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Generalized hypergeometric systems and the fifth and sixth Painlevé equations

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Abstract. This paper concerns (generalized) hypergeometric systems associated with the fifth and sixth Painlevé equations, which are the second order nonlinear ordinary differential equations. The Painlevé equations govern monodromy preserving deformations of certain second order linear scalar equations. We reduce these scalar equations to generalized hypergeometric systems.

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1 Introduction

In some problems of the general theory of ordinary differential equations (ODEs) it is very efficient to study systems of ODEs rather than single scalar equations. The benefit is that the problem can be studied by using the matrix calculus and most likely can easily be generalized. Thus, the methods of reduction of a linear differential equation with a finite number of regular and irregular singularities to a system of linear differential equations of some canonical form are needed. In general, the reduction problems are difficult and only partial results are available in this direction, see for instance [1, 9].

The current paper concerns the study of the second order linear differential equation

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \tag{1}$$

where $p_1(x)$ and $p_2(x)$ are certain rational functions (exact formulas are given in the sections below). The isomonodromy deformations of equation (1) with such choice of coefficients lead to the famous fifth and sixth Painlevé equations [12]. The solutions of these equations, the Painlevé transcendents, are nonlinear special functions which appear in many areas of modern mathematics and mathematical physics (random matrix theory, algebraic geometry, integrable systems, topological field theories and many others). The Painlevé equations are second order nonlinear differential equations of the form

$$\frac{d^2\lambda}{dt^2} = R\left(t, \lambda, \frac{d\lambda}{dt}\right),$$

where R is a rational function, having the Painlevé property, which is that their general solutions possess no movable critical points (see, for instance, [7] for definitions). Moreover, the Painlevé transcendents are not expressible in terms of classical linear special functions. Nowadays, the interest in the Painlevé equations is growing due to numerous applications.

There are many systems one can associate with the scalar differential equation (1). In this paper we are interested in the systems of the form

$$(x - B)\frac{dY}{dx} = AY, \tag{2}$$

where the matrix A does not depend on x and the diagonal elements of the matrix B include all singularities of equation (1). We call such systems generalized hypergeometric systems. If the matrix B is diagonal, then we call the system a hypergeometric system following [9]. We remark that the systems we consider can also be viewed as generalized Okubo systems, but we want to distinguish apparent singularities (there is a holomorphic basis of solutions at such points) and include them as elements of the matrix B. Apparent singularities, as will be discussed later on, play a special role in monodromy preserving deformations of equation (1), and hence we are interested in studying the problem of reduction (1) to generalized hypergeometric systems. Systems of the type (2) recently appeared in the study of the Heun equation [3].

In this paper, we first consider equation (1) with 4 regular singularities $x = 0, 1, \infty, t$ and one apparent singularity λ. The scalar equation (1) is Fuchsian in this case, and the algorithm of reduction is known [9]. We explicitly compute the hypergeometric system (2), where the 4×4 matrix B is diagonal $(0, 1, t, \lambda)$ and the constant matrix A is the sum of a lower triangular matrix and a nilpotent matrix having elements i, $i + 1$ equal to 1 and all others equal to zero. If the parameter t moves in the complex plane, the isomonodromy deformations of (1) (deformations which preserve the monodromy group of the equation) lead to the sixth Painlevé equation (P_{VI}) for the function $\lambda(t)$. From the works of Okamoto, Noumi and others it is known that the parameter space of the sixth Painlevé equation admits the action of the extended affine Weyl group. The corresponding action of the group on solutions of (P_{VI}) is known as the action of the group of Bäcklund transformations. One of such Bäcklund transformations was recently rederived in [4] from the integral transformation of 2×2 system. Thus, we are interested to understand the action of this transformation on the hypergeometric system. This gives a new insight into the nature of the Painlevé equations and their Bäcklund transformations. In particular, the action of the Bäcklund transformation gives a new hypergeometric system with a new apparent singularity and different eigenvalues and diagonal elements.

It is also possible [5] to consider other 4×4 systems, called Okubo systems, equivalent to equation (1), but the apparent singularity is not singled out there in the diagonal matrix B as in the hypergeometric system we consider. Other types of systems of differential equations associated with the sixth Painlevé equation and the action of the Bäcklund transformations on them are considered in [10, 11]. Other

Painlevé equations are also studied from this perspective, see for instance the paper [2] concerning the fourth Painlevé equation. We also remark that equation (1) gives the Heun equation as the result of the confluence process when the apparent singularity tends to one of 4 other regular singularities of (1) and the 3×3 hypergeometric system associated with the Heun equation was useful in finding the integral transformations between its solutions [3].

