f^2 + 3f - 3,\ f_2 = 3a + 2f$. If $f_1 = 0$, then Lemma 1, v). Let $f_1 \neq 0$ and $f_2 = 0$, then $a = (-2f)/3$. We calculate $L_3 = h_1 h_2$, where $h_1 = 5f + 6,\ h_2 = f^2 - 3f - 9$. If $h_1 = 0$, then $L_4 \neq 0$ and if $h_2 = 0$, then Lemma 3, i).

In the case **6)** we denote $a_1 = \beta t$, $a_{02} = \alpha t^2 + a$ and calculate L_1.

Let $\alpha = \beta^2$, then $L_1 = 0$ yields $a = (-f^2 - 6f - 3)/3$. The second Liapunov quantity is $L_2 = f_1 f_2$, where $f_1 = 3(f+5)\beta^2 t^2 + (f^2 + 6f + 6)(f+1),\ f_2 = 3(f+1)\beta^2 t^2 - (3f^3 + 19f^2 + 33f + 15)$.

If $f_1 = 0$, then Lemma 1, vi). Assume $f_1 \neq 0$ and let $f_2 = 0$, then we find t^2 and replacing into the expression for L_3, we obtain $L_3 = h_1 h_2$, where $h_1 = f^3 + 9f^2 + 18f + 9,\ h_2 = 5f^3 + 30f^2 + 54f + 24$. If $h_1 = 0$, then Lemma 3, ii), if $h_2 = 0$, then $L_4 \neq 0$ and therefore the origin is a focus.

Let now $\alpha \neq \beta^2$ and $\alpha = -(f+3)\beta^2$, then $L_1 = 0$ yields $a = (-f^2 - 4f - 5)/(f+1)$ and $L_2 = 3(f+3)(f+1)\beta^2 t^2 + 3f^2 + 20f + 27$. We find t^2 from $L_2 = 0$ and replacing into the expression for L_3, we obtain $L_3 = u_1u_2$, where $u_1 = 8f + 15,\ u_2 = 7f + 12$. If $u_1 = 0$, then Lemma 3, ii); if $u_2 = 0$, then $L_4 \neq 0$ and therefore $O(0,0)$ is a focus.

Assume now $(\alpha + (f + 3)\beta^2)(\alpha - \beta^2) \neq 0$. Then from $L_1 = 0$ we find t^2 and replacing into the expression for L_2, we obtain $L_2 = f_1f_2$, where $f_1 = [(f + 4)(f + 1)+2a](f+3)(f+2)\beta^4+(af^2+10af+15a+3f^3+18f^2+28f+9)\alpha\beta^2+(af+3a+2f^2+3f-3)\alpha^2,\ f_2 = (f^2+6f+3+6a)(f+2)\beta^4-\alpha\beta^2(3af+6a+f^2+4f+3)-(3a+2f)\alpha^2$.

If $f_1 = 0$, then Lemma 1, vii). Assume $f_1 \neq 0,\ f_2 = 0$ and calculate L_3. The resultant of f_2 and L_3 by β is

$$Res(f_2, L_2, \beta) = v_1v_2v_3v_4v_5v_6,$$

where $v_1 = f + 1,\ v_2 = 3a + f^2 + 6f + 3,\ v_3 = 6a + f^2 + 6f + 3,\ v_4 = (f + 1)a + f^2 + 4f + 5,\ v_5 = af^2 + 12af + 18a + f^3 + 7f^2 + 9f,\ v_6 = 75(f + 2)a^2 + 150af^2 + 390af + 180a + 75f^3 + 240f^2 + 209f + 54$.

Let $v_1 = 0$, then $L_2 = 0$ yields $\alpha = [2(1-3a)\beta^2]/(3a-2)$ and $L_3 = w_1w_2$, where $w_1 = 7a - 3,\ w_2 = 15a^2 - 12a + 2$. If $w_1 = 0$, then $L_1 \equiv 28\beta^2t^2 + 25 \neq 0$ and if $w_2 = 0$, then $L_4 \neq 0$. Therefore the origin is a focus.

Assume $v_1 \neq 0,\ v_2 = 0$, i.e. $a = (-f^2 - 6f - 3)/3$. Then $L_2 = 0$ yields $\alpha = -[(f^2 + 6f + 3)(f + 2)\beta^2]/[(f + 3)(f + 1)]$ and $L_3 = w_1w_2$, where $w_1 = f+6,\ w_2 = 5f^2+10f+3$. If $w_1 = 0$, then Lemma 3, ii) and if $w_2 = 0$, then $L_4 \neq 0$.

Let $v_1v_2 \neq 0,\ v_3 = 0$, then $a = (-f^2 - 6f - 3)/6$. The vanishing of the second Liapunov quantity gives $\alpha = -[f(f^2 + 6f + 7)\beta^2]/(f^2 + 2f + 3)$ and $L_3 = w_1w_2$, where $w_1 = f^2 + 6f + 6,\ w_2 = 5f^3 - 9f + 6$. If $w_1 = 0$, then Lemma 3, ii) and if $w_2 = 0$, then $L_4 \neq 0$.

Assume $v_1v_2v_3 \neq 0,\ v_4 = 0$, then $a = (-f^2 - 4f - 5)/(f + 1)$. In this case from $L_2 = 0$, we find $\alpha = -[(f^2+13f+18)\beta^2]/(f^2+10f+15)$ and $L_3 = 31f^2+122f+121$ has not real roots.

