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On stability of Pareto-optimal solution of portfolio optimization problem with Savage's minimax risk criteria

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Abstract. A multicriteria Boolean optimization problem consisting in an efficient choice of a Pareto-optimal portfolio of investor's assets that uses the Savage's minimax risk criteria is considered. Upper and lower attainable bounds of the stability radius of such portfolio with regard to independent changes of elements of a risk matrix are obtained.

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1 Introduction

In recent years, interest towards multi-objective decision-making processes under uncertainty and risk has grown dramatically. It can be explained by numerous applications of such problems in game theory, mathematical economics, optimal control, investment analysis, banking, insurance business, etc. Widespread occurrence of discrete optimization models has conditioned the interest of many experts to the study of various types of stability aspects, parametric and post-optimal analysis problems of both scalar (one-criterion) and vector (multicriteria) discrete optimization (see, for example, monographs [1–3], reviews [4–6], and annotated bibliographies [7, 8]).

One of the well-known approaches to investigation of the stability of discrete optimization problems is aimed at obtaining the so-called quantitative stability characteristics. This approach consists in finding the limit level of perturbations of initial problem data which do not change the studied original solution. As a rule, the perturbed parameters are the vector criterion coefficients. The majority of results in this research area are related to stability radius formula for Pareto-optimal (efficient) solutions of vector linear optimization problems [9, 10], in particular, Boolean problems [11], game theory problems [12, 13], and also for the stability radius of a lexicographic optimum of certain Boolean problems with linear criteria [14, 15].

This paper deals with obtaining upper and lower attainable bounds of the stability radius of a Pareto-optimal solution of portfolio optimization problem with Savage's minimax risk criteria.

2 Basic definitions and auxiliary statements

Let us consider the vector variant of the portfolio optimization problem, i.e. the problem of financial investments management, based on Markovitz's "portfolio theory" [16, 17] (see also the bibliography in [18]). To this end, we introduce the following notations:

$N_n = \{1,\ 2,\ ...,\ n\}$ – assets (shares, companies' bonds, real estate etc.),

N_m – economic strategies of an investor,

R – three-dimensional risk matrix (missed opportunities) of $m \times n \times s$ size with elements r_{ijk} from $\mathbf{R}$,

r_{ijk} – risk quantity of an investor choosing strategy $i \in N_m$ and asset $j \in N_n$ with criterion $k \in N_s$,

$x = (x_1,\ x_2,\ ...,\ x_n)^T \in X \subset \{0,\ 1\}^n$ – investor's portfolio of assets.

$$x_j = \begin{cases} 1, & \text{if the investor chooses an asset } j, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Presumably, each investor's portfolio x from a given portfolio set X assures expected total profit p and does not exceed the total amount of available capital c, i. e. for each portfolio $x = (x_1,\ x_2,\ ...,\ x_n)^T \in X$ the conditions

$$\sum_{j \in N_n} p_j\ x_j \geq p, \quad \sum_{j \in N_n} c_j x_j \leq c,$$

hold, where p_j is the expected profit of asset j, c_j is the cost of asset j.

Along with three-dimensional matrix $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$ we use its two-dimensional sections $R_k \in \mathbf{R}^{m \times n}, \quad k \in N_s$.

Let the following vector function

$$f(x,\ R) = (f_1(x,\ R_1),\ f_2(x,\ R_2),\ ...,\ f_s(x,\ R_s))$$

be defined over the set X with *Savage's minimax risk* (extreme pessimism) *criteria* [19, 20], (see also [21–23])

$$f_k(x,\ R_k) = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk} x_j \rightarrow \min_{x \in X}, \qquad k \in N_s.$$

