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On quasiidenties of torsion free nilpotent loops

Alexandru Covalschi

Abstract. It is proved that any loop which contains an infinite cyclic group and does not contain infinite number of relative prime periodic elements has an infinite and independent basis of quasiidentities. In particular, any torsion free nilpotent loop has an infinite and independent basis of quasiidentities.

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One of the classical directions of investigation of algebraic systems in their general theory is the quasivariety theory of algebraic systems, founded by A.I. Malcev [1–3].

This paper studies the problem of existence of an independent basis of quasiidentities for certain loops. It is proved that if L is a loop which contains an infinite cyclic group and does not contain an infinite number of prime periodic elements, then the quasiidentities of L have an independent and infinite basis of quasiidentities. In particular, every torsion-free nilpotent loop has an infinite and independent basis of quasiidentities and the quasivariety generated by it has infinity of coverages.

1 Main notions and denotations

A quasigroup is an algebra with the basic set Q and with three basic binary operations $\cdot$, $/$, $\backslash$ defined on it which satisfy the identities

$$x \cdot (x\backslash y) = x\backslash(x \cdot y) = (y/x) \cdot x = (y \cdot x)/x = y.$$

If a quasigroup Q has such an element e that $e \cdot x = x \cdot e = x$ for all $x \in Q$, then Q is called a *loop* and e is called its unity (see [4] or [5]). Therefore, we consider a loop Q as an algebra with three basic operations of the quasigroup Q and one null basic operation e.

Let a be a non-unity element of a loop L. If some product of m factors, each equal to the element a, is equal to the unity element $e \in L$, then a is called *relative m-periodic*. In particular, if m is a prime number then the relative m-periodic element a is called *relative prime periodic*. If the loop L does not contain a periodic element, then they say that it is *torsion free*.

A quasiidentity (quasigroupoid quasiidentity) of variables $x_1, \ldots, x_n$ is a universal formula which has the form

$$(\forall x_1 \ldots x_n)\&_{i\in I} u_i(x_1,\ldots,x_n) = u_i'(x_1,\ldots,x_n) \Rightarrow$$

$$u(x_1,\ldots,x_n) = u'(x_1,\ldots,x_n),$$

where I is a finite set of indices, u_i, u_i', u, u' are quasigrupoid words of variables $x_1,\ldots,x_n$. When writing quasiidenties, the symbols $\forall x_1 \ldots x_n$ are usually omitted. As the equality of two words $u = v$ in the loop class is equivalent to $u/v = e$, the quasiidentities written in the loop signature are studied as $\&_{i\in I} u_i = e \Rightarrow u = e$.

A quasigroup class formed only of quasigroups in which the quasiidentities of a given system of quasiidentities are true is called a *quasivariety*.

A system Σ of quasiidentities is called *independent* if no quasiidentity of Σ results from all the rest. A *basis* of the system Σ is such a subsystem $\Sigma' \subseteq \Sigma$ that any quasiidentity from Σ results from the overall of the quasiidentities from Σ'.

A quasivariety N is called a *coverage* of quasivariety M if $M \subset N$ and for any quasivariety K the inclusions $M \subseteq K \subset N$ imply $M = K$.

As usual, prime numbers are denoted by $p_i, i \in \Sigma = \{0, 1, 2, \ldots\}$, the infinite cyclic group - by Z, the cyclic group of order p_i - by Z_{p_i} , the quasivariety generated by a quasigroup Q - by qQ. The set of all natural numbers will be denoted by N.

2 The basic results

We shall say that the quasiidentity $\Phi(x_1,\ldots,x_n) = \&_{i=1}^m u_i(x_1,\ldots,x_n) = u_i'(x_1,\ldots,x_n) \Rightarrow u(x_1,\ldots,x_n) = u'(x_1,\ldots,x_n))$ is compatible in the quasigroup Q if the formula $\varphi(x_1,\ldots,x_n) = (\&_{i=1}^m u_i = u_i' \& u = u')$ is compatible in Q, that is, there are such values $x_1 = a_1,\ldots,x_n = a_n$ of the variables in Q that the following equalities are true:

$$u_1(a_1,\ldots,a_n) = u_1'(a_1,\ldots,a_n), \ldots, u_m(a_1,\ldots,a_n) = u_m'(a_1,\ldots,a_n),$$

$$u(a_1,\ldots,a_n) = u'(a_1,\ldots,a_n).$$

Lemma 1. *The conjunction of a finite number of quasiidentities compatible in any quasigroup is equivalent to one quasiidentity.*