As the result of the confluence process when one of regular singularities of (1) associated with (P_{VI}) coalesces with another regular singularity, we get a linear equation the isomonodromy deformations of which give the fifth Painlevé equation (P_V). In this case, equation (1) has two regular singularities $x=0,\infty$, one irregular singularity $x=1$ and one apparent singularity λ (which becomes the solution of (P_V) viewed as a function of the deformation parameter t). We introduce a generalized hypergeometric system and compute it explicitly for the linear system associated with the fifth Painlevé equation. The system we present encodes the information of the generalized Riemann scheme (information about the singularities of the scalar equation) in elements of the matrix B, which is not diagonal in this case, and diagonal elements and the eigenvalues of the matrix A. We remark that the generalized Okubo type systems have been recently studied in [8], but as remarked above, the apparent singularity does not appear in the diagonal elements of the matrix B.

The paper is organized as follows. In the following two sections we consider the problems outlined above in detail. The main results and open problems are summarized in the last section.

2 A hypergeometric system associated with the sixth Painlevé equation

Equation (1) with

$$p_1(x)=\frac{1-\theta_0}{x}+\frac{1-\theta_1}{x-1}+\frac{1-\theta_2}{x-t}-\frac{1}{x-\lambda}, \tag{3}$$

$$p_2(x)=\frac{k_1(k_2+1)}{x(x-1)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}-\frac{t(t-1)H_{VI}}{x(x-1)(x-t)}, \tag{4}$$

$$\begin{aligned} t(t-1)H_{VI} &= k_1(k_2+1)(\lambda-t)+\lambda(\lambda-1)(\lambda-t)\mu^2- \\ &-(\theta_0(\lambda-1)(\lambda-t)+\theta_1\lambda(\lambda-t)+(\theta_2-1)\lambda(\lambda-1))\mu \end{aligned} \tag{5}$$

and

$$k_1+k_2+\theta_0+\theta_1+\theta_2=0 \tag{6}$$

is a Fuchsian equation with 4 regular singularities $x=0,1,\infty,t$ and one apparent singularity λ.

The sixth Painlevé equation is the following nonlinear ordinary differential equation of second order for the unknown function $\lambda(t)$:

$$\lambda'' = \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)(\lambda')^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\lambda'+$$

$$\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right),$$

where $'$ stands for the derivation with respect to the independent variable t and α, β, γ, δ are complex parameters.

One of standard ways to derive the sixth Painlevé equation is to study monodromy preserving deformations of a second order Fuchsian differential equation on $\mathbb{P}^1$ with four regular singular points and one apparent singularity [12], i.e., to consider deformations of equation (1) with (3)–(6). This leads to a system of partial differential equations, and the compatibility condition gives a Hamiltonian system

$$\frac{d\lambda}{dt}=\frac{\partial H_{VI}}{\partial\mu},\quad \frac{d\mu}{dt}=-\frac{\partial H_{VI}}{\partial\lambda} \tag{7}$$

and, hence, (P_{VI}) for the function $\lambda(t)$ with

$$\alpha=\frac{(2k_1+\theta_0+\theta_1+\theta_2-1)^2}{2},\ \ \beta=-\frac{\theta_0^2}{2},\ \ \gamma=\frac{\theta_1^2}{2},\ \ \delta=\frac{1-\theta_2^2}{2}.$$

The reader is referred to [7, 12] for further details.

Each element of the hypergeometric system (2) is written as

$$(x-\lambda_j)y_j'=\sum_{k=1}^{4}\alpha_{j,k}y_k,\ \ j\in\{1,\dots,4\},$$

where $\lambda_1=0,\ \lambda_2=1,\ \lambda_3=t,\ \lambda_4=\lambda$. The matrix B in (2) is diagonal with finite singularities of (1) on the diagonal. The matrix A in (2) is independent of x and we impose condition that it is the sum of a lower triangular matrix and a nilpotent matrix having elements i, $i+1$ equal to 1 and all others equal to zero. Hence, $\alpha_{i,j}=0,\ j>i+1$, and $\alpha_{i,i+1}=1$. Because of the special form of the system, we can find successively

$$\begin{aligned}
y_2&=xy_1'+f_0(x)y_1,\\
y_3&=x(x-1)y_1''+g_1(x)y_1'+g_0(x)y_1,\\
y_4&=x(x-1)(x-t)y_1'''+h_2(x)y_1''+h_1(x)y_1'+h_0(x)y_1
\end{aligned}$$

with some functions f, g, h of x depending on the coefficients of the matrix A and, thus, we can easily find the fourth order differential equation for the first component y_1 of the vector Y. Next, we can find conditions on the coefficients when it is reduced to equation (1) with (3), (4). The elements below the diagonal are extremely cumbersome and we do not write them here[1]. However, by direct computations and using the algorithm of [1] it can be verified that the following statement holds true.