We consider the problem of finding Pareto set $P^s(R)$, where a *Pareto-optimal (efficient) portfolios (solutions)* is regarded as *portfolio optimization problem* $Z^s(R)$:

$$P^s(R) = \{x \in X :\ P^s(x,\ R) = \emptyset\},$$

where $P^s(x,\ R) = \{x' \in X :\ x \underset{R}{\succ} x'\}$, whereas symbol $\underset{R}{\succ}$ is a binary relation defined over the set X as follows:

$$x \underset{R}{\succ} x' \quad \Leftrightarrow \quad g(x,\ x',\ R) \geq \mathbf{0} \ \ \&\ \ g(x,\ x',\ R) \neq \mathbf{0},$$

where $\mathbf{0} = (0,\ 0,\ \dots,\ 0) \in \mathbf{R}^s$, $g(x,\ x',\ R) = (g_1(x,\ x',\ R_1),\ \ g_2(x,\ x',\ R_2),\ \dots,\ g_s(x,\ x',\ R_s))$, $g_k(x,\ x',\ R_k) = f_k(x,\ R_k) - f_k(x',\ R_k) = \max\limits_{i\in N_m} R_{ik}x - \max\limits_{i\in N_m} R_{ik}x'$, $k \in N_s$, and $R_{ik} = (r_{i1k},\ r_{i2k},\ \dots,\ r_{ink})$ is row i of matrix $R_k \in \mathbf{R}^{m\times n}$.

In space $\mathbf{R}^d$ of an arbitrary dimension $d \in \mathbf{N}$ we set the l_∞-metric, i.e. as the norm of vector $z = (z_1,\ z_2,\ ...,\ z_d) \in \mathbf{R}^d$ we understand the number

$$\|z\| = \max\{|z_j| : \quad j \in N_d\},$$

and as the norm of matrix we understand the norm of a vector composed of all matrix elements. Thus the inequalities $\|R\| \geq \|R_k\| \geq \|R_{ik}\|$ holds for any $i \in N_m$ and $k \in N_s$.

As usual (see, for example,[6, 9–12]), *stability radius* of portfolio $x^0 \in P^s(R)$ is defined as follows:

$$\rho^s(x^0, R) = \begin{cases} \sup\ \Xi, & \text{if } \ \Xi \neq \emptyset, \\ 0, & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 :\ \forall R' \in \Omega(\varepsilon) \quad (x^0 \in P^s(R + R'))\},$$

$$\Omega(\varepsilon) = \{R' \in \mathbf{R}^{m\times n\times s} :\ \|R'\| < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is the set of *perturbing matrices*, and $Z^s(R + R')$ is the *perturbed problem*.

The following lemma is evident.

Lemma. *Let $x^0 \in P^s(R)$, $\varphi > 0$. If for any perturbing matrix $R' \in \Omega(\varphi)$ and any solution $x \in X \setminus \{x^0\}$ index $q \in N_s$ exists, such that the inequality $g_q(x,\ x^0,\ R_q + R'_q) > 0$ holds, then $x^0 \in P^s(R + R')$ for any $R' \in \Omega(\varphi)$.*

It is also quite evident that for any matrix $R_k \in \mathbf{R}^{m\times n}$ and any solutions $x^0,\ x \in X$ the following inequalities are true:

$$R_{ik}x - R_{i^0k}x^0 \geq -\|R_k\|\ \|x + x^0\|^*, \qquad i,\ \ i^0 \in N_m, \qquad k \in N_s, \tag{1}$$

where $\|z\|^* = \sum\limits_{j\in N_n} |z_j|$, $z = (z_1,\ z_2,\ ...,\ z_n)^T$.

3 Stability radius bounds

For portfolio $x^0 \in P^s(R)$ we introduce the following notations:

$$\varphi = \min_{x\in X\setminus\{x^0\}}\ \max_{k\in N_s}\ \min_{i^0\in N_m}\ \max_{i\in N_m}\ \frac{R_{ik}x - R_{i^0k}x^0}{\|x + x^0\|^*},$$

$$\psi = \min_{x\in X\setminus\{x^0\}}\ \max_{k\in N_s}\ \min_{i^0\in N_m}\ \max_{i\in N_m}\ \frac{R_{ik}x - R_{i^0k}x^0}{\|x - x^0\|^*}.$$

Theorem. *For stability radius $\rho^s(x^0,\ R)$, $\ s \geq 1$, of a Pareto-optimal portfolio x^0 of problem $Z^s(R)$ the following bounds are true:*

$$\varphi \leq \rho^s(x^0,\ R) \leq \psi.$$

Proof. Let $x^0 \in P^s(R)$. The formula

$$\forall x \in X \setminus \{x^0\} \quad (x^0 \notin P^s(x^0,\ R))$$

obviously holds. Hence with account of inequality $\|x + x^0\|^* \geq \|x - x^0\|^* > 0$, this results in $\psi \geq \varphi \geq 0$.