Proof. It is sufficient to prove the lemma for the conjunction of two quasiidentities φ_1 and φ_2. Let the following equalities be:

$$\varphi_1 = (\&_{i=1}^m u_i(x_1,...,x_k) = u_i'(x_1,\ldots,x_k) \Rightarrow u(x_1,\ldots,x_k) = u'(x_1,\ldots,x_k));$$

$$\varphi_2 = (\&_{i=1}^m v_i(y_1,\ldots,y_s) = v_i'(y_1,\ldots,y_s) \Rightarrow v(y_1,\ldots,y_s) = v'(y_1,\ldots,y_s)).$$

We shall show that the formula φ_1 & φ_2 is equivalent to the quasiidentity $\varphi = (\&_{i=1}^m u_i(x_1,\ldots,x_k) = u_i'(x_1,\ldots,x_k)$ & $\&_{j=1}^m v_j(y_1,\ldots,y_s) = v_j'(y_1,\ldots,y_s) \Rightarrow u(x_1,\ldots,x_k)v(y_1,\ldots,y_s) = u'(x_1,\ldots,x_k)v'(y_1,\ldots,y_s))$.

Indeed, let the formula φ_1 & φ_2 be true in the quasigroup Q. We assume that the left side of the quasiidentity φ is true in Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, $y_j \to b_j$ $(j = 1, \ldots, s)$, where $a_1, \ldots, a_k,\ b_1, \ldots, b_s \in Q$. As the quasiidentities φ_1 and φ_2 are true in Q, we have $(u(a_1, \ldots, a_k) = u'(a_1, \ldots, a_k))$ and $(v(b_1, \ldots, b_s) = v'(b_1, \ldots, b_s))$. Therefore $u(a_1, \ldots, a_k)v(b_1, \ldots, b_s) = u'(a_1, \ldots, a_k)v'(b_1, \ldots, b_s)$. Thus, the quasiidentity φ is a consequence of the formula φ_1 & φ_2.

Conversely, let the quasiidentity φ be true in the quasigroup Q. We show that the quasiidentity φ_1 is true in the quasigroup Q. We assume that the left side of the quasiidentity φ_1 is true in Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, where $a_1, \ldots, a_k \in Q$. As the quasiidentity φ_2 is compatible in any quasigroup, and thus in the quasigroup Q, then for certain substitutions $y_j \to b_j$ $(j = 1, \ldots, s)$, where $b_1, \ldots, b_s \in Q$, we have the equalities: $v_j(b_1, \ldots, b_s) = v'_j(b_1, \ldots, b_s)$ $(j = 1, \ldots, s)$, $v(b_1, \ldots, b_s) = v'(b_1, \ldots, b_s)$.

As a result, the left side of the quasiidentity φ is true in the quasigroup Q for the substitutions $x_i \to a_i$ $(i = 1, \ldots, k)$, $y_j \to b_j$ $(j = 1, \ldots, s)$. As the quasiidentity φ is true in the quasigroup Q, then from $u(a_1, \ldots, a_k)v(b_1, \ldots, b_s) = u'(a_1, \ldots, a_k)v'(b_1, \ldots, b_s)$ it follows that $u(a_1, \ldots, a_k) = u'(a_1, \ldots, a_k)$. Similarly, we can show that φ_2 is true in the quasigroup Q. Thus, the formula φ_1 & φ_2 is a consequence of the formula φ. This completes the proof of Lemma 1. $\square$

As any quasiidentity is compatible in any loop then from Lemma 1 follows.

Corollary 1. *In the class of loops the conjunction of a finite number of quasiidentities is equivalent to one quasiidentity.*

Lemma 2. *Let quasiidentity φ be true in a quasigroup Q and let the quasivariety qQ, generated by the quasigroup Q, contain an infinite cyclic group Z. Then the set of all prime cyclic groups Z_{p_i} in which φ is not true is finite.*

Proof. Let's assume that the statement of the lemma is not true, and thus, the set

$$I = \{i \in \Sigma | Z_{p_i} \vdash \urcorner \varphi\}$$

is infinite. Let

$$\varphi = (\&_{i=1}^{m} u_i(x_1, \ldots, x_k) = u'_i(x_1, \ldots, x_k) \Rightarrow u(x_1, \ldots, x_k) = u'(x_1, \ldots, x_k)).$$

We study the finite representative quasigroup

$$L = lp(x_1, \ldots, x_n \| u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n),\ i = 1, \ldots, m$$

from qQ generated by elements $x_1, \ldots, x_n$ with the defining relations $u_i(x_1, \ldots, x_n) = u'_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$.