Proposition 1. *The diagonal elements of the hypergeometric system associated with equation (1) with (3)–(6) are* $\theta_0,\ \theta_1-1,\ \theta_2-2,\ -1$ *and the eigenvalues are given by* $-2,\ -1,\ -k_1,\ k_1+\theta_0+\theta_1+\theta_2-1$.

[1]The pdf file with the matrix of the hypergeometric system is available at www.mimuw.edu.pl/∼filipuk/files/ForPaper.pdf

This proposition shows that each diagonal element of the matrix A is equal to the characteristic exponent at the respective regular singular point modulo integers. Also we have that two of the eigenvalues of the matrix A are equal to the characteristic exponents at infinity of equation (1). This, in turn, implies that the local and global behaviour of solutions of the scalar equation and the system does not change.

It is well known that the parameter space of (P_{VI}) admits the action of the extended affine Weyl group of type $D_4^{(1)}$ (see [11] and references therein). It is generated by several basic transformations. By a Bäcklund transformation we mean a transformation of dependent variables and parameters that leaves system (7) invariant. The following transformation is one of generators of the group of Bäcklund transformations. Let us define new variables $\tilde{\lambda}$, $\tilde{\mu}$ as follows:

$$\tilde{\lambda} = \lambda + \frac{k_1}{\mu},\ \tilde{\mu} = \mu,\ \tilde{k}_1 = -k_1,\ \tilde{\theta}_0 = k_1 + \theta_0,\ \tilde{\theta}_1 = k_1 + \theta_1,\ \tilde{\theta}_2 = k_1 + \theta_2. \tag{8}$$

Then one can verify directly that, if the pair (λ, μ) satisfies the Hamiltonian system (7), then the pair $(\tilde{\lambda}, \tilde{\mu})$ again satisfies the same system with new parameters $\tilde{\theta}_0,\ \tilde{\theta}_1,\ \tilde{\theta}_2, \tilde{k}_1$.

As shown in [4], this transformation appears in the result of the integral transformation of the 2×2 linear Fuchsian system. Other generators of the group of Bäcklund transformations appear in the result of simple gauge transformations [6].

Next we study the action of transformation (8) on the hypergeometric system.

Theorem 1. *The Bäcklund transformation (8) induces a new hypergeometric system associated with* (P_{VI}) *with* $B = \mathrm{diag}(0, 1, t, \lambda + k_1/\mu)$ *and a new matrix* A *which has elements* $\theta_0 + k_1,\ \theta_1 + k_1 - 1,\ \theta_2 + k_1 - 2,\ -1$ *on the diagonal and eigenvalues equal to* $-2,\ -1,\ k_1,\ 2k_1 + \theta_0 + \theta_1 + \theta_2 - 1$.

3 A generalized hypergeometric system associated with the fifth Painlevé equation

We consider equation (1) with

$$p_1(x) = \frac{1 - k_0}{x} + \frac{\eta_1 t}{(x-1)^2} + \frac{1 - \theta_1}{x - 1} - \frac{1}{x - \lambda}, \tag{9}$$

$$p_2(x) = \frac{k}{x(x-1)} - \frac{tH_V}{x(x-1)^2} + \frac{\lambda(\lambda - 1)\mu}{x(x-1)(x-\lambda)}, \tag{10}$$

$$tH_V = \lambda(\lambda-1)^2\mu^2 - (k_0(\lambda-1)^2 + \theta_1\lambda(\lambda-1) - \eta_1 t\lambda)\mu + k(\lambda - 1) \tag{11}$$

and

$$4k = (k_0 + \theta_1)^2 - k_\infty^2. \tag{12}$$

The generalized Riemann scheme giving local exponents at regular and irregular singularities of equation (1) with (9)–(12) is given in [12]. The monodromy preserving deformations lead to the Hamiltonian system (7) for the Hamiltonian H_V and, hence, to the fifth Painlevé equation given by

$$\lambda'' = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)(\lambda')^2 - \frac{1}{t}\lambda' + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda + \beta\frac{1}{\lambda}\right) + \frac{\gamma}{t}\lambda + \delta\frac{\lambda(\lambda+1)}{\lambda-1}$$

with

$$\alpha = \frac{k_\infty^2}{2},\ \beta = -\frac{k_0^2}{2},\ \gamma = \eta_1(1+\theta_1),\ \delta = -\frac{\eta_1^2}{2}.$$

for the function $\lambda(t)$.