To prove Theorem, firstly it is necessary to prove that $\rho^s(x^0,\ R) \geq \varphi$, which is evident if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for any portfolio $x \in X \setminus \{x^0\}$, there is such index $q \in N_s$ that

$$\min_{i^0 \in N_m} \max_{i \in N_m} (R_{iq}x - R_{i^0q}x^0) \geq \varphi\ \|x + x^0\|^*. \tag{2}$$

Further, taking into account (1), for any perturbing matrix $R' \in \Omega(\varphi)$ and any $k \in N_s$, we have:

$$g_k(x,\ x^0,\ R_k + R'_k) = \max_{i \in N_m}(R_{ik} + R'_{ik})x - \max_{i \in N_m}(R_{ik} + R'_{ik})x^0 =$$

$$= \min_{i^0 \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i^0k}x^0 + R'_{ik}x - R'_{i^0k}x^0) \geq$$

$$\geq \min_{i^0 \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i^0k}x^0) - \|R'_k\| \|x + x^0\|^*.$$

Hence, in view of $\varphi > \|R'\| \geq \|R'_q\|$ inequality (2) implies

$$g_q(x,\ x^0,\ R_q + R'_q) > 0.$$

Therefore, due to Lemma we have $x^0 \in P^s(R + R')$ for any perturbing matrix $R' \in \Omega(\varphi)$, i.e. the inequality $\rho^s(x^0,\ R) \geq \varphi$ is true.

Further, we prove the inequality $\rho^s(x^0,\ R) \leq \psi$. In accordance with the definition of ψ there is such portfolio $x \in X \setminus \{x^0\}$ that the following inequalities are true:

$$\psi \|x - x^0\|^* \geq \min_{i^0 \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i^0k}x^0), \qquad k \in N_s. \tag{3}$$

Now, setting $\varepsilon > \psi$, we consider the perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m \times n \times s}$ whose elements are defined as follows:

$$r^0_{ijk} = \begin{cases} \delta, & \text{if } \ i \in N_m,\ x^0_j \geq x_j,\ k \in N_s, \\ \\ -\delta, & \text{if } \ i \in N_m,\ x^0_j < x_j,\ k \in N_s, \end{cases}$$

where $\psi < \delta < \varepsilon$. Then $\|R^0\| = \|R^0_k\| = \|R^0_{ik}\| = \delta$ where $i \in N_m, \quad k \in N_s$. In addition, all rows $R^0_{ik}, \quad i \in N_m$, of matrix R^0_k are equal and consist of components δ and $-\delta$ for any index $k \in N_s$. Therefore, denoting this row by B (it only depends on x and x^0), we have

$$B(x - x^0) = -\delta\|x - x^0\|^*, \qquad \|B\| = \delta.$$

Hence, in view of (3), for any index $k \in N_s$, we obtain

$$g_k(x,\ x^0,\ R_k + R^0_k) = \max_{i\in N_m}(R_{ik} + B)x - \max_{i\in N_m}(R_{ik} + B)x^0 =$$

$$= \max_{i\in N_m} R_{ik}x - \max_{i\in N_m} R_{ik}x^0 + B(x - x^0) = \min_{i^0\in N_m} \max_{i\in N_m}(R_{ik}x - R_{i^0k}x^0) + B(x - x^0) \leq$$

$$\leq (\psi - \delta)\ \|x - x^0\|^* < 0.$$

Thus, the binary relation $x^0 \underset{R+R^0}{\succ} x$ holds. Therefore, for any $\varepsilon > \psi$ there is such perturbing matrix $R^0 \in \Omega(\varepsilon)$ that Pareto-optimal portfolio x^0 of problem $Z^s(R)$ looses its Pareto-optimality in the perturbed problem $Z^s(R + R^0)$, i.e. $x^0 \notin P^s(R + R^0)$. Therefore $\rho^s(x^0,\ R) \leq \psi$. □