Let $v_1v_2v_3v_4 \neq 0,\ v_5 = 0$, then $a = [f(-f^2-7f-9)]/(f^2+12f+18)$. We calculate $L_2 = z_1z_2$, where $z_1 = (f^2-3f-9)\alpha+\beta^2(f^3+8f^2+21f+18),\ z_2 = (f^2+6f+6)\beta^2+f\alpha$. If $z_1 = 0$, then Lemma 3, ii); if $z_2 = 0$, then $L_3 = 107f^3 + 426f^2 + 540f + 216$. Let $L_3 = 0$, then $L_4 \neq 0$.

Assume $v_1v_2v_3v_4v_5 \neq 0,\ v_6 = 0$ and calculate L_3 and L_4. Solve the system of equations $\{L_3 = 0, L_4 = 0\}$ by α and a, then $v_6 = 0$ has not real solutions.

In the case **7)** we calculate the first two Liapunov quantities and the resultant of them by a_{11}, then we get

$$Res(L_1, L_2, a_{11}) = f_1f_2f_3f_4,$$

where $f_1 = 2a - 1,\ f_2 = 2aa_{02} + 2a - 3a_{02} - 1,\ f_3 = 4a^2 - 4aa_{02} - 4a + 3a_{02} + 1$, $f_4 = 4a^2a_{02} + 20a^2 + 2aa_{02}^2 - 26aa_{02} - 24a + a_{02}^2 + 16a_{02} + 7$.

If $f_1 = 0$, then $a = 1/2$ and $a_{02} = a_{11}^2$. In this case $L_1 = L_2 = 0$ and $L_3 \neq 0$.

If $f_2 = 0$, then $a_{02} = (1 - 2a)/(2a - 3)$ and $L_1 = a(2a - 3)a_{11}^2 + 2a^2 - 3a + 1$. Let $L_1 = 0$, then $a_{11}^2 = (2a^2 - 3a + 1)/[a(3 - 2a)]$. In this case $L_2 \neq 0$ and therefore the origin is a focus.

If $f_3 = 0$, then $a_{02} = (2a - 1)^2/(4a - 3)$. The first Liapunov quantity is $L_1 = 32a^4 - 80a^3 + 32a^2a_{11}^2 + 72a^2 - 36aa_{11}^2 - 28a + 9a_{11}^2 + 4$. Let $L_1 = 0$ and express a_{11}^2, then $L_2 \neq 0$.

Assume $f_4 = 0$. This equation admits the parametrization $a = (18t - 24 - t^2)/[2t(t+2)]$, $a_{02} = (16t - 20 - 3t^2)/[2(t+2)]$. In this case L_1 looks $L_1 = g_1 g_2$, where $g_1 = 4(5t-6)(t+2)^2 a_{11}^2 + (3t^2 - 14t + 24)(3t-10)^2(t-2)$, $g_2 = 2(t+2)^2(t-4)(t-6)a_{11}^2 - (3t^2 - 14t + 24)(3t-10)^2(t-2)$. If $g_1 = 0$, then Lemma 1, viii). Let $g_1 \neq 0$ and $g_2 = 0$, then express a_{11}^2 from $g_2 = 0$ and calculate L_2. We obtain that $L_2 \neq 0$.

In the case **8)** we calculate the first Liapunov quantity and denote $w \equiv (2a^2 f + 2a^2 + 2af^3 + 14af^2 + 26af + 16a + 2f^4 + 13f^3 + 28f^2 + 24f + 6)a_1^2 + (2af + 2a + 2f^2 + 3f)(2a+f)^2$. If $w = 0$, then $L_1 = f_1 f_2$, where $f_1 = f + 1$, $f_2 = (f+1)a^2 - 4af - 6a - f^3 - 6f^2 - 9f - 3$. If $f_1 = 0$, then $L_2 = 0$ yields $a_{02} = (2a^2 - 3a + 1)/(1 - 3a)$ and Lemma 2, ix).

Assume $f_1 \neq 0$ and $f_2 = 0$. The equation $f_2 = 0$ admits the parametrization $a = (t^3 - 3t - 1)/[t(t^2 - 1)]$, $f = (1 - 2t - 2t^2)/(t^2 - 1)$. We calculate the second Liapunov quantity $L_2 = [(t^2 + 3t - 1)(t-1)t]a_{02} - (t^2 - t - 3)(t+2)(t+1)$ and if $L_2 = 0$, then Lemma 1, ix).

Let $w \neq 0$, then from $L_1 = 0$ we find a_{02} and substituting in L_2 we get $L_2 = g_1 g_2 g_3$, where $g_1 = 2a + f - (f+2)a_1^2$, $g_2 = 2a + 2f + 1$, $g_3 = (2a - f^2 - 2f - 2)(a-1)(f+2)(f+1)a_1^6 - v(2a^2 f^2 - 6a^2 - 2af^3 - 28af^2 - 66af - 40a - f^4 - 12f^3 - 30f^2 - 22f - 2)a_1^4 + v^3(3af + 3a - 5f - 7)a_1^2 - v^5$.

If $g_1 = 0$, then Lemma 2, ix). If $g_1 \neq 0$, $g_2 = 0$, then $a = (-2f-1)/2$ and $L_3 \neq 0$. Assume $g_1 g_2 \neq 0$ and $g_3 = 0$, then Lemma 1, x). $\square$

Acknowledgments. The author of this work is supported by the Slovenian Human Resources Development and Scholarship Fund and thanks the Center for Applied Mathematics and Theoretical Physics, University of Maribor for its hospitality and support during the stay at the Center.

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Received 15.06.2010