As for $i \in I$ the quasiidentity φ is false in the cyclic group Z_{p_i}, then there are such elements $a_1, \ldots, a_n \in Z_{p_i}$ that $u_i(a_1, \ldots, a_k) = u_i^{'}(a_1, \ldots, a_k)$ $(i = 1, \ldots, m)$, but $u(a_1, \ldots, a_n) \neq u'(a_1, \ldots, a_n)$, that is $u(a_1, \ldots, a_n)^{-1}u'(a_1, \ldots, a_n) \neq e$ or written simpler still $u^{-1}u' \neq e$. According to Dik's Theorem [3], there is a homomorphism

$\theta_i : L \to Z_{p_i}$ for which $\theta_i(u^{-1}u') \neq e$, $(i \in I)$. By Theorem 1 [1, p.73] there is such a homomorphism $\theta_i : L \to \prod_{i \in I} Z_{p_i}$ that $\theta(a)(i) = \theta_i(a)$, for any $a \in L$ and any $i \in I$. The set I is infinite. Then $\theta_i(u^{-1}u')$ is an element of infinite order of group $\theta(L)$. As $\theta(L)$ is a finitely generated abelian group, $\theta(L)$ can be decomposed in the direct product of cyclic groups. As the element $\theta(u^{-1}(u')$ has a finite order, we conclude that there is a homomorphism $\Psi : \theta(L) \to Z$ so $e \neq \psi\theta(u^{-1}u') = u(\psi\theta(x_1), \ldots, \psi\theta(x_n))^{-1}u^{'}(\psi\theta(x_1, \ldots, \psi\theta(x_n))$, that is $u(\psi\varphi(a_1), \ldots, \psi\varphi(a_n))^{-1} \neq u'(\psi\varphi(a_1, \ldots, \psi(x_a))$.

Therefore, for values of variables $x_1 = \psi\varphi(a_1), \ldots, x_n = \psi\varphi(a_n)$ we obtained that the quasiidentity φ is false in the infinite cyclic group Z.Contradiction. This completes the proof of Lemma 2. $\square$

Let Σ be an independent system of quasiidentities. Then for any formula $\varphi \in \Sigma$ there is a quasigroup Q_φ, so $Q_\varphi| =\neg\varphi$, but $Q_\varphi| = \psi$ for any formula $\psi \in \Sigma\backslash\{\varphi\}$ by the definition of independent system of quasiidentities. We call the set $\{Q_\varphi|\varphi \in \Sigma\}$ *the system corresponding* to the independent system Σ.

Lemma 3. *Suppose there is a quasivariety N of quasigroups definted by an infinite and independent system of compatible quasiidentities $\{\varphi_i|i \in I \subseteq \Sigma\}$ with the corresponding system of quasigroups $\{Q_i|i \in I\}$. If a subquasivariety $M \subseteq N$ can be defined in the quasivariety N such that for some bijective application $\alpha : I \to \Sigma$ we have $Q_i \vdash \psi_{\alpha(j)}$ for all $j \in I\backslash\{i\}$. Then the quasivariety M has an infinite and independent basis of quasiidentities in the class of all quasigroups.*

Proof. Let α be a bijective application from I on Σ. Let's denote $\Sigma = \{\varphi_i \& \psi_{\alpha(i)}|i \in I\}$. Obviously, any quasiidentity from Σ is true in any quasigroup from M. Conversely, if in the quasigroup Q all formulas from Σ are true, then $Q \in M$. Therefore, the set Σ defines the quasivariety M in the class of all quasigroups. As all formulas from $\Sigma\backslash\{\varphi_i \& \psi_{\alpha(i)}\}$ are true in Q and the formula $\varphi_i \& \psi_{\alpha(i)}$ is false in the quasigroup Q_i, then Σ is an independent system of quasiidentities. By Lemma 1 each formula from Σ is equivalent to a quasiidentity. Hence the system Σ is equivalent to a system Σ' of quasiidentities. As Σ is independent and infinite, it results that $\Sigma\prime$ is also independent and infinite. This completes the proof of Lemma 3. $\square$

Theorem. *If the loop L contains an infinite cyclic group and does not contain an infinity of p_i-periodic elements, then the quasiidentity qL generated by the loop L has an infinite and independent basis of quasiidentities.*

Proof. Denote by I the set of all indices $i \in \Sigma$ of prime numbers for which the loop L does not contain relative p_i-periodic elements. According to the hypothesis, the set I is infinite and for any $i \in I$ the quasiidentity $x^{p_i} = e \Rightarrow x = e$ is true in the loop L, where by u^{p_i} we understand the p^i fold product of the element u written as $(\ldots(uu \cdot u)\ldots u)u$. Let $\Sigma = \{\psi_i|i \in \Sigma\}$ be a set of quasiidentities (some of them may coincide) which defines the quasivariety qL and N - the quasivariety of loops defined by the independent system $\{x^{pi} = e \Rightarrow x = e|i \in I\}$ of quasiidentities. As every quasiidentity of this system is true in the loop L, then there is the inclusion $qL \subseteq N$. Let ψ_i $(i \in \Sigma)$ be an arbitrary quasiidentity from Σ:

$$\psi_i = (\&_{i=1}^m u_i(x_1, \dots, x_k) = e \Rightarrow u(x_1, \dots, x_k) = e);$$

we shall denote $M_i = \{Z_{p_k}| =^\neg \psi_i \in I | Z_{p_k}| =^\neg \psi_i\}$. By Lemma 2 the set M_i is finite. We construct the quasiidentity ψ_i', corresponding to the quasiidentity ψ_i, as follows: if $M_i = \varnothing$ then we consider $\psi_i' = \psi_i$ and if $M_i \neq \varnothing$ then we consider

$$\psi_i' = (\&_{i=1}^m u_i(x_1, \dots, x_k) = e \Rightarrow (\dots (u(x_1, \dots, x_k))^{p_1}) \dots)^{p_m} = e,$$

where $M_i = \{p_{i_1}, \dots, p_{i_m}\}$.

We show that the quasiidentities ψ_i' and ψ_i are equivalent in the class N. Obviously, ψ_i' is a consequence of the quasiidentity ψ_i. In particular, this results in the quasiidentity ψ_i' be true in each of the cyclic groups Z_{p_j}, $j \in I \backslash \{i_1, \dots, i_m\}$. Obviously, if $j \in \Sigma \backslash \{i_1, \dots, i_m\}$ then the quasiidentity ψ_i' is true in the cyclic group Z_{p_j}. Hence for every $j \in \Sigma$ the quasiidentity ψ_i' is true in the cyclic group Z_{p_j}.

Let there be the loop $Q \in N$ and assume that the quasiidentity ψ_i' is true in the loop Q. Let the left side of the quasiidentity ψ_i be true in Q for the substitutions $x_i \to a_i, i = 1, \dots, n$. As ψ_i' is true in Q, we have $(\dots (u(a_1, \dots, a_n)^{p_i}) \dots)^{p_{i_m}} = e$.

Now, applying the quasiidentities $x^{p_{ik}} = e \Rightarrow x = e, k = i_1, \dots, i_m$, which are true in every loop from the quasivariety N, from the last equality we obtain $u(a_1, \dots, a_n) = e$. Therefore, the quasiidentity ψ_i is true in the loop Q. Hence in the class N the quasiidentities ψ_i and ψ_i' are equivalent and ψ_i' is true in the cyclic group Z_{p_j} for any $j \in I$. The set $\{Z_{p_j} | j \in I\}$ is the system corresponding to the independent system of quasiidentities $\{x^{p_i} = e \Rightarrow x = e) | i \in I\}$.

From here by Lemma 3 it results that the quasivariety qL has an infinite and independent basis of quasiidentities. This completes the proof of Theorem. □

Corollary 2. *Every torsion-free nilpotent loop has an infinite and independent basis of quasiidentities.*

3 Applications

1. From local Malcev Theorem's the following *coverage criterion* of quasivarieties results: *If the quasivariety M has an independent and infinite basis of quasiidentities, then M has an infinity of coverages.* The detailed proof of this statement can be found, for instance, in [6].

According to Corollary 2 and the coverage criterion of quasivarieties, we obtain the following statement.

If L is a torsion free nilpotent loop of any rank, then the quasivariety qL has infinity of coverages in the latices of loop quasivarieties.

2. Let $M_{2\times 2}(K)$ be the vector space of square matrices with elements from associative ring K. We define multiplication and division in $M_{2\times 2}(K)$ by formulas:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} a + x & b + y \\ c + z & d + t + (x - a)(yc - bz) \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} / \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ c-z & d-t+(a-2x)(yc-bz) \end{pmatrix}.$$

It is easy to see that the set $M_{2\times 2}(K)$ forms a commutative loop with respect to multiplication and division. The unity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ $= \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Denote that loop by L. As the ring K satisfies the identity $0 \cdot x = x$ then from formulas which defined the operations it follows that elements of the form $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ belong to the centre of loop L. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $C = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ be arbitrary elements of L. We compute its associator $(A,B,C) = (AB \cdot C)/(A \cdot BC) = \begin{pmatrix} 0 & 0 \\ 0 & (anz - ayp + 2mbz - 2myc + xbp - xnc) \end{pmatrix}$. Hence the associator (A,B,C) belongs to the centre of L. Consequently, the loop L is nilpotent of class 2. As K is a ring of characteristic zero then it is easy to see that L is a torsion free loop. Then any subloop of cartesian product of loop L is torsion free. From here it follows by Corollary 2 that any free loop of quasivariety generated by L has an infinite and independent basis of quasiidentities.

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Alexandru Covalschi
State Pedagogical University ”Ion Creangă”
Chisinău, Moldova
E-mail: *Alexandru_Covalschi@yahoo.com*

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