The upper bound ψ of the stability radius $\rho^s(x^0,\ R)$ indicated in Theorem is attainable, since for $m = 1$ our problem $Z^s(R)$ is transformed into a vector (s-criteria) Boolean programming problem with linear criteria:

$$R_k x \to \min_{x\in X}, \qquad k \in N_s, \tag{4}$$

whereas the upper bound turns into the form

$$\rho^s(x^0,\ R) \leq \psi = \min_{x\in X\setminus\{x^0\}} \max_{k\in N_s} \frac{R_k(x - x^0)}{\|x - x^0\|^*},$$

where R_k is k-th row of matrix $R \in \mathbf{R}^{s\times n}$. It is known [6, 10] that the right-hand side of this ratio is the expression of the stability radius of $x^0 \in P^s(R)$ of problem (4). Therefore, if $m = 1$, we have $\rho^s(x^0,\ R) = \psi$, that assures the attainability of this upper bound.

It is also quite evident that the lower bound φ is also attainable. Indeed, let the equality $\|x+x^0\|^* = \|x-x^0\|^*$ be true for any $x \in X\setminus\{x^0\}$, then $\rho^s(x^0,\ R) = \varphi = \psi$.

So we have the following corollary of Theorem, which shows that the radius of stability of Pareto-optimal portfolio $x^0 \in P^s(R)$ can be equal to the lower positive bound φ and may not coincide with the upper bound ψ.

Corollary 1. *There exists a class of problems $Z^s(R)$ such that for the solution $x^0 \in P^s(R)$ the following correlations are true:*

$$0 < \rho^s(x^0,\ R) = \varphi < \psi. \tag{5}$$

Proof. Let $\varphi > 0$. The inequality $\varphi < \psi$ is true if $\|x + x^0\|^* > \|x - x^0\|^*$ holds for any vector $x \in X \setminus \{x^0\}$. To prove the equality $\rho^s(x^0,\ R) = \varphi$ in accordance with Theorem, it is sufficient to identify the class of problems for which the inequality $\rho^s(x^0,\ R) \leq \varphi$ is true. Further exposition is devoted to this.

The definition of $\varphi > 0$ entails such vector $\widehat{x} \in X \setminus \{x^0\}$ that

$$\varphi \|\widehat{x} + x^0\|^* \geq g_k(\widehat{x},\ x^0,\ R_k), \qquad k \in N_s. \tag{6}$$

Further exposition will be for any index $k \in N_s$.

We introduce the following notations:

$$i(x^0) = \arg\max\{R_{ik}x^0 :\ i \in N_m\},$$

$$i(\widehat{x}) = \arg\max\{R_{ik}\widehat{x} :\ i \in N_m\},$$

$$\Delta = \|\widehat{x} + x^0\|^* - \|\widehat{x} - x^0\|^* > 0.$$

Further, we assume that the inequality holds:

$$(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} > \varphi\Delta, \tag{7}$$

which entails the inequality $i(x^0) \neq i(\widehat{x})$, since $\varphi\Delta > 0$ holds.

For any number $\varepsilon > \varphi$ we define the elements of the section R_k^0 of the perturbing matrix R^0 by the rule

$$r_{ijk}^0 = \begin{cases} \delta, & \text{if} \quad i = i(x^0), \quad x_j^0 = 1, \\[2ex] -\delta, & \text{if} \quad i = i(x^0), \quad x_j^0 = 0, \\[2ex] -\delta, & \text{if} \quad i \in N_m \setminus \{i(x^0)\}, \quad \widehat{x}_j = 1, \\[2ex] 0 & \text{otherwise}, \end{cases}$$

where

$$\min\left\{\varepsilon,\ \frac{1}{\Delta}(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x}\right\} > \delta > \varphi. \tag{8}$$

Noteworthy, the last inequalities are correct because of (7).