Since equation (1) is not Fuchsian as in (P_{VI}) case above, the algorithm of [1] is not applicable and we need to find a new type of system to reduce the equation. We introduce the following generalized hypergeometric system.

Theorem 2. *The generalized hypergeometric system of equation (1) with (9)–(12) is given by*

$$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & \eta_1 t & x-1 & 0 \\ 0 & 0 & 0 & x-\lambda \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} k_0 & 1 & 0 & 0 \\ \alpha_{2,1} & \theta_1 & 1 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & -2 & 1 \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

with

$$\alpha_{2,1} = \lambda\mu - \mu + k_0(\theta_1 + \eta_1 t + 1/\lambda - 1) - k - tH_V,$$

$$\lambda(\lambda-1)\alpha_{4,3} = k_0(\lambda-1)^2 - \lambda(1 - \theta_1 + \eta_1 t - \lambda - \theta_1\lambda + (\lambda-1)^2\mu) =: q_1,$$

$$\lambda^2(\lambda-1)\alpha_{4,2} = q_1(k_0(\lambda-1) + \lambda(1+\theta_1+\mu-\lambda\mu)),$$

$$\alpha_{4,1} = \frac{q_1}{\lambda^2(\lambda-1)}\Big\{k_0^2(\lambda-1) + k_0\lambda\Big[\theta_1 + \eta_1 t + (\lambda-2)(\lambda-1)\mu\Big] -$$

$$-\lambda^2\Big[k + \mu\big(\theta_1 + \eta_1 t - \theta_1\lambda + (\lambda-1)^2\mu\big)\Big]\Big\},$$

$$(\lambda-1)\alpha_{3,2} = 1 + \theta_1 + (1+k_0)\eta_1 t + k(\lambda-1)^2 - \lambda - q_2\lambda -$$

$$-q_3(\lambda-1)\mu + \lambda(\lambda-1)^3\mu^2,$$

$$q_2 = \theta_1 + k_0\eta_1 t,\quad q_3 = k_0 - (2k_0 + \theta_1 + \eta_1 t)\lambda + (k_0+\theta_1)\lambda^2,$$

$$\lambda^2\alpha_{3,1} = \lambda^2(2k(\lambda-1) - q_4\mu + (\lambda-1)^2(2\lambda-1)\mu^2) -$$

$$-k_0^2(1+q_5\lambda) + k_0\lambda(q_6 + \eta_1 t(1 + \eta_1(1+\lambda(\lambda\mu-2)) + q_7(\lambda-1))),$$

$$q_4 = \theta_1 + \eta_1 t + \lambda - 3\theta_1\lambda - 2\eta_1 t\lambda + 2\theta_1\lambda^2 - 1,$$

$$q_5 = \eta_1 t\lambda + \lambda(\lambda-1)^2\mu - 1,\quad q_6 = \theta_1(1-\lambda)(1+\lambda^2\mu),$$

$$q_7 = 1 - 2\mu + \lambda(k + \mu(3 - 2\lambda + \lambda(\lambda-1)\mu)).$$

Substituting $y_1 = y$ into the system we require that y solves equation (1) with (9)–(12). A routine calculation shows that the matrix A in the system has eigenvalues $-2,\ -1,\ (k_0+\theta_1-k_\infty)/2,\ (k_0+\theta_1+k_\infty)/2$ which encode the information of the generalized Riemann scheme in [12]. We note that the action of the Bäcklund transformations of (P_V) on the sytem can also be studied similarly to (P_{VI}) case.

4 Conclusions

We have computed the hypergeometric system associated with the sixth Painlevé equation via (1) and studied the action of a particular Bäcklund transformation on it. We introduced a new type of systems, the generalized hypergeometric system, and reduced equation (1) associated with the fifth Painlevé equation to it. The generalized hypergeometric systems give a new type of reduction problems and are worth of further study. The generalized hypergeometric systems for other Painlevé equations and the confluence process are currently under investigation and will be published elsewhere. We expect that the hypergeometric systems could be applied to other problems concerning the Painlevé equations. It is an open (and computationally difficult) problem to study the (generalized) hypergeometric systems for the (degenerate) Garnier systems and examine their symmetries. There is some evidence [10] that new symmetries of the Garnier systems may not exist and, so, the hypergeometric systems could shed some more light on this problem.

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