Due to the structure of the section R_k^0 we have

$$R_{ik}^0\widehat{x} = -\delta\|\widehat{x}\|^*, \qquad i \in N_m \setminus \{i(x^0)\}, \tag{9}$$

$$R_{i(x^0)k}^0 x^0 = \delta\|x^0\|^*, \tag{10}$$

$$\|R_k^0\| = \|R^0\| = \delta, \quad R^0 \in \Omega(\varepsilon).$$

Moreover, the equality holds:

$$R_{i(x^0)k}^0\widehat{x} = \delta(\Delta - \|\widehat{x}\|^*). \tag{11}$$

Indeed, let us denote the sets:

$$Q_1 = \{j \in N_n : \quad \widehat{x}_j = x_j^0 = 1\},$$
$$Q_2 = \{j \in N_n : \quad \widehat{x}_j = 1, \quad x_j^0 = 0\}.$$

Then the following equalities are obvious:

$$|Q_1| = \Delta/2,$$
$$|Q_2| = \|\widehat{x}\|^* - \Delta/2,$$
$$R^0_{i(x^0)k}\widehat{x} = \delta|Q_1| - \delta|Q_2|,$$

from which the inequality (11) ensues.

Further, we will prove that $g_k(\widehat{x},\ x^0,\ R_k + R_k^0) < 0$. In line with (10) we have

$$f_k(x^0,\ R_k + R_k^0) = \max_{i \in N_m}(R_{ik} + R_{ik}^0)x^0 = f_k(x^0,\ R_k) + \delta\|x^0\|^*. \tag{12}$$

We will prove that the equality is true:

$$f_k(\widehat{x},\ R_k + R_k^0) = f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^*. \tag{13}$$

Using (9), we have

$$f_k(\widehat{x},\ R_k + R_k^0) = \max\Big\{(R_{i(\widehat{x})k} + R^0_{i(\widehat{x})k})\widehat{x}, \quad \max_{i \neq i(\widehat{x})}(R_{ik} + R_{ik}^0)\widehat{x}\Big\} =$$
$$= \max\Big\{(f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^*), \quad \max_{i \neq i(\widehat{x})}(R_{ik} + R_{ik}^0)\widehat{x}\Big\}.$$

Thus, taking into account the obvious inequalities

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* \geq (R_{ik} + R_{ik}^0)\widehat{x}, \qquad i \in N_m \setminus \{i(x^0),\ i(\widehat{x})\},$$

to prove (13) we must prove that

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* \geq (R_{i(x^0)k} + R^0_{i(x^0)k})\widehat{x}.$$

To this end, using (8) and (11), we have

$$f_k(\widehat{x},\ R_k) - \delta\|\widehat{x}\|^* - (R_{i(x^0)k} + R^0_{i(x^0)k})\widehat{x} = (R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} - \delta\|\widehat{x}\|^* -$$
$$- R^0_{i(x^0)k}\widehat{x} > \delta(\Delta - \|\widehat{x}\|^*) - R^0_{i(x^0)k}\widehat{x} = 0.$$

At last, consistently applying (12), (13), (6) and (8), we obtain

$$g_k(\widehat{x},\ x^0,\ R_k + R_k^0) = g_k(\widehat{x},\ x^0,\ R_k) - \delta\|\widehat{x} + x^0\|^* \leq (\varphi - \delta)\|\widehat{x} + x^0\|^* < 0.$$

Because of that such inequality is true for any $k \in N_s$, that $x^0 \underset{R+R^0}{\succ} \widehat{x}$.

Therefore, the formula

$$\forall \varepsilon > \varphi \quad \exists R^0 \in \Omega(\varepsilon) \qquad (x^0 \notin P^s(R + R^0))$$

holds, which because of the vector $x^0 \in P^s(R)$ results in the inequality $\rho^s(x^0,\ R) \leq \varphi$. In summary, we get proof that correlation (5) is valid. $\square$

We give a numeric example proving Corollary 1.

Example. Let $m = 2$, $n = 3$, $k = 1$; $X = \{x^0,\ x^1\}$, $x^0 = (1,\ 1,\ 0)^T$, $\widehat{x} = (0,\ 1,\ 1)^T$;

$$R = \begin{pmatrix} -5 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

Then $f(x^0,\ R) = 0$, $f(\widehat{x},\ R) = 4$, i. e. x^0 is the optimal portfolio of the problem $Z^1(R)$; $\|\widehat{x} + x^0\|^* = 4$, $\|\widehat{x} - x^0\|^* = 2$, $i(x^0) = 2$, $i(\widehat{x}) = 1$. So $\varphi = 1$, $\psi = 2$, $(R_{i(\widehat{x})k} - R_{i(x^0)k})\widehat{x} = 5 > 2 = \varphi(\|\widehat{x} + x^0\|^* - \|\widehat{x} - x^0\|^*)$.

By Theorem $\rho^1(x^0,\ R) \geq 1$. On the other hand, if

$$R^0 = \begin{pmatrix} 0 & -\delta & -\delta \\ \delta & \delta & -\delta \end{pmatrix},$$

where $1 < \delta < 2.5$, then $\|R^0\| = \delta$ and $f(x^0,\ R+R^0) = 2\delta > 4-2\delta = f(\widehat{x},\ R+R^0)$.

As a result we have that $x^0 \notin P^1(R + R^0)$. Hence $\rho^1(x^0,\ R) \leq 1$. Thus, by Theorem we have $\rho^1(x^0,\ R) = \varphi = 1 < \psi = 2$.

Pareto-optimal portfolio $x^0 \in P^s(R)$ is called *stable*, if $\rho^s(x^0,\ R) > 0$. In addition, let us introduce the traditional Smale set $Sm^s(R)$ [24], i.e. the set of *strongly efficient portfolios*:

$$Sm^s(R) = \{x \in X : \ \ \forall x' \in X \setminus \{x\} \ \ \exists q \in N_s \quad (f_q(x',\ R_q) > f_q(x,\ R_q))\}.$$

Apparently, $Sm^s(R) \subseteq P^s(R)$ for any matrix $R \in \mathbf{R}^{m\times n\times s}$ and $Sm^s(R)$ can be empty.

Corollary 2. *Pareto-optimal portfolio $x^0 \in P^s(R)$ is stable iff $x^0 \in Sm^s(R)$.*

Proof. Sufficiency. Let Pareto-optimal portfolio x^0 of problem $Z^s(R)$ be strongly efficient. Then for any $x \in X \setminus \{x^0\}$ we have

$$\xi(x) = \max_{k\in N_s}\ \min_{i^0\in N_m}\ \max_{i\in N_m}\ \frac{R_{ik}x - R_{i^0k}x^0}{\|x + x^0\|^*} = \max_{k\in N_s}\ \frac{f_k(x,\ R_k) - f_k(x^0,\ R_k)}{\|x + x^0\|^*} > 0.$$

Therefore, by Theorem, we have $\rho^s(x^0,\ R) \geq \varphi = \min\limits_{x\in X\setminus\{x^0\}} \xi(x) > 0$, i.e. portfolio $x^0 \in P^s(R)$ is stable.

Necessity. Let portfolio $x^0 \in P^s(R)$ be stable. Then, according to Theorem, we obtain $\psi \geq \rho^s(x^0,\ R) > 0$. Therefore, for any portfolio $x \in X \setminus \{x^0\}$ we have

$$\max_{k\in N_s}\ \frac{f_k(x,\ R_k) - f_k(x^0,\ R_k)}{\|x - x^0\|^*} > 0.$$

It means that for any $x \in X \setminus \{x^0\}$ there is such index $q \in N_s$, that $f_q(x,\ R_q) > f_q(x^0,\ R_q)$, i.e. $x^0 \in Sm^s(R)$. □

Since from the equality $\varphi = 0$ the equality $\psi = 0$ ensues, then the following corollary results from Theorem:

Corollary 3. *If $x^0 \in P^s(R)$, then $\rho^s(x^0,\ R) = 0$ if $\varphi = 0$.